A Universal Spinor Bundle and Applications to the Calculus of Variations of Spinorial Equations

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Outline

Introduction

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A Universal Spinor Jet Bundle

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Appendix

Setup of Spin Geometry

- M closed spin manifold of dimension m
- $\tau_M : TM \to M$ tagent bundle of M
- $\mathcal{S}_{r,s}(M)$ space of pseudo-Riemannian metrics of signature (r, s) with $\mathcal{C}^1$-topology, $g \in \mathcal{S}_{r,s}(M)$
- $\Theta^g : \mathrm{Spin}^g M \to \mathrm{SO}^g M$ a metric spin structure for M
- $D\!\!\!/^g : H^1(\Sigma^g M) \subset L^2(\Sigma^g M) \to L^2(\Sigma^g M)$ Dirac operator

Dirac equation:

$$D\!\!\!/^g \psi = \lambda\psi, \qquad 0 \neq \psi \in \Gamma(\Sigma^g M), \qquad \lambda \in \mathbb{R}$$

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How to formulate if many metrics are involved?

Einstein-Dirac Equation

Definition: For any $\lambda, \varepsilon \in \mathbb{R}$, the system of equations

$$\not{D}^g \psi = \lambda\psi$$
$$\mathrm{Ricc}^g - \tfrac{1}{2}\,\mathrm{scal}^g\, g = \frac{\varepsilon}{4} T_{(g,\psi)},$$

is called *Einstein-Dirac equation*.

Here, $T_{(g,\psi)} \in T^2(M)$ is the *energy momentum tensor of* ψ defined by

$$\forall X, Y \in \mathcal{T}(M) : T_{(g,\psi)}(X,Y) := \tfrac{1}{2}\, Re\langle X \cdot \nabla^g_Y \psi + Y \cdot \nabla^g_X \psi, \psi\rangle,$$

Einstein-Dirac Functional

Definition: The *Einstein-Dirac functional* $\mathcal{L}$ is given by

$$(g,\psi) \mapsto \int_M \mathrm{scal}^g + \varepsilon\lambda\langle\psi,\psi\rangle - \varepsilon\langle \not{D}^g \psi,\psi\rangle dv^g,$$

where g is a Riemannian metric and $\psi \in \Gamma(\Sigma^g M)$ is a spinor field.

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What is the domain of definition for $\mathcal{L}$?

A known way out

Theorem: The map

$$\begin{array}{rcl} L^2(\Sigma M) := \coprod_{g\in\mathcal{S}_{m,0}(M)} L^2(\Sigma^g M) & \to & S_{m,0}(M) \\ \psi \in L^2(\Sigma^g M) & \mapsto & g \end{array}$$

is a continuous bundle of Hilbert spaces such that, the *identification isomorphisms*

$$\bar{\beta}_{g,h} : L^2(\Sigma^g M) \to L^2(\Sigma^h M), \qquad g, h \in S_{m,0}(M)$$

(constructed in Bourguignon/Gauduchon '92 for the Riemannian case and in Bär/Gauduchon/Moroianu '03 for Lorentzian metrics) provide a global trivialization
$\bar{\beta}_{g,\_} : L^2(\Sigma M) \to L^2(\Sigma^g M) \times S_{m,0}(M)$.

Pros and Cons

- As $L^2(\Sigma M)$ has an infinite-dimensional fibre over an infinite dimensional base, the topology is very "large".
- Identification isomorphisms $\beta_{g,h}$ are well known and often sufficient to carry out local computations.
- Hard to study global questions.
- Can we find a natural finite dimensional bundle that captures the spinor bundles of a spin manifold with respect to all metrics including their Dirac structure?

Recall: Dirac Structure

On a metric spinor bundle $\pi_M^g : \Sigma^g M \to M$ there exists

- an extention of the Levi-Civita connection $\nabla^g : \Gamma(\pi_M^g) \to \Gamma(\tau_M^* \otimes \pi_M^g)$,
- an extension of the metric g to the spinor bundle,
- a Clifford multiplication $\mathfrak{m}^g : TM \otimes_{\mathbb{R}} \Sigma^g M \to \Sigma^g M$, $V \otimes \psi \mapsto V \cdot_g \psi$

such that for all $\psi, \psi' \in \Gamma(\pi_M^g)$, $V, W \in \Gamma(\tau_M)$

$$\begin{aligned} -2g(V,W)\psi &= V \cdot_g W \cdot_g \psi + W \cdot_g V \cdot_g \psi, \\ \nabla_V^g g(\psi, \psi') &= g(\nabla_V^g \psi, \psi') + g(\psi, \nabla_V^g \psi'), \\ \nabla_V^g (W \cdot_g \psi) &= \nabla_V^g W \cdot_g \psi + W \cdot_g \nabla_V^g \psi, \\ g(V \cdot_g \psi, \psi') &= (-1)^{s+1} g(\psi, V \cdot_g \psi'). \end{aligned}$$

Main Theorem I

Main Theorem: There exists a finite dimensional vector bundle $\bar{\pi}^{\Sigma}_{SM} : \bar{\Sigma}M \to J^1\pi^{r,s}$ such that for each pseudo-Riemannian metric $g \in \mathcal{S}_{r,s}(M)$, the associated metric spinor bundle $\pi^g : \Sigma^g M \to M$ can be recovered from it (including the Dirac structure), i.e. there exists there exists a morphism of (generalized) Dirac bundles $\bar{\iota}_g$ such that

$$\begin{array}{ccc} \Sigma^g M & \overset{\bar{\iota}_g}{\dashrightarrow} & \bar{\Sigma}M \\ \downarrow{\scriptstyle \pi^g_M} & & \downarrow{\scriptstyle \bar{\pi}^{\Sigma}_{SM}} \\ M & \xrightarrow{j^1(g)} & J^1\pi^{r,s} \end{array}$$

commutes. Here, $J^1\pi^{r,s}$ denotes the first jet bundle of $\pi^{r,s} : S_{r,s}M \to M$. In addition, $\bar{\pi}^{\Sigma}_{SM}$ is *natural with respect to spin diffeomorphisms*.

Main Theorem II

Main Theorem: There exists a maximal Cauchy development for the Einstein-Dirac equation on Lorentzian manifolds.

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On Vector Spaces

Definition: Let V be an oriented real m-dimensional vector space, $\mathrm{GL}^+ V$ be the set of oriented bases of V and $S_{r,s}V$ the set of non-degenerate symmetric bilinear forms of signature (r,s) on V. The canonical map

$$\begin{array}{rcl} \kappa^V : \mathrm{GL}^+ V & \to & S_{r,s}V \\ b & \mapsto & g_b \end{array}$$

such that g_b is b-pseudo-orthonormal is surjective and there exists a diffeomorphism such that the follwing diagram commutes:

$$\begin{array}{ccccc} \widetilde{\mathrm{GL}}^+ V & \xrightarrow[2:1]{\theta} & \mathrm{GL}^+ V & \xrightarrow{\kappa^V} & S_{r,s}V \\ \downarrow & & \downarrow\!\!\downarrow & \overset{\cong}{\nearrow} & \\ \widetilde{\mathrm{GL}}^+ V/\operatorname{Spin}_{r,s} & \xrightarrow{\cong} & \mathrm{GL}^+ V/\mathrm{SO}_{r,s}\,. & & \end{array}$$

We set $\tilde{\kappa}^V := \kappa^V \circ \theta$.

Connection

For any two bases $b, b' \in \mathrm{GL}^+ V$, let $\tau_b(b') \in \mathbb{R}^{m\times m}$ be the coordinate matrix defined by $b'_j = \tau_b(b')^i_j b_i$, $1 \leq j \leq m$. For any $X, Y \in T_b \mathrm{GL}^+ V$, we define

$$\langle X, Y\rangle_b := \langle d\tau_b X, d\tau_b Y\rangle := \mathrm{tr}((d\tau_b X)^\dagger d\tau_b Y),$$

where $A^\dagger := I_{r,s} A^T I_{r,s}$. Then the $\langle \_, \_ \rangle_b$ assemble to a natural pseudo-Riemannian metric on $\mathrm{GL}^+ V$ such that $\mathrm{SO}_{r,s}$ acts by isometries. In particular,

$$T^v_b \mathrm{GL}^+ V := \ker d_b \kappa^V, \qquad T^h \mathrm{GL}^+ V := (T^v_b \mathrm{GL}^+ V)^\perp,$$

defines an orthogonal decomposition such that $T^h \mathrm{GL}^+ V$ is a connection on $\mathrm{GL}^+ V$, the *Bourguignon-Gauduchon horizontal distribution*. (Again, lifts to universal cover $\tilde{\kappa}^V : \widetilde{\mathrm{GL}}^+ V \to S_{r,s} V$.)

Universal Spinor Bundle

Theorem: There exists a finite-dimensional vector bundle π^{Σ}_{SM},

$$\Sigma M := \widetilde{\mathrm{GL}}^+ M \times_{\rho_{r,s}} \Sigma_{r,s} \xrightarrow{\pi^{\Sigma}_{SM}} S_{r,s}M \xrightarrow{\pi^{r,s}} M, \qquad \pi^{\Sigma}_{M} : \Sigma M \to M$$

natural w.r.t. spin diffeomorphisms, together with

- a *vertical connection* $\boldsymbol{\nabla} : \Gamma(\tau^{v}_{S_{r,s}M}) \times \Gamma(\pi^{\Sigma}_{SM}) \to \Gamma(\pi^{\Sigma}_{SM})$,
- a *universal spinorial metric* $\boldsymbol{\eta}$ on π^{Σ}_{SM}
- a *universal Clifford multiplication* $\mathfrak{m} : (\pi^{r,s})^*(TM) \otimes \Sigma M \to \Sigma M$

compatible with the *universal pseudo-Riemannian*

- *metric* $\boldsymbol{g}$ on $\tau^{r,s}_{M} := (\pi^{r,s})^*(\tau_M)$.
- *vertical connection* $\nabla : \Gamma(\tau^{v}_{\pi^{r,s}}) \times \Gamma(\tau^{r,s}_{M}) \to \Gamma(\tau^{r,s}_{M})$.

Universal Structures

Definition: Let $\phi, \phi' \in \Sigma M|_{g_x}$, $X^* = (g_x, V) \in (\pi^{r,s})^*(TM)$, $V \in T_x M$. We define the *universal pseudo-Riemannian metric* by

$$\mathbf{g}(X^*, X^*) := g_x(V, V),$$

the *universal spinorial metric* by

$$\boldsymbol{\eta}(\phi, \phi') := g_x(\phi, \phi'),$$

and the *universal Clifford multiplication*

$$\mathfrak{m}(X^* \otimes \phi) := X^* \bullet \phi := V \cdot_{g_x} \phi.$$

Properties of universal structures

Lemma: The universal structures satisfy the compatibility conditions

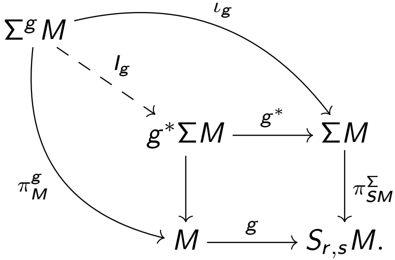
$$
\begin{aligned}
-2\boldsymbol{g}(X^*, X^*) &= X^* \bullet Y^* \bullet \psi + Y^* \bullet X^* \bullet \psi \\
\boldsymbol{\nabla}_X(\boldsymbol{\eta}(\phi, \phi')) &= \boldsymbol{\eta}(\boldsymbol{\nabla}_X \phi, \phi') + \boldsymbol{\eta}(\phi, \boldsymbol{\nabla}_X \phi'), \\
\boldsymbol{\nabla}_X(Y^* \bullet \phi) &= \nabla_X Y^* \bullet \phi + Y^* \bullet \boldsymbol{\nabla}_X \phi, \\
\boldsymbol{\eta}(X^* \bullet \phi, \phi') &= (-1)^{s+1} \boldsymbol{\eta}(\phi, X^* \bullet \phi).
\end{aligned}
$$

where $X^*, Y^* \in \Gamma(\tau_M^{r,s})$, $X \in \Gamma(\tau^v_{S_{r,s}M})$, $\phi, \phi' \in \Gamma(\pi^{\Sigma}_{SM})$.

We call this a *generalized Dirac structure*.

Pullback Theorem

Theorem: For any metric g, there exists a morphism I_g of vector bundles such that



commutes. In addition, I_g is an isometric isomorphism with respect to the spinorial metric on π^g_M and $g^*\boldsymbol{\eta}$ and it is compatible with the Clifford multiplications $\mathfrak{m}^g$ and $g^*\mathfrak{m}$.

Problem:

But I_g is **not** compatible with the vertical connections, since we would have to check that

$$(g^*\boldsymbol{\nabla})_V(I_g(\psi)) = \boldsymbol{\nabla}_{dgV}(\iota_g(\psi)) = \nabla^g_{dgV}\psi,$$

which makes absolutely no sense, since a **horizontal** lift dgV is certainly **not** vertical, so $\boldsymbol{\nabla}_{dgV}$ is **not defined**.

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Jet Spaces

Definition: Let $\pi^X : X \to M$ be a smooth fibre bundle. For any $p \in M$, denote by $\Gamma_p(\pi^X)$ the space of sections defined on a neighbourhood near p. Two such sections s_1, s_2 *have the same* 1*-jet at* $p \in U$, if $s_1(p) = s_2(p) \in X$ and $ds_1|_{T_pM} = ds_2|_{T_pM}$. The equivalence class $j^1_p(s)$ of a local section $s \in \Gamma_p(\pi^X)$ is the 1*-jet of* s *at* p. The set

$$J^1\pi^X := \{j^1_p(s) \mid p \in M, s \in \Gamma_p(\pi^X)\}$$

is the *first jet space of* π. The space $J^1\pi^X$ comes along with two canonical projections called the *source* respectively *target projection*:

$$\begin{aligned} \pi^X_1 : J^1\pi^X &\to M & \qquad \pi^X_{1,0} : J^1\pi^X &\to X \\ j^1_p(s) &\mapsto p & j^1_p(s) &\mapsto s(p) \end{aligned}$$

Relevance to Geometry

Let $g \in \mathcal{S}_{r,s}(M)$ be a pseudo-Riemannian metric on M. This metric induces a *Levi-Civita connection* and its Christoffel symbols Γ^i_{jk} are given by

$$2\Gamma^i_{jk} = g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}),$$

which depends only on the 1-jet of g. Consequently, the Levi-Civita connection and therefore the spinorial Levi-Civita connection depend only on the 1-jet of g.

Holonomic Lifts

Theorem: For any fibre bundle $\pi^X : X \to M$ be a fibre bundle. Consider the pull-back diagram

$$\begin{array}{ccccc} \pi_{1,0}^*(TX) & \longrightarrow & TX & \hookleftarrow & T^vX \\ \downarrow & & \downarrow{\scriptstyle \tau_X} & \swarrow{\scriptstyle \tau_X^v} & \\ J^1\pi^X & \xrightarrow{\pi_{1,0}^X} & X. & & \end{array}$$

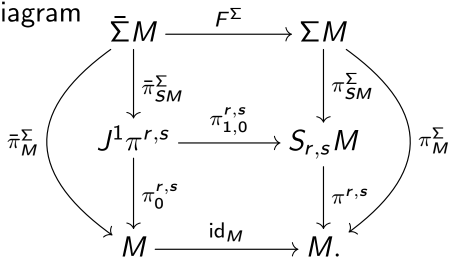
For any $j_p^1(s) \in J^1\pi^X$, there exists a natural decomposition

$$\pi_{1,0}^*(TX)|_{j_p^1(s)} = \pi_{1,0}^*(T^vX)|_{j_p^1(s)} \oplus \underbrace{(j_p^1(s), ds(T_pM))}_{\in \pi_{1,0}^*(TX)|_{j_p^1(s)}}.$$

This decomposition is well-defined (i.e. does not depend on the choice of s for a given $j_p^1(s)$). For any $Y \in T_pM$ the tuple $(j_p^1(s), ds(Y))$ is called a *holonomic lift*.

Universal Spinor Jet Bundle

Theorem: The universal spinor bundle can be extended to a commutative diagram



Moreover, the vector bundle $\bar{\pi}^{\Sigma}_{SM}$ carries a connection $\bar{\boldsymbol{\nabla}}$ satisfying

$$F^{\Sigma}(\bar{\boldsymbol{\nabla}}_{\bar{X}}\bar{\phi}|_{j^1_x(g)}) = \boldsymbol{\nabla}_{X^v}\phi|_{g(x)} + \nabla^{g}_{X_h}(\phi \circ g)|_x,$$

where $\bar{\phi} := (\mathrm{id}, \pi^{r,s}_{1,0} \circ \phi) \in \Gamma(\bar{\pi}^{\Sigma}_{SM})$, $\phi \in \Gamma(\pi^{\Sigma}_{SM})$, $\bar{X} \in T_{j^1_x(g)}(J^1\pi^{r,s})$, $X := d\pi^{r,s}_{1,0}\bar{X}$ and X^v and X_h are the vertical and horizontal part. Here, $\pi^{r,s}_{1,0} = j^1_{1,0}\pi^{r,s}$ and $\pi^{r,s}_{0} := j^1_0\pi^{r,s}$.

Universal Dirac structure

Definition: Consider the vector bundle $\bar{\pi}^{\Sigma}_{SM} : \bar{\Sigma}M \to J^1\pi^{r,s}$ We define

$$\begin{aligned} \bar{\mathbf{g}}(\bar{X}^*, \bar{Y}^*) &:= g_x(V, W), \\ \bar{\boldsymbol{\eta}}_{j^1_x(g)}(\bar{\phi}, \bar{\phi}') &:= \boldsymbol{\eta}_{g_x}(\phi, \phi'), \\ \bar{\mathfrak{m}}(\bar{X}^* \otimes \bar{\phi}) &:= \bar{X}^* \,\bar{\bullet}\, \bar{\phi} := V \cdot_{g_x} \phi, \end{aligned}$$

where $j^1_x(g) \in J^1\pi^{r,s}$,
$\bar{X}^* = (j^1_x(g), V)$, $\bar{Y}^* = (j^1_x(g), W) \in (\pi^{r,s}_0)^*(TM)$, $\bar{\phi}, \bar{\phi}' \in \Gamma(\bar{\pi}^{\Sigma}_{SM})$,
$\phi := F^{\Sigma}(\bar{\phi})$, $\phi' := F^{\Sigma}(\bar{\phi}')$.

Compatibility Relations

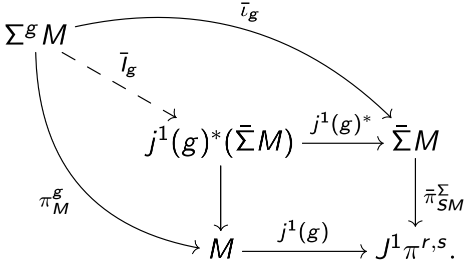
Lemma: The universal structures satisfy the following compatibility relations:

$$\begin{aligned}
-2\bar{\mathbf{g}}(\bar{X}^*, \bar{Y}^*)\bar{\phi} &= \bar{X}^* \mathbin{\bar{\bullet}} \bar{Y}^* \mathbin{\bar{\bullet}} \bar{\phi} + \bar{Y}^* \mathbin{\bar{\bullet}} \bar{X}^* \mathbin{\bar{\bullet}} \bar{\phi}, \\
\bar{\boldsymbol{\nabla}}_{\bar{X}} \bar{\boldsymbol{\eta}}(\bar{\phi}, \bar{\phi}') &= \bar{\boldsymbol{\eta}}(\bar{\boldsymbol{\nabla}}_{\bar{X}} \bar{\phi}, \bar{\phi}') + \bar{\boldsymbol{\eta}}(\bar{\phi}, \bar{\boldsymbol{\nabla}}_{\bar{X}} \bar{\phi}'), \\
\bar{\boldsymbol{\nabla}}_{\bar{X}}(\bar{Y}^* \mathbin{\bar{\bullet}} \bar{\phi}) &= \bar{\nabla}_{\bar{X}} \bar{Y}^* \mathbin{\bar{\bullet}} \bar{\phi} + \bar{Y}^* \mathbin{\bar{\bullet}} \bar{\boldsymbol{\nabla}}_{\bar{X}} \bar{\phi}, \\
\bar{\boldsymbol{\eta}}(\bar{X} \mathbin{\bar{\bullet}} \bar{\phi}, \bar{\phi}') &= (-1)^{s+1} \bar{\boldsymbol{\eta}}(\bar{\phi}, \bar{X} \mathbin{\bar{\bullet}} \bar{\phi}'),
\end{aligned}$$

where $\bar{X} \in TJ^1\pi^{r,s}$, $\bar{X}^*, \bar{Y}^* \in (\pi_0^{r,s})^*(TM)$, $\bar{\phi}, \bar{\phi}' \in \Gamma(\bar{\pi}^{\Sigma}_{SM})$.

Pullback Property

Theorem: For every metric g on M, there exists a morphism $\bar{I}_g$ of vector bundles such that



commutes. In addition, $\bar{I}_g$ is isometric with respect to the spinorial metric on π^g_M and $j^1(g)^*\bar{\boldsymbol{\eta}}$, it is compatible with the Clifford multiplication $\mathfrak{m}^g$ and $j^1(g)^*\overline{\mathfrak{m}}$ and it is compatible with the spinorial Levi-Civita connection on π^g_M and $j^1(g)^*\bar{\boldsymbol{\nabla}}$.

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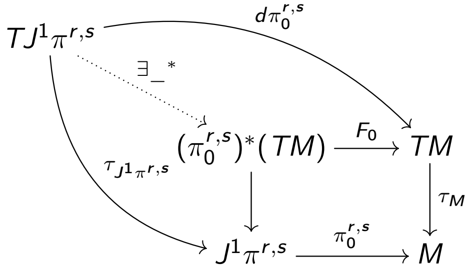
A technicality on Jet Spaces

There exists $\_^*$ such that



Universal Killing Operator

$$\begin{array}{rcl} \bar{K}_\lambda := \bar{\boldsymbol{\nabla}} - \lambda \bar{\mathfrak{m}}^* : \Gamma(\bar{\pi}^{\Sigma}_{SM}) & \to & \Gamma(\tau^*_{J^1\pi^{r,s}} \otimes \bar{\pi}^{\Sigma}_{SM}) \\ \bar{\phi} & \mapsto & (\bar{X} \mapsto \bar{\boldsymbol{\nabla}}_{\bar{X}}\bar{\phi} - \lambda \bar{X}^* \,\bar{\bullet}\, \bar{\phi}) \end{array}$$

For a universal spinor field $F^{\Sigma}(\bar{\phi}) = \phi \circ g =: \psi$, we obtain $\bar{K}_\lambda(\bar{\phi})|_{j^1(g)} = 0$ if and only if

$$\forall x \in M : \forall \bar{X} \in T_{j^1_x(g)} J^1\pi^{r,s} : \boldsymbol{\nabla}_{X^v}\phi|_{g(x)} + \nabla^g_{X_h}\psi|_x = \lambda X_h \cdot_{g_x} \psi$$

Some Observations / Questions

Assume $\bar{\phi}$ satisfies

$$\forall x \in M : \forall \bar{X} \in T_{j^1_x(g)} J^1\pi^{r,s} : \boldsymbol{\nabla}_{X^v}\phi|_{g(x)} + \nabla^g_{X_h}\psi|_x = \lambda X_h \cdot_{g_x} \psi.$$

- If ϕ is vertically parallel, then ψ is a g-Killing spinor. Conversely, can one always extend a Killing spinor to a vertically parallel $\bar{\phi}$?
- Does $\bar{\phi}$ have any interesting properties even if its not vertically parallel?
- Can we use this framework to obtain results about pseudo-Riemannian Killing spinors?

Thank you for your attention!

- Get in touch: mail@nikno.de or www.nikno.de
- Get the PDF slides: http://bit.ly/2tloYAg
- Talk to Olaf: olaf.mueller@uni-regensburg.de
- Download the paper: http://arxiv.org/abs/1504.01034
- Some references:

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Some more References

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Outline

Universal Levi Civita Connection

Definition: Via pullback

$$\begin{array}{ccc} (\pi^{r,s})^*(\mathrm{GL}^+ M) & \longrightarrow & \mathrm{GL}^+ M \\ \Big\downarrow \pi^+_{r,s} & & \Big\downarrow \pi^+ \\ S_{r,s}M & \xrightarrow{\pi^{r,s}} & M \end{array}$$

we obtain a vertical distribution on $\pi^+_{r,s}$. Let $\gamma : \mathrm{GL}^+_m \to \mathrm{GL}(\mathbb{R}^m)$ be the standard representation (given by matrix multiplication). Recall that $\mathrm{GL}^+ M \times_\gamma \mathbb{R}^m = TM$, so $(\pi^{r,s})^*(TM) = (\pi^{r,s})^*(\mathrm{GL}^+ M) \times_\gamma \mathbb{R}^m$ and therefore, we obtain a vertical connection on $\tau^{r,s}_M : (\pi^{r,s})^*(TM) \to S_{r,s}M$, which is denoted by

$$\nabla : \Gamma(\tau^v_{\pi^{r,s}}) \times \Gamma(\tau^{r,s}_M) \to \Gamma(\tau^{r,s}_M).$$

We call ∇ the *vertical universal Levi-Civita connection*.

Universal Levi-Civita connection

Definition: For any $\bar{X} \in T_{j_x^1(g)} J^1\pi^{r,s}$,
$\bar{Y}^* = (j_x^1(g), V) \in (\pi_0^{r,s})^*(TM)$, $\bar{Y} = (g_x, V) \in (\pi^{r,s})^*(TM)$,
$V \in T_x M$, we set

$$\bar{\nabla}_{\bar{X}} \bar{Y}^*|_{j_x^1(g)} := \nabla_{X^v} Y^* + \nabla^g_{X_h} V,$$

where $d\pi_{1,0}^{r,s} \bar{X} =: X = X^v \oplus d_x g X_h$ is decomposed into its vertical part and horizontal lift. The connection $\bar{\nabla}$ is called *universal Levi-Civita connection*, c.f. Pérez/Masqué 08.

Details on Vertical Connection

Definition: For any $x \in M$, we define

$$T_x^{vv} \operatorname{GL}^+ M := T^v \operatorname{GL}^+(T_x M), \quad T_x^{vh} \operatorname{GL}^+ M := T^h \operatorname{GL}^+(T_x M),$$

and analogously for $\widetilde{\operatorname{GL}}^+ M$. The resulting decomposition $T^v \operatorname{GL}^+ M = T^{vv} \operatorname{GL}^+ M \oplus T^{vh} \operatorname{GL}^+ M$ is called a *vertical distribution* on $\kappa^M : \operatorname{GL}^+ M \to S_{r,s} M$ (and analogously on $\tilde{\kappa}^M : \widetilde{\operatorname{GL}}^+ M \to S_{r,s} M$). We denote by

$$\boldsymbol{\nabla} : \Gamma(\tau^v_{S_{r,s}M}) \times \Gamma(\pi^\Sigma_{SM}) \to \Gamma(\pi^\Sigma_{SM})$$

the induced *vertical connection*, i.e. the connection induced on the associated bundle $\pi^\Sigma_{SM} : \Sigma M \to S_{r,s} M$ that is only defined for all directions in the vertical space $\tau^v_{S_{r,s}M} : T^v S_{r,s} M \to M$.