Schwarzschild Spacetimes and Black Holes

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Figure 1: The Black Hole at the Battle of Hephaistos, 9784 CY

1 Definition of the Schwarzschild Spacetime

1.1 Physical Motivation

With Einstein's theory of General Relativity in the starting block as possible replacement of the Newtonian theory of gravitation, the first natural question is of course if it correctly describes the motion of the planets around the sun. In order to test this, one needs to find the solution for the gravitational field (in the sense of General Relativity) of the sun. To a good approximation, this is given by the *Schwarzschild solution*.

The Schwarzschild solution is a static, spherically symmetric solution to the vacuum Einstein equations. As said above, it can be considered as one half of the model of a spherically symmetric (and static) massive object, namely the part describing the exterior gravitational field. The other half of the model has to describe the interior of the star, hence it is *not* a vacuum solution. One has to take a (spherically symmetric) solution for a suitable matter model (e.g., a fluid), and match the two solutions at the radius of the star. We will not think about the interior solution at all, though we might make some comments along the way.

1.1 Remark. One can also ask for solutions of (spherically symmetric) charged objects, or objects which rotate. This leads to the so-called *Reissner-Nordström solution*, the *Kerr solution*, and the *Kerr-Newman solution*.

1.2 Spacetime Symmetries in General Relativity

We start with a short discussion about symmetries in General Relativity. In particular, we want to find a way to express *spherical symmetry* and the notion of a spacetime to be "time-independent".

1.2 Definition (G-symmetric)**.** Let G be a Lie group. We say a semi-Riemannian manifold (M, g) is *G-symmetric* if there exists an injective Lie group homomorphism $G \hookrightarrow \mathrm{Iso}(M, g)$.

1.3 Example.

(i) $(\mathbb{R}^n, g_{\mathrm{Euclid}})$ is $\mathrm{SO}(n)$-symmetric.

(ii) $(\mathbb{S}^{n-1}, g^\circ)$ is also $\mathrm{SO}(n)$-symmetric.

(iii) $(\mathbb{R}^{1,n}, g_{\mathrm{Mink}})$ is $\mathrm{SO}(1, n)$-symmetric.

(iv) $(\mathbb{R}^{1,n}, g_{\mathrm{Mink}})$ is also $\mathrm{SO}(n)$-symmetric via $A : (t, x) \mapsto (t, Ax)$ for any $A \in \mathrm{SO}(n)$.

(v) $(\mathbb{R}^{1,n}, g_{\mathrm{Mink}})$ is $(\mathbb{R}, +)$-symmetric via : $s(t, x) \mapsto (t + s, x)$ for any $s \in \mathbb{R}$.

First we start by explaining how to model "time-independent" spacetimes, since this is simpler than spherical symmetry.

1.4 Definition (Stationary and Static)**.** Let (M, g) be a Lorentzian manifold.

(i) (M, g) is said to be *stationary* if it is $(\mathbb{R}, +)$-symmetric, and if all orbits are timelike curves. Equivalently, (M, g) is stationary if there exists a complete timelike Killing vector field (the generator of the $\mathbb{R}$-action).

(ii) (M, g) is said to be *static*, if it is stationary with some complete timelike Killing vector field U which additionally is *irrotational*, i.e. satisfies $U^\flat \wedge \mathrm{d}U^\flat = 0$.

1.5 Example. The Minkowski spacetime is static. A suitable $\mathbb{R}$-action is the one of Example 1.3 (v). The corresponding Killing vector field is ∂_t. This is irrotational since $\partial_t^\flat = -\,\mathrm{d}t$, hence alrady $\mathrm{d}\partial_t^\flat = -\,\mathrm{d}^2 t = 0$.

Intuitively, the notion of static means that an observer traveling through the spacetime along the flow of the Killing field does not observe any changes in his surrounding space. This is made more precise by the following theorem.

1.6 Theorem. Let (M, g) be a simply connected static spacetime. Then there exists a spacelike hypersurface $\Sigma \subset M$ and a diffeomorphism $M \cong \mathbb{R}_t \times \Sigma_x$ such that the metric is given by

$$g = -\beta(x)^2 \,\mathrm{d}t^2 + h\,, \tag{1.1}$$

where $h = g|_\Sigma$ is the Riemannian metric on Σ induced by g, and where $\beta \in C^\infty(\Sigma)$ is a smooth, positive function on Σ.

Proof. Let U be the static vector field. Using that $U^\flat \wedge \mathrm{d}U^\flat = 0$, one can show that the one-form $\omega = -\frac{1}{g(U,U)} U^\flat$ is closed, i.e. $\mathrm{d}\omega = 0$. Since M is simply connected, every closed form is exact, thus there exists a smooth function $\tau : M \to \mathbb{R}$ such that $\omega = \mathrm{d}\tau$. Since we can always add a constant to τ, we may assume that $0 \in \mathrm{im}(\tau)$. Now one verifies that $\Sigma := \tau^{-1}(0)$ is a spacelike hypersurface, that the flow of the vector field U induces a diffeomorphism between M and $\mathbb{R} \times \Sigma$, and that the pullback of g under this diffeomorphism is given by (1.1) with $\beta(x) = -g(U(x), U(x))$ for any $x \in \Sigma$. □

1.7 Remark. If M is not simply connected, then the statement of the previous theorem still holds locally, i.e. in small neighborhoods. This means that one can always introduce local coordinates in which the metric is of the form (1.1).

Now we turn to spherical symmetry, which is more involved than staticity.

1.8 Definition (spherically symmetric)**.** We say a Lorentzian manifold (M, g) is *spherically symmetric in the weak sense* if it is $\mathrm{SO}(n)$-symmetric. We say it is *spherically symmetric in the strong sense* if additionally all orbits of the $\mathrm{SO}(n)$-action are spacelike submanifolds isometric to $(\mathbb{S}^{n-1}(r), g^\circ)$ for some $r > 0$.

1.9 Example.

(i) Minkowski spacetime is spherically symmetric in the weak sense for the usual $\mathrm{SO}(n)$-action on the spatial part described in Example 1.3 (iv). Notice that the quotient by this group action is given by

$$\mathbb{R}^{1,n} / \,\mathrm{SO}(n) = \mathbb{R}_t \times [0, \infty)_r \times \mathbb{S}^{n-1}\,,$$

which is a manifold with boundary.

(ii) Minkowski spacetime with the spatial origins removed, i.e. $M = \mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$, is spherically symmetric in the strong sense for the same $\mathrm{SO}(n)$-action as in (i). This time the quotient is

$$\mathbb{R} \times (\mathbb{R}^n \setminus \{0\}) / \mathrm{SO}(n) = \mathbb{R}_t \times (0, \infty)_r \times \mathbb{S}^{n-1} ,$$

which is an ordinary manifold without boundary.

The reason for distinguishing between weak and strong spherical symmetry is precisely that one would like split the spatial part as $(a, b) \times \mathbb{S}^2$, i.e. one would like to be able to introduce spherical coordinates.

Now we put the notions of staticity and spherical symmetry together.

1.10 Definition. A Lorentzian manifold (M, g) is said to be *spherically symmetric and static in a compatible sense* if it is spherically symmetric and static, and if moreover the two group actions $\Phi : (\mathbb{R}, +) \to \mathrm{Iso}(M, g)$ and $\theta : \mathrm{SO}(n) \to \mathrm{Iso}(M, g)$ are compatible in the sense that $\Phi_s \circ \theta_A = \theta_A \circ \Phi_s$ for all $s \in \mathbb{R}$ and $A \in \mathrm{SO}(n)$.

The meaning of this definition is that an observer traveling along the flow of the static Killing vector field observes a spherically symmetic geometry in his restspaces. This is made more precise by the following theorem.

1.11 Theorem. Let (M^4, g) be four dimensional, simply connected, and spherically symmetric and static in a compatible sense. Let $M \cong \mathbb{R}_t \times \Sigma_x$ and $g \cong -\beta(x)^2 \, \mathrm{d}t^2 + h$ be the static splitting as done in Theorem 1.11. Then the isometric $\mathrm{SO}(3)$-action on (M, g) maps $\{t\} \times \Sigma$ to itself for every $t \in \mathbb{R}$. Therefore, it restricts to an isometric $\mathrm{SO}(3)$-action on (Σ, h).

Proof. If one differentiates $\Phi_s \circ \theta_A = \theta_A \circ \Phi_s$ with respect to s, one obtains that the static vector field U is invariant under θ in the sense that $(\theta_A^* U)|_p = U|_{\theta_A(p)}$ for all $p \in M$. From this it now follows that U must be orthogonal to the orbits of the $\mathrm{SO}(n)$-action. Otherwise the tangential projection of U onto an orbit would yield a nonvanishing (since invariant) vector field on this orbit, and this is not possible since the orbits are diffeomorphic to $\mathbb{S}^2$.

This orthogonality implies that the $\mathrm{SO}(n)$-action leaves invariant the splitting of M in space and time in the sense that it leaves invariant every hypersurface $\{t\} \times \Sigma$. To see this, notice that for any $\xi \in \mathfrak{so}(3)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \tau(\theta_{\mathrm{e}^{\lambda\xi}(p)}) = \mathrm{d}\tau(\frac{\mathrm{d}}{\mathrm{d}\lambda} \theta_{\mathrm{e}^{\lambda\xi}(p)}) = -\frac{1}{g(U, U)} g(U, \frac{\mathrm{d}}{\mathrm{d}\lambda} \theta_{\mathrm{e}^{\lambda\xi}(p)}) = 0 .$$

Here we used that $\mathrm{d}\tau = \omega = -\frac{1}{g(U,U)} U^\flat$, and that $\frac{\mathrm{d}}{\mathrm{d}\lambda} \theta_{\mathrm{e}^{\lambda\xi}(p)}$ is tangential to the orbit. This shows that the $\mathrm{SO}(n)$-action leaves $\{t\} \times \Sigma$ invariant, since the function τ stays unchanged. □

The next task in line would be to discuss in more detail how the spatial part (Σ, h) can look like. In particular, one would like to know under which conditions it splits as

$$\Sigma = (a, b)_r \times \mathbb{S}^2 , \qquad h = f_1(r) \, \mathrm{d}r^2 + f_2(r) g^\circ .$$

This seems to be a rather delicate matter, therefore we will not go into details here. We only make the following definition:

1.12 Definition. The *standard static and spherically symmetric* spacetime is

$$M = \mathbb{R}_t \times (a,b)_r \times \mathbb{S}^2 \,, \qquad g = -\mathrm{e}^{-2a(r)}\,\mathrm{d}t^2 + \mathrm{e}^{-2b(r)}\,\mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2)\,, \quad (1.2)$$

where $a, b \in C^\infty((a,b))$ are smooth functions of r.

1.3 The Schwarzschild Spacetime

1.13 Definition (Einstein Tensor)**.** For any Lorentz manifold (M,g) the tensor field

$$G := \mathrm{Ric} - \tfrac{1}{2} g\,\mathrm{scal} \in \mathcal{T}^2(M)$$

is called *Einstein Tensor*. Here $\mathrm{Ric} = \mathrm{Ric}^g$ denotes the Ricci curvature and $\mathrm{scal} = \mathrm{scal}^g$ denotes the scalar curvature of (M,g). In coordinates this equation reads $G_{ik} = R_{ik} - \frac{1}{2} g_{ik}\,\mathrm{scal}$. The manifold (M,g) *satisfies the Vacuum Einstein Equation (VEE)*, if $G = 0$.

1.14 Lemma (Characterization of (VEE))**.** A spacetime (M,g) satisfies (VEE) if and only if (M,g) is Ricci flat, i.e. $\mathrm{Ric}^g = 0$.

Proof.
"$\Longrightarrow$": Assume $G = 0$. This implies

$$0 = \mathrm{tr}(G) = \mathrm{scal} - \tfrac{4}{2}\,\mathrm{scal} = -\,\mathrm{scal},$$

thus $\mathrm{Ric} = G + \frac{1}{2} g\,\mathrm{scal} = G = 0$. Therefore (M,g) is Ricci flat.
"$\Longleftarrow$": Assume $\mathrm{Ric} = 0$. This implies $\mathrm{scal} = \mathrm{tr}(\mathrm{Ric}) = 0$. Thus $G = \mathrm{Ric} - \frac{1}{2} g\,\mathrm{scal} = 0$. Consequently (M,g) solves (VEE). □

1.15 Definition (Schwarzschild spacetime)**.** Let g° be the round metric on $\mathbb{S}^2$. Then The manifold (M,g) defined by

$$\begin{aligned} M_S &:= \mathbb{R} \times]2m, \infty[\times \mathbb{S}^2 \\ g_S &:= -(1 - \tfrac{2m}{r}) dt^2 + \tfrac{1}{(1-\frac{2m}{r})} dr^2 + r^2 g^\circ \\ &= -\tfrac{r-2m}{r} dt^2 + \tfrac{r}{r-2m} dr^2 + r^2 g^\circ \end{aligned}$$

is called *exterior Schwarzschild spacetime (expressed in Schwarzschild coordinates)* of mass $m \in \mathbb{R}_{>0}$. The quantity $r_s := 2m$ is the *Schwarzschild radius*. Analogously, the spacetime

$$\mathbb{R} \times]0, 2m[\times \mathbb{S}^2$$

with the same metric as above is called *interior Schwarzschild spacetime.*

1.16 Remark. Notice that due to the singularity of the metric g_S at $r = 2m$ we cannot simply patch together the exterior and the interior Schwarzschild spacetime into one globally well-defined Semi-Riemannian manifold.

1.17 Remark (Schwarzschild radius for our sun)**.** In physical units the Schwarzschild radius is given by $r_s = 2mG/c^2$, where G is the gravitaional constant. If we plug in the mass of our sun, we obtain $r_s \approx 3km$. Notice that the radius of the sun ($\sim 700.000km$) is a lot larger.

1.18 Lemma. The Schwarzschild spacetime (M_S, g_S) is static and spherically symmetric.

Proof. ∂_t is a timelike Killing vector field with respect to g_S. The corresponding restspaces are $]2m, \infty[\times \mathbb{S}^2$. Thus (M_S, g_S) is static. The canonical action of SO_3 on $]2m, \infty[\times \mathbb{S}^2 \subset \mathbb{R}^3$ is an isometric group action and the orbits are spacelike spheres. Thus (M_S, g_S) is spherically symmetric. □

1.19 Theorem. Let $a, b \in \mathcal{C}^\infty(]2m, \infty[, \mathbb{R})$. Define the Lorentz manifold

$$\begin{aligned} M &:= \mathbb{R} \times]2m, \infty[\times \mathbb{S}^2 \\ g_{a,b} &:= -e^{2a}dt^2 + e^{2b}dr^2 + r^2 g^\circ. \end{aligned}$$

Then $(M, g_{a,b})$ is a *physically resonable solution of the Vacuum Einstein Equation*, i.e. satisfies

$$\lim_{r\to\infty} a(r) = \lim_{r\to\infty} b(r) = 0, \qquad G = 0, \tag{1.3}$$

if and only if

$$a = \tfrac{1}{2}\ln(1 - \tfrac{2m}{r}), \qquad b = -\tfrac{1}{2}\ln(1 - \tfrac{2m}{r}), \tag{1.4}$$

i.e. if and only if $(M, g_{a,b})$ is the Schwarzschild spacetime.

Proof. For simplicity, we set $g := g_{a,b}$ in this proof.

Step 1 (construct ON frame): The Schwarzschild coordinates $(t, r, \vartheta, \varphi)$ induce a global coordinate frame $(\partial t, \partial r, \partial\vartheta, \partial\varphi)$ on TM independently of a,b. In these coordinates the metric is given by

$$g = -e^{2a}dt^2 + e^{2b}dr^2 + r^2 d\vartheta^2 + r^2 \sin(\vartheta)^2 d\varphi^2.$$

The coordinate frame can easily be g-orthonormalized to a frame

$$e_0 := e^{-a}\partial t, \qquad e_1 := e^{-b}\partial r, \qquad e_2 := r^{-1}\partial\vartheta, \qquad e_3 := r^{-1}\sin(\vartheta)^{-1}\partial\varphi.$$

The corresponding dual coframe is given by

$$\theta^0 := e^a dt, \qquad \theta^1 := e^b dr, \qquad \theta^2 := r d\vartheta, \qquad \theta^3 := r\sin(\vartheta)d\varphi.$$

are also orthonormal, if we extend the metric to T^*M, c.f. Theorem 5.2. Consequently, the metric is given by

$$g = g_{\nu\mu}\theta^\nu \otimes \theta^\mu, \qquad g_{\nu\mu} = \varepsilon_\nu \delta_{\nu\mu}, \qquad \varepsilon = (-1, +1, +1, +1).$$

STEP 2 (connection 1-forms): Let ω^i_j be the globally defined connection 1-forms with respect to the frame $(e_0, \ldots, e_3)$ above. Since the Levi civita connection ∇^g induced by g will always be metric by definition (regardless of the functions a, b), we obtain from (5.1)

$$\forall 1 \leq i, j \leq m : \omega_{ij} + \omega_{ji} = dg_{ij} = 0, \qquad \omega_{ij} = g_{ik}\omega^k_j.$$

From this it follows immediately that the matrix of the (ω_{ij}) is skew-symmetric. This implies

$$(\omega^i_j) = \begin{pmatrix} 0 & \omega^0_1 & \omega^0_2 & \omega^0_3 \\ \omega^0_1 & 0 & \omega^1_2 & \omega^1_3 \\ \omega^0_2 & -\omega^1_2 & 0 & \omega^2_3 \\ \omega^0_3 & -\omega^1_3 & -\omega^2_3 & 0 \end{pmatrix}. \tag{1.5}$$

This symmetry condition can be expressed by

$$\omega^i_j = -\varepsilon_i \varepsilon_j \omega^j_i. \tag{1.6}$$

To determine the remaining six unknowns, we calculate the exterior derivatives $d\theta^\alpha$ of θ^α, express them in terms of the basis $\theta^\sigma \wedge \theta^\rho$ and compare them with Cartans first structure equation

$$\boxed{\mathrm{d}\theta^\alpha = -\omega^\alpha_\beta \wedge \theta^\beta,}$$

c.f. Theorem 5.3:

$$\begin{aligned} -\omega^0_\beta \wedge \theta^\beta &= d\theta^0 = a' e^a dr \wedge dt = -a' e^{-b} \theta^0 \wedge \theta^1, \\ -\omega^1_\beta \wedge \theta^\beta &= d\theta^1 = 0, \\ -\omega^2_\beta \wedge \theta^\beta &= d\theta^2 = dr \wedge d\vartheta = -r^{-1} e^{-b} \theta^2 \wedge \theta^1, \\ -\omega^3_\beta \wedge \theta^\beta &= d\theta^3 = \sin(\vartheta) dr \wedge d\varphi + r\cos(\vartheta) d\vartheta \wedge d\varphi \\ &= -r^{-1}(e^{-b}\theta^3 \wedge \theta^1 + \cot(\vartheta)\theta^3 \wedge \theta^2). \end{aligned} \tag{1.7}$$

One verifies that

$$\begin{aligned} &\omega^0_1 = a' e^{-b} \theta^0, \ \omega^0_2 = \omega^0_3 = 0 \\ &\omega^1_2 = -r^{-1} e^{-b} \theta^2, \ \omega^1_3 = -r^{-1} e^{-b} \theta^3 \\ &\omega^2_3 = -r^{-1} \cot(\vartheta) \theta^3 \end{aligned} \tag{1.8}$$

together with the symmetries (1.5) solve (1.7). By Theorem 5.3(iv) this impiles that the ω^α_β are the connection 1-forms of g.

STEP 3 (curvature 2-forms): By Cartans second structure equation (5.4) the associated curvature 2-forms Ω^i_j satisfy

$$\boxed{\Omega^i_j = \mathrm{d}\omega^i_j + \omega^i_k \wedge \omega^k_j.}$$

From this and (1.6) it follows that

$$\begin{aligned} \Omega^j_i &= \mathrm{d}\omega^j_i + \omega^j_k \wedge \omega^k_i = -\varepsilon_i \varepsilon_j \, \mathrm{d}\omega^i_j - \omega^k_i \wedge \omega^j_k \\ &= -\varepsilon_i \varepsilon_j \, \mathrm{d}\omega^i_j - \varepsilon_k^2 \varepsilon_i \varepsilon_j \omega^i_k \wedge \omega^k_j = -\varepsilon_i \varepsilon_j \Omega^i_j, \end{aligned}$$

thus the matrix of curvature 2-forms Ω^i_j satisfies the same symmetry as (1.5). Therefore we calculate the following entries using (1.8) and express the result in terms of the basis $\theta^\alpha \wedge \theta^\beta$:

$$\begin{aligned}
\Omega^0_1 &= -e^{-2b}((a')^2 - a'b' + a'')\theta^0 \wedge \theta^1 \\
\Omega^0_2 &= -e^{-2b} r^{-1} a' \theta^0 \wedge \theta^2 \\
\Omega^0_3 &= -a' e^{-2} r^{-1} \theta^0 \wedge \theta^3 \\
\Omega^1_2 &= b' e^{-2b} r^{-1} \theta^1 \wedge \theta^2 \\
\Omega^1_3 &= b' e^{-2b} r^{-1} \theta^1 \wedge \theta^3 \\
\Omega^2_3 &= (1 - e^{-2b}) r^{-2} \theta^2 \wedge \theta^2.
\end{aligned} \tag{1.9}$$

This completely determines the curvature tensor.

Step 4 (Ricci tensor): We calculate the Ricci tensor Ric using eq. (5.5)

$$\mathrm{Ric}_{ik} = \sum_j \Omega^j_i(E_j, E_k).$$

This calculation can be simplified as follows: Due to the symmetries of Ω^j_i we only sum over all j such that $j \neq i$ (this holds in general). It follows from (1.9) that our specific curvature 2-forms satisfy $\Omega^i_i = f_{ji}\theta^j \wedge \theta^i$ for some functions $f_{ji} \in \mathcal{C}^\infty$. Consequently, we only have to sum over all $j \neq i$ such that in addition $\{j, i\} = \{j, k\}$. But this set is empty unless $i = k$. In other words,

$$\forall 1 \leq i \neq k \leq 3 : R_{ik} = 0.$$

For the remaining terms we calculate:

$$\begin{aligned}
\mathrm{Ric}_{00} &= -\tfrac{1}{2}(e^{-2b}(a'b' - a'' - (a')^2) + \tfrac{2}{r} a' e^{-2b}), \\
\mathrm{Ric}_{11} &= \tfrac{1}{2}(e^{-2b}(a'b' - a'' - (a')^2) - \tfrac{2}{r} b' e^{-2b}), \\
\mathrm{Ric}_{22} &= \tfrac{1}{2}(-\tfrac{a'e^{-2b}}{r} + \tfrac{b'e^{-2b}}{r} + \tfrac{1-e^{-2b}}{r^2}), \\
\mathrm{Ric}_{33} &= \tfrac{1}{2}(-\tfrac{a'e^{-2b}}{r} + \tfrac{b'e^{-2b}}{r} + \tfrac{1-e^{-2b}}{r^2}).
\end{aligned} \tag{1.10}$$

Step 5 (Final Argument): By Lemma 1.14 the manifold $(M, g_{a,b})$ solves the (VEE) if and only if it is Ricci flat.
"$\Longrightarrow$": Assume Ric $= 0$. Then in particular

$$0 = R_{00} + R_{11} \overset{(1.10)}{\Longrightarrow} -a' - b' = 0 \Longrightarrow (a+b)' = 0.$$

This implies that $a + b$ is a constant function. Due to the asymptotics in (1.3) this implies $a + b = 0$, i.e. $a = -b$. Furthermore

$$0 = R_{33} \Longrightarrow -1 = e^{-2b}(2b'r - 1) = (-e^{-2b} r)'$$

Consequently, by choosing $-2m$ as an integration constant, we obtain

$$e^{-2b} r = r - 2m \Longrightarrow e^{-2b} = 1 - \tfrac{2m}{r} \Longrightarrow -2b = \ln(1 - \tfrac{2m}{r}),$$

which together with $a = -b$ finally implies (1.4).
"$\Longleftarrow$": Apparently the functions a and b specified in (1.4) satisfy

$$\lim_{r\to\infty} a(r) = \lim_{r\to\infty} b(r) = 0$$

and $a = -b$. Using (1.10), we calculate

$$\begin{aligned} a' &= mr^{-2}(1 - \tfrac{2m}{r})^{-1}, \\ a'' &= -2mr^{-3}(1 - \tfrac{2m}{r})^{-2} - 4m^2r^{-4}(1 - \tfrac{2m}{r})^{-2}. \end{aligned}$$

Hence

$$\begin{aligned} a'b' - a'' - (a')^2 + \tfrac{2}{r}a' &= \tfrac{2}{r}a' - 2(a')^2 - a'' \\ &= 2mr^{-3}(1 - \tfrac{2m}{r})^{-3} - 2m^2r^{-4}(1 - \tfrac{2m}{r})^{-2} \\ &+ 2mr^{-3}(1 - \tfrac{2m}{r})^{-2} + 4m^2r^{-4}(1 - \tfrac{2m}{r})^{-2} \\ &= 0. \end{aligned}$$

By (1.10) this implies

$$\mathrm{Ric}_{00} = 0 = \mathrm{Ric}_{11}\,.$$

Furthermore

$$\begin{aligned} 2r^2e^{2b}\,\mathrm{Ric}_{22} = 2r^2e^{2b}\,\mathrm{Ric}_{33} &= 2b'r + e^{2b} - 1 \\ &= -2mr^{-1}(1 - \tfrac{2m}{r})^{-1} + (1 - \tfrac{2m}{r})^{-1} - 1 \\ &= (1 - \tfrac{2m}{r})^{-1}(1 - 2mr^{-1}) - 1 = 0. \end{aligned}$$

Thus $\mathrm{Ric}_{22} = \mathrm{Ric}_{33}$.

□

2 Geodesics in the Schwarzschild Spacetime

In this section, we study timelike and null geodesics in the exterior Schwarzschild spacetime.

Timelike geodesics are potential wordlines of massive particles (such as planets) moving in the exterior gravitational field of a spherically symmetric (static) mass distribution (such as our sun). Therefore they provide an experimental test of general relativity by testing, e.g., whether the orbits of planets in our solar system can be described by timelike Schwarzschild geodesics. As a matter of fact, the first experimental confirmation of general relativity was the explanation of the *perihelion advance of Mercury* by Schwarzschild geodesics.

Null geodesics can be thought of as describing the motion of photons, i.e. of light in the geometric optics approximation. A famous general relativistic effect is *gravitational lensing*, i.e the bending of light by gravitational fields. This effect, as caused by our sun, can also be explained by studying null geodesics in the Schwarzschild spacetime.

After studying geodesics in the Schwarzschild spacetime in general, we shall come back to these connections to experimentally observable effects later.

2.1 General Properties of Geodesics

In general, the geodesic equation cannot be solved explicitly, i.e. by actually writing down the solutions. Therefore, in order to study the behaviour of geodesics, one has to rely on other methods. A very convenient method, which one might be familiar with already from classical mechanics, is to exploit *symmetries*. The reason for this is the relation between symmetries and *conserved quantities*, as is generally explained by Noether's theorem. Here we will not go into the Lagrangian formalism (which is necessary for Noether's theorem), but rather prove directly how symmetries lead to conserved quantities. The intuitive idea about the usefulness of conserved quantities is that each conserved quantity reduces the degrees of freedom (the number of coordinates we need to describe our system) by one, hence allows us to decrease the number of equations we have to solve by one.

Now we show how each symmetry, described infinitesimally by a Killing vector field, leads to a conserved quantity of the geodesic equation.

2.1 Lemma. Let (M, g) be a Lorentzian manifold and let $X \in \mathcal{T}(M)$ be a Killing field and γ be a geodesic. Then

$$\frac{\mathrm{d}}{\mathrm{d}\tau} g(X(\gamma(\tau)), \dot{\gamma}(\tau)) = 0\,,$$

hence $g(X(\gamma(\tau)), \dot{\gamma}(\tau))$ is constant.

Proof. Since X is Killing, it satisfies $g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$ for any $Y, Z \in \mathcal{T}(M)$. Therefore, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau} g(X, \dot{\gamma}) &= g(\nabla_{\dot{\gamma}} X, \dot{\gamma}) + g(X, \nabla_{\dot{\gamma}} \dot{\gamma}) \\ &= g(\nabla_{\dot{\gamma}} X, \dot{\gamma}) \\ &= -g(\dot{\gamma}, \nabla_{\dot{\gamma}} X) \\ &= -g(\nabla_{\dot{\gamma}} X, \dot{\gamma}) = -\frac{\mathrm{d}}{\mathrm{d}\tau} g(X, \dot{\gamma})\,.\end{aligned}$$

Clearly this implies that this quantity must vanish. □

The following Lemma will also be needed.

2.2 Lemma. Let (M, g) be a SR manifold and let $\varphi : M \to M$ be an isometry. Let

$$\Sigma := \{p \in M \mid \varphi(p) = p\} \subset M$$

be the submanifold of fixed points of φ. Then Σ is a totally geodesic submanifold.

Proof. Let $(p, v) \in \Sigma \times T_p\Sigma$ and let $\gamma : I \to M$ be the geodesic with these initial values. Then $\varphi \circ \gamma$ is again a geodesic satisfying $(\varphi \circ \gamma)(0) = \gamma(0)$. Using the fact that $\varphi|_\Sigma = \mathrm{id}_\Sigma$, we obtain

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(\varphi \circ \gamma)(0) = \mathrm{d}\varphi|_p \dot{\gamma}(0) = \mathrm{d}(\mathrm{id})|_p \dot{\gamma}(0) = \dot{\gamma}(0).$$

By uniqueness of geodesics, we obtain $\varphi \circ \gamma = \gamma$. Consequently

$$\forall t \in I : \varphi(\gamma(t)) = \gamma(t) \Longleftrightarrow \gamma(t) \in \Sigma.$$

□

Now we come to the study of geodesics in the Schwarzschild spacetime. The main content of the following theorem is the derivation of an equation for the radial coordinate of a geodesic which resembles the equation of a Newtonian particle in one spatial dimension which moves in an "effective" potential.

2.3 Theorem. Let (M_S, g) be the exterior Schwarzschild spacetime with Schwarzschild coordinates $(t, r, \vartheta, \varphi)$. Let $\tilde{\gamma}$ be a geodesic. There exists an isometry $\Lambda : M \to M$ such that $\gamma := \Lambda \circ \tilde{\gamma}$ satisfies $\vartheta(\gamma) = \frac{\pi}{2}$. Moreover, $r(\tau) = r(\gamma(\tau))$ satifies

$$\boxed{\underbrace{\tfrac{1}{2}\dot{r}^2}_{E_{\text{kin}}} + \underbrace{\tfrac{1}{2}(1 - \tfrac{2m}{r})(\tfrac{L^2}{r^2} + \kappa)}_{E_{\text{pot}}} = \underbrace{\tfrac{1}{2}E^2}_{E_{\text{ges}}}}, \tag{2.1}$$

where

$$\kappa := -g(\dot{\gamma}, \dot{\gamma}), \qquad E := g(\dot{\gamma}, \partial_t), \qquad L := g(\dot{\gamma}, \partial_\varphi).$$

are all constants.

Proof.

Step 1: First of all, we define the isometry

$$\begin{aligned} F : M &\to M \\ (t, r, \vartheta, \varphi) &\mapsto (t, r, \pi - \vartheta, \varphi) \end{aligned}$$

The fixed point set Σ of this isometry is the hypersurface defined by $\vartheta = \frac{\pi}{2}$. By Lemma 2.2 Σ is totally geodesic. Now let $\gamma : I \to M$ be any geodesic with initial values (p, v). Since the $\mathbb{S}^2$-component of M carries the round metric, there exists an isometry of M which carries (p, v) into $\Sigma \times T\Sigma$. Since Σ is totally geodesic, we may assume that γ lies completely in Σ, i.e. $\vartheta(\gamma(\tau)) = \frac{\pi}{2}$ for all $\tau \in I$.

Step 2: Any geodesic has constant speed, thus κ is a constant. Since ∂_t and ∂_φ are Killing vector fields the quantities E and L are also constants by Lemma 2.1. They are explicitely given by

$$\begin{aligned} E &= g(\dot{\gamma}, \partial_t) = -(1 - \tfrac{2m}{r})dt(\dot{\gamma}) = -(1 - \tfrac{2m}{r})\dot{t}. \\ L &= g(\dot{\gamma}, \partial_\varphi) = r^2 \sin(\vartheta)^2 \dot{\varphi} \end{aligned}$$

Therefore

$$\begin{aligned} -\kappa &= g(\dot{\gamma}(\tau), \dot{\gamma}(\tau)) \\ &= -(1 - \tfrac{2m}{r(\tau)})\dot{t}(\tau)^2 + \tfrac{1}{(1-\frac{2m}{r(\tau)})}\dot{r}(\tau)^2 + r(\tau)^2(\dot{\vartheta}(\tau)^2 + \sin(\vartheta(\tau))^2\dot{\varphi}(\tau)^2) \\ &= -\tfrac{E^2}{(1-\frac{2m}{r})} + \tfrac{1}{(1-\frac{2m}{r(\tau)})}\dot{r}(\tau)^2 + r(\tau)^2\dot{\vartheta}(\tau)^2 + \tfrac{L^2}{r^2\sin(\vartheta)^2} \\ &= -\tfrac{E^2}{(1-\frac{2m}{r})} + \tfrac{1}{(1-\frac{2m}{r})}\dot{r}(\tau)^2 + \tfrac{L^2}{r^2}. \end{aligned}$$

In the last step we use that $\vartheta = \frac{\pi}{2}$. Rearranging the last equality, we obtain

$$\tfrac{1}{2}E^2 = \tfrac{1}{2}\dot{r}^2 + \tfrac{1}{2}(1 - \tfrac{2m}{r})(\tfrac{L^2}{r^2} + \kappa).$$

$\square$

2.4 Remark. The first step in the previous proof, i.e. the reduction to $\theta = \frac{\pi}{2}$ is the analogue of the exploitation of the conservation of the total angular momentum in the Newtonian Kepler problem. There, also as a first step, one uses the conservation of the direction of the angular momentum vector to see that the particle must move in the plane orthogonal to the angular momentum. Without loss one may then assume that this plane is given by $\theta = \frac{\pi}{2}$. One might also be able to introduce the total angular momentum here and exploit its conservation in a similar way (hopefully it is also conserved).

2.5 Remark. One sees that (2.1) looks like the Newtonian equation of a particle moving in one dimension in the effective potential $V(r) := \frac{1}{2}(1 - \frac{2m}{r})(\frac{L^2}{r^2} + \kappa)$. And this is cool.

2.2 Timelike Geodesics and Perihelion Advance

In this paragraph we discuss the behaviour of timelike geodesics. First we make some general observations about their behaviour via a qualitative study of the radial equation. Afterwards, we want to compare the behaviour of timelike geodesics in the Schwarzschild spacetime to the behaviour of Newtonian particles in the classical Kepler potential.

We parametrize the geodesics by proper time (i.e. arc-length), then $\kappa = -g(\dot{\gamma}, \dot{\gamma}) = 1$. Therefore, our effective equation is

$$\tfrac{1}{2}\dot{r}^2 + \underbrace{\tfrac{1}{2}(1 - \tfrac{2m}{r})(1 + \tfrac{L^2}{r^2})}_{=V(r)} = \frac{1}{2}\dot{r}^2 + \left(\frac{1}{2} - \frac{m}{r} + \frac{L^2}{2r^2} - \frac{mL^2}{r^3}\right) = E^2\,. \tag{2.2}$$

The effective potential satisfies $V(r) \to -\infty$ for $r \to 0$ and $V(r) \to \frac{1}{2}$ for $r \to \infty$. Moreover, we always have $V(2m) = 0$. The further shape of the effective potential depends crucially on the size of L and m. To see this, one calculates the extremal points of $V(r)$, for which one finds

$$R_{\pm} = \frac{L^2 \pm \sqrt{L^4 - 13L^2m^2}}{2m}\,.$$

One sees that for $L^2 < 12m^2$, there are no real solutions. Hence $V(r)$ has no extremal points and looks like this. This means that (just as in Newtonian physics), every particle whose angular momentum is too small in comparison to m crashes into the source of gravity.

For $L^2 > 12m^2$, we have two real solutions, and it is easy to check that the smaller one R_- is a maximum and the larger one R_+ a minimum, see. Hence R_- corresponds to an unstable circular orbit, whereas R_+ corresponds to a stable circular orbit. The orbits of planets correspond to small oscillations around the stable orbit. Qualitatively this is similar as for the classical Kepler problem, but since the effective potential has a different form, there is a quantitative difference as we shall see in the following.

The classical Kepler problem is given by $\ddot{x} = m\nabla \frac{1}{|x|}$. Similar as we did above, it follows from conservation of the angular momentum vector (more precisely, of its direction) that the motion must lie in a plane. Introducing polar coordinates (r, φ) in this plane and using conservation of the angular momentum $L = r^2\dot{\varphi}$, the following equation is obtained from conservation of energy

$$\frac{1}{2}\dot{r}^2 + \left(\frac{L^2}{r^2} - \frac{m}{r}\right) = E\,. \tag{2.3}$$

One sees that the effective potential appearing here differs from the Schwarzschildean one by a constant term and by the $\frac{1}{r^3}$-term, which is only present in the Schwarzschildean potential. The constant term is unimportant since it does not change the shape of the potential. The $\frac{1}{r^3}$-term, however, really changes the shape of the potential, and therefore also the motion of particles. We will illustrate next how this leads to a precision of the perihelion that is not present for Kepler orbits.

We consider a particle with L large enough and E accordingly such that it moves on a closed orbit in either the Schwarzschild spacetime or in the Kepler potential. For the following considerations, it is best if we paramtrize the motion of the particle by its angular coordinate φ. Since $\dot{\varphi} = \frac{L}{r^2} > 0$, this is indeed possible. Be aware here that we let φ range over all of $\mathbb{R}$ and do not explicitly calculate modulo 2π for the moment. A *perihel* of the motion is a an extremal point of the function r. See figure for an illustration. Next, we consider $u = \frac{1}{r}$ as a function of φ. Then we have

$$\dot{r} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{u} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\varphi}\dot{\varphi} = -L\frac{\mathrm{d}u}{\mathrm{d}\varphi}\,, \tag{$*$}$$

and hence also

$$\ddot{r} = -L\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2}\dot{\varphi} = -L^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2}\,. \tag{$**$}$$

So far this holds for both the classical and the relativistic case, but now we have to treat the two separately.

Newtonian: From the classical equation (2.3) it follows easily that

$$\ddot{r} = \frac{2L^2}{r^3} - \frac{m}{r^2} = 2L^2u^3 - mu^2\,.$$

If we equate this with $(**)$, we obtain the equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = \frac{m}{L^2}\,. \tag{2.4}$$

This linear ODE is easy to solve, and the solution is

$$u(\varphi) = a\cos(\varphi) + \frac{m}{L^2}\,. \tag{2.5}$$

It is not very difficult to see that this describes an *ellipse*.

Relativistic: This time, it follows from (2.2) that

$$\ddot{r} = -2L^3mu^4 + L^2u^3 - mu^2\,.$$

Equating this with $(**)$ again, we obtain the equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = \frac{m}{L^2} + 3mu^2\,. \tag{2.6}$$

We cannot explicitly solve this equation anymore, therefore we make a (very) naive perturbation argument. Since r is much larger compared to $m$it follows that the quadratic term only makes a small contribution. Hence the Keplerian solution $u_0(\varphi) = \frac{m}{L^2}(1+\epsilon\cos\varphi)$ is an approximate solution to (2.6). The perturbative argument is now to replace the quadratic term by $3mu_0^2$ and study the resulting equation, which is given by

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = \frac{m}{L^2} + \frac{3m^3}{L^4}\left(1+\frac{\epsilon^2}{2}+2\epsilon\cos\varphi+\frac{\epsilon^2}{2}\cos 2\varphi\right)\,.$$

A particular solution of this equation is given by

$$u(\varphi) = \frac{m}{L^2}(1+\epsilon\varphi) + \frac{3m^3}{L^4}\left(1+\frac{\epsilon^2}{2}-\frac{\epsilon^2}{6}\cos 2\varphi+\epsilon\varphi\sin\varphi\right)\,.$$

For some unknown reasons one likes this function as an approximate solution to (2.6), therefore we use it to calculate the perihels. To this end, we must calculate the positions of its maxima. We have

$$u'(\varphi) = -\frac{m\epsilon}{L^2}\sin\varphi + \frac{3m^2\epsilon}{L^4}\left(\frac{\epsilon}{3}\sin 2\varphi+\sin\varphi+\varphi\cos\varphi\right)\,.$$

The first perihel is at $\varphi = 0$. What one sees, however, is that the next perihel is *not* at $\varphi = 2\pi$ (this would be needed for an ellipse), but at $\varphi = 2\pi+\delta$, where δ is given by the approximate equation

$$0 \approx -\frac{m\epsilon}{L^2}\sin\delta\frac{3m^2\epsilon}{L^4}(2\pi+\delta)\cos\delta\,.$$

Hence it follows that

$$\delta \approx \tan\delta \approx \frac{3m^2}{L^2}(2\pi+\delta) \approx \frac{6\pi m^2}{L^2}\,. \tag{2.7}$$

If one evaluates this for Mercury, one obtains the famous 43" (be aware that the mass of the planet was absorbed into the units).

2.6 Remark. The *Perihelion advance* of Mercury was the first famous prediction of General Relativity. It was observed long before that Mercury does not move on an ellipse, but that its perihel moves. Before general relativity, one had tried to account for this effect by treating the perturbation to Mercury's motion caused by the other planets. Although this could be used to predict a rotating ellipse, the predicted precession of the ellipse did not agree with experimental observations. It was a great success for Einstein's theory that the additional precession caused by relativistic effects brought the theoretical predictions to agreement with the observations. Indeed, as Einstein told his former colleague Fokker, this first successful prediction caused him to have heart palpitations. However, one should also say that by far the largest effect on Mercury's perihel motion is caused by the other planets (this can be taken into account by an effective change to the potential), and only the second largest effect is the one coming from General Relativity. Still, the relativistic effect makes up around 5-10%.

2.3 Lightlike Geodesics and Bending of light

Now we analyze the behaviour of lightlike geodesics, i.e. $\kappa = -g(\dot\gamma, \dot\gamma) = 0$. In this case the radial equation reads

$$\tfrac{1}{2}\dot r^2 + \underbrace{\tfrac{1}{2}(1-\tfrac{2m}{r})\tfrac{L^2}{r^2}}_{=V(r)} = E^2 \,.$$

Hence the effective potential is

$$V(r) = \frac{L^2}{2r^3}(r-2m)\,.$$

One sees that this time, the general shape of the effective potential is independent of L (unless $L = 0$). We have $V(r) \to -\infty$ for $r \to 0$ and $V(r) \to 0$ for $r \to \infty$. Moreover, now we have a single extremum, namely a maximum, which happens to be at $r = 3m$. It is interesting to observe that hence there is a circular lightlike orbit, i.e. the orbit of a photon that neither escapes to infinity, nor falls into the gravitational source, but stays at the same distance $r = 3m$.

If we believe for now that lightlike geodesics indeed describe the propagation of photons (in the geometric optics approximation), then all this already shows that gravity has a significant influence on the propagation of photons.

Another famous observation is *gravitational lensing*, i.e. the bending of light around a massive object. This can also be observed for lightlike Schwarzschild geodesics if one studies the behaviour of the angular variable $\varphi(\tau)$. One can do similar calculations as we did for the perihelion advance, we refer for to the literature [Oloff] for this.

3 The Kruskal Extension

In this section, we want to understand the (geometric) meaning of the so-called *event horizon* $\{r = 2m\}$. At first sight it seems that the metric becomes singular at the horizon. However, as we shall see in this section, this is not an actual singularity in the metric/geometry, but comes from the fact that our coordinate system $(t, r, \vartheta, \varphi)$ becomes invalid at the horizon. You may compare this to the way in which the well-known spherical coordinates on $\mathbb{R}^3$ become invalid at the origin. What we will do in this section is to suitably *continue* the Schwarzschild spacetime beyond the horizon. Although at the moment this is a question of rather mathematical than physical interest, the continuation we will be crucial for the definition of *black holes* in the following section.

3.1 Remark. In the motivational section 1.1 it was said that the exterior Schwarzschild solution is thought to model the exterior gravitational field of a spherically symmetric, static object. Moreover, it was said that the gravitational field in the interior of the object has to be modeled by a solution of the Einstein equation for some suitable matter model. As a matter of fact, for many usual matter models (e.g. perfect fluid) it turns out that one has to match the two solutions *outside* of the event horizon. Therefore, for any object whose interior can be described by such a matter model (as should be the case for planets and usual stars), the continuation we obtain in the following has no actual physical meaning.

However, since it is not expected that the interior of black holes can be described by a classical matter model (due to "quantum effects" of some sort which should play a role due to extreme energy densities), the continuation of the Schwarzschild solution beyond the event horizon is of interest for such objects. Of course, whether such objects actually exist in real life may be questioned, but at the moment it seems that the generally accepted opinion is that they do in fact exist.

3.1 Geodesics approaching the horizon

3.2 Kruskal-Szekeres coordinates

3.2 Theorem. The Schwarzschild spacetime M can be isometrically embedded as a proper subset into the *Kruskal-Szekeres spacetime*

$$\begin{aligned} M_K &:= \{(U,V) \in \mathbb{R}^2 \mid V^2 - U^2 < 1\} \times \mathbb{S}^2, \\ g_K &= \tfrac{32m^3}{r} e^{-\frac{r}{2m}} (-dV^2 + dU^2) + r^2 d\omega^2 \,. \end{aligned}$$

Here r is now a smooth function of U and V. More explicitly, the embedding is given by

$$\begin{aligned} F : M = \mathbb{R} \times]2m, \infty[\times \mathbb{S}^2 &\to M_K \times \mathbb{S}^2 \\ (t, r, \omega) &\mapsto (V(t,r), U(t,r), \omega), \end{aligned}$$

where the functions U and V are defined by

$$\begin{aligned} V &:= \sqrt{\tfrac{r}{2m} - 1} e^{\frac{r}{4m}} \sinh\left(\tfrac{t}{4m}\right) = \sqrt{\tfrac{r-2m}{2m}} e^{\frac{r}{4m}} \sinh\left(\tfrac{t}{4m}\right), \\ U &:= \sqrt{\tfrac{r}{2m} - 1} e^{\frac{r}{4m}} \cosh\left(\tfrac{t}{4m}\right) = \sqrt{\tfrac{r-2m}{2m}} e^{\frac{r}{4m}} \cosh\left(\tfrac{t}{4m}\right). \end{aligned}$$

This embedding provides an extension of the original Schwarzschild spacetime beyond the event horizon, as can be seen from figure .

Proof. First we check that F maps indeed into M_K and onto

$$F(M) = \{(u,v) \in \mathbb{R}^2 \mid v^2 - u^2 < 0, u > 0\} \times \mathbb{S}^2.$$

To that end we calculate

$$V^2 - U^2 = \left(1 - \tfrac{r}{2m}\right) e^{\frac{r}{2m}} =: h(r)$$

Notice that $h \in \mathcal{C}^\infty(]2m, \infty[,] - \infty, 0[)$ and from this it follows that the range of F is correctly specified. In addition

$$h'(r) = -\tfrac{1}{2m} e^{\frac{r}{2m}} + \tfrac{1}{2m}\left(1 - \tfrac{r}{2m}\right) e^{\frac{r}{2m}} = -\tfrac{r}{4m^2} e^{\frac{r}{2m}} < 0$$

and is therefore a diffeomorphism. Since $U > 0$, we obtain

$$\tfrac{V}{U} = \tanh\left(\tfrac{t}{2m}\right) \Longrightarrow \operatorname{artanh}\left(\tfrac{V}{U}\right) 2m = t.$$

Since $\tanh : \mathbb{R} \to]-1,1[$ is a diffeomorphism, we conclude that F is indeed a diffeomorphism. We calculate

$$\begin{aligned} dV &= \tfrac{1}{4m} U\,dt + \left(\tfrac{1}{2}\left(\tfrac{r}{2m}-1\right)^{-\frac{1}{2}} \tfrac{1}{2m} e^{\frac{r}{4m}} + \sqrt{\tfrac{r}{2m}-1}\, e^{\frac{r}{4m}} \tfrac{1}{4m} \right) \sinh\left(\tfrac{t}{4m}\right) dr \\ &= \tfrac{1}{4m} U\,dt + \left(1 + \tfrac{r}{2m} - 1\right) \sinh\left(\tfrac{t}{4m}\right) \left(\tfrac{r}{2m}-1\right)^{-\frac{1}{2}} \tfrac{1}{4m} e^{\frac{r}{4m}} dr \\ &= \tfrac{1}{4m} U\,dt + \sinh\left(\tfrac{t}{4m}\right) \left(\tfrac{r}{2m}-1\right)^{-\frac{1}{2}} \tfrac{r}{8m^2} e^{\frac{r}{4m}} dr\,, \end{aligned}$$

and analogously

$$dU = \tfrac{1}{4m} V\,dt + \cosh\left(\tfrac{t}{4m}\right) \left(\tfrac{r}{2m}-1\right)^{-\frac{1}{2}} \tfrac{r}{8m^2} e^{\frac{r}{4m}} dr\,.$$

Therefore

$$\begin{aligned} -dV^2 + dU^2 &= \tfrac{1}{16m^2}(-U^2+V^2)dt^2 + \left(\tfrac{r}{2m}-1\right)^{-1} \tfrac{r^2}{64m^4} e^{\frac{r}{2m}} dr^2 \\ &= -\tfrac{1}{16m^2}\left(1-\tfrac{r}{2m}\right) e^{\frac{r}{2m}} dt^2 + \left(\tfrac{r}{2m}-1\right)^{-1} \tfrac{r^2}{64m^4} e^{\frac{r}{2m}} dr^2 \\ &= -\tfrac{1}{16m^2}\left(1-\tfrac{r}{2m}\right) e^{\frac{r}{2m}} dt^2 + \left(\tfrac{r}{2m}-1\right)^{-1} \tfrac{r^2}{64m^4} e^{\frac{r}{2m}} dr^2 \\ &= -\tfrac{1}{16m^2} \tfrac{2m-r}{2m} e^{\frac{r}{2m}} dt^2 + \tfrac{2m}{r-2m} \tfrac{r^2}{64m^4} e^{\frac{r}{2m}} dr^2 \end{aligned}$$

thus

$$\begin{aligned} 16m^2 \tfrac{2m}{r} e^{-\frac{r}{2m}}(-dV^2+dU^2) &= -\tfrac{2m}{r} e^{-\frac{r}{2m}} \tfrac{2m-r}{2m} e^{\frac{r}{2m}} dt^2 + \tfrac{2m}{r} e^{-\frac{r}{2m}} \tfrac{2m}{r-2m} \tfrac{r^2}{4m^2} e^{\frac{r}{2m}} dr^2 \\ &= -\tfrac{2m-r}{r} dt^2 + \tfrac{r}{r-2m} dr^2 = g - r^2 d\omega^2\,. \end{aligned}$$

We obtain the formula

$$g_K = \tfrac{32m^3}{r} e^{-\frac{r}{2m}} (-dV^2 + dU^2) + r^2 d\omega^2.$$

□

3.3 Remark. singularity at $r = 2m$ is removed

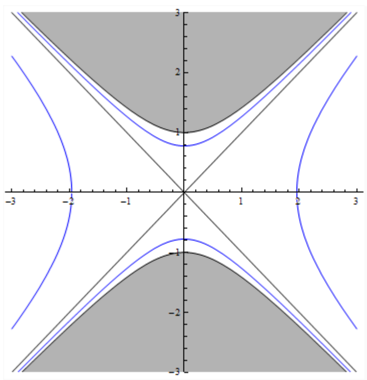

Figure 2: The Kruskal Spacetime

4 The Penrose Diagram of Schwarzschild and Black Holes

In order to discuss the concept of black holes, it is suitable to represent the Schwarzschild solution in yet another way, namely by its *Penrose diagram.* Generally, Penrose diagrams are an important tool for qualitative studies of spherically symmetric spacetimes.

4.1 Theorem (Penrose Diagram)**.** There exists an open, bounded embedding of the Kruskal-Szekeres spacetime into $\mathbb{R}_T \times \mathbb{R}_X \times \mathbb{S}^{n-1}_\omega$ such that the metric looks like

$$g_P = \frac{32m^3}{r} e^{-\frac{r}{2m}} \frac{1}{4\cos\left(\frac{T+X}{2}\right)^2 \cos\left(\frac{T-X}{2}\right)^2} (-dT^2 + dX^2) + r^2 d\omega^2.$$

The image of the embedding is pictured in:

Proof. We introduce the null coordinates $\tilde{U}$, $\tilde{V}$ by means of the diffeomorphism

$$\begin{array}{rcl} \tilde{G} : M_K & \to & \tilde{M}_P \\ (U, V, \omega) & \mapsto & (V - U, V + U, \omega) =: (\tilde{U}, \tilde{V}, \omega) \end{array}$$

$$\tilde{M}_p = \{(\tilde{U}, \tilde{V}) \in \mathbb{R}^2 \mid \tilde{U}\tilde{V} < 1\} \times \mathbb{S}^2.$$

In these coordinates the metric looks like

$$\tilde{g}_P = \frac{32m^3}{r} e^{-\frac{r}{2m}} \frac{1}{2}(-d\tilde{V}d\tilde{U} - d\tilde{U}d\tilde{V}) + r^2 d\omega^2,$$

since

$$dV = \frac{1}{2}(d\tilde{V} + d\tilde{U}), \qquad dU = \frac{1}{2}(d\tilde{V} - d\tilde{U}).$$

Next we introduce finite null coordinates $\tilde{u}$, $\tilde{v}$ by the diffeomorphisn

$$\begin{array}{rcl} G : M_P & \to & \tilde{M}_P \\ (\tilde{u}, \tilde{v}, \omega) & \mapsto & (\tan(\tilde{u}), \tan(\tilde{v}), \omega), \end{array}$$

where

$$M_P = \{(\tilde{u}, \tilde{v}) \in] - \tfrac{\pi}{2}, \tfrac{\pi}{2}[^2 | -\tfrac{\pi}{2} + \tilde{v} \le \tilde{u} \le \tfrac{\pi}{2} - \tilde{v}\} \times \mathbb{S}^2$$

In these coordinates the metric looks like

$$g_P = \frac{32m^3}{r} e^{-\frac{r}{2m}} \frac{1}{2\cos(\tilde{u})^2 \cos(\tilde{v})^2} (-d\tilde{v}d\tilde{u} - d\tilde{u}d\tilde{v}) + r^2 d\omega^2$$

In a last step we change to

$$\begin{array}{rcl} M_P & \to & M_P \\ (T, X) & \mapsto & \frac{1}{2}(T + X, T - X), \end{array}$$

which yields a metric in coordinates:

$$g_P = \frac{32m^3}{r} e^{-\frac{r}{2m}} \frac{1}{4\cos\left(\frac{T+X}{2}\right)^2 \cos\left(\frac{T-X}{2}\right)^2} (-dT^2 + dX^2) + r^2 d\omega^2$$

□

4.2 Remark (Physical Interpretation)**.** conformally equivalent to $\mathbb{R}^{1,1} \times \mathbb{S}^2$, conformal invariance of null geodesics, future null infinity $\mathscr{J}^+$, the fate of null geodesics, future timelike infinity i^+, spacelike infinity i^0, definition black hole region, horizon $r = 2m$

4.3 Lemma (conformal invariance of null geodesics)**.** Let (M, g) be a Lorentz manifold and $h = e^{2f}g$ be conformal to g. Let γ be a g null geodesic in M. Then γ is an h null pregeodesic (i.e. can be reparametrized to an h-geodesic).

Proof. □

5 Appendix: Some Notions from Differential Geometry

5.1 Definition (Lie derivative)**.** Let $\tau \in \mathcal{T}^k(M)$ be any tensor field and $X \in \mathcal{T}(M)$ be a vector field with local flow $\theta_t(p)$, $t \in \mathbb{R}$, $p \in M$. Then the *Lie derivative of* τ *with respect to* X is defined by

$$\forall p \in M : (\mathcal{L}_X \tau)|_p := \tfrac{d}{dt}(\theta_t^* \tau)|_p|_{t=0} = \lim_{t \to 0} \tfrac{\theta_t^*(\tau|_{\theta_t(p)}) - \tau_p}{t}.$$

5.2 Theorem (Musicial Isomorphisms)**.** Let (M, g) be a Semi-Riemannian m-manifold of signature (r, s) (i.e. r-fold negative definite and s-fold positive definit). For any vector field $X \in \mathcal{T}(M)$ denote by $X^\flat \in \mathcal{T}^1(M)$ the 1-form defined by

$$\forall Y \in \mathcal{T}(M) : X^\flat(Y) := g(X, Y).$$

Then the following hold:

(i) The map $\flat : TM \to T^*M$ resp. $\flat : \mathcal{T}(M) \to \mathcal{T}^1(M)$ is an isomorphism. Its inverse is denoted by $\sharp : \mathcal{T}^1(M) \to \mathcal{T}(M)$.

(ii) Setting

$$\forall \omega, \eta \in \mathcal{T}^*(M) : g(\omega, \eta) := g(\omega)$$

extends the metric g to a fibre metric on T^*M.

(iii) If $E_0, \ldots, E_m$ is a g-orthonormal local frame, then its dual coframe $E^0, \ldots, E^m$ is orthonormal with respect to the extended metric.

5.3 Theorem (Cartan Calculus)**.** Let (M, g) be a Semi-Riemannian m-manifold. Let ∇ be a linear connection on M, $U \subset M$, $E_1, \ldots, E_m$ be a local frame and $\varphi^1, \ldots, \varphi^m$ be the dual coframe.

(i) There exists a uniquely determined matrix of forms $\omega_i^j \in \Omega^1(U)$, $1 \leq i, j \leq m$ such that

$$\forall 1 \leq i \leq m : \forall X \in \mathcal{T}(U) : \nabla_X E_i = \omega_i^j(X) E_j.$$

The forms ω_i^j are called *connection* 1*-forms* with respect to $(E_1, \ldots, E_m)$.

(ii) The connection ∇ is metric on U if and only if

$$\omega_{ij} + \omega_{ji} = dg_{ij}, \qquad \omega_{ij} := g_{ik}\omega_j^k \tag{5.1}$$

(iii) Let τ be the *torsion tensor* of ∇, i.e.

$$\forall X, Y \in \mathcal{T}(X,Y) : \tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

Expand $\tau(X,Y) = \tau^j(X,Y)E_j$ locally in U, where $\tau^j \in \Omega^2(U)$. Then *Cartan's first structure equation* holds:

$$\boxed{\forall 1 \leq j \leq m : d\varphi^j = \varphi^i \wedge \omega_i^j + \tau^j.} \tag{5.2}$$

(iv) Let $\tilde{\omega}_i^j \in \Omega^1(U)$ be a matrix of 1-forms that satisfies the symmetries (5.1) and (5.2) for $\tau = 0$. Then the local connection $\tilde{\nabla}$ defined by

$$\forall X \in \mathcal{T}_U : \tilde{\nabla}_X E_i := \tilde{\omega}_i^j(X)E_j, \quad \forall f \in \mathcal{C}^\infty(U) : \tilde{\nabla}_X(fE_i) = X(f)E_i + f\tilde{\nabla}_X E_i,$$

agrees with the Levi-Civita connection of g. Alltogether, a matrix of 1-forms ω_i^j are the connection 1-forms of the Levi civita connection of a metric g if and only if they satisfy the symmetries (5.1) with respect to g and Cartans first structure equation (5.2) for $\tau = 0$.

(v) Let ∇ be the Levi-Civita connection on M and $R \in \mathcal{T}_1^3(M)$ be the induced curvature tensor. Expanding locally to

$$\forall 1 \leq k \leq m : R(X,Y)E_k = \Omega_k^l(X,Y)E_l\Omega_i^j \tag{5.3}$$

we obtain the *curvature* 2-*forms* $\Omega_k^l \in \Omega^2(U)$. They satisfy *Cartan's second structure equation*

$$\boxed{\forall 1 \leq k, l \leq m : \Omega_k^l = d\omega_k^l + \omega_\nu^l \wedge \omega_k^\nu} \tag{5.4}$$

(vi) The Ricci tensor Ric satisfies

$$\boxed{\mathrm{Ric}_{ik} = \Omega_i^j(E_j, E_k)} \tag{5.5}$$

Nomenclature

g° round metric on the sphere $\mathbb{S}^2$, page 5

$\mathcal{L}$ Lie derivative, page 19

(M_S, g_S) Schwarzschild spacetime, page 5

$\mathbb{N}$ the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$, page 21

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