Dirac Eigenvalues of higher Multiplicity

Nikolai Nowaczyk

Outline

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Introduction
- Classical Spin Geometry
- Main Result
- Main Idea

Proof of Main Theorem
- Construction of the Set A
- Construction of the Bundle E
- Construction of the Loop $\mathbf{g}$
  - Loops of Metrics via Loops of Diffeomorphisms
  - Surgery Stability

General Setup

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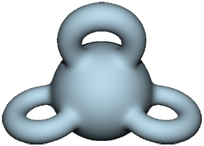
- spec $\not{D}^g_{\mathbb{K}} \subset \mathbb{R}$ spectrum of $\not{D}^g_{\mathbb{K}}$

Dirac Spectral Geometry



(M, g, Θ^g)

Riemannian spin manifold

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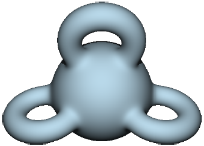

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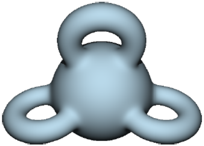

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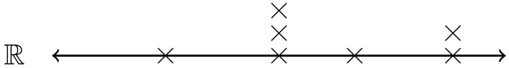

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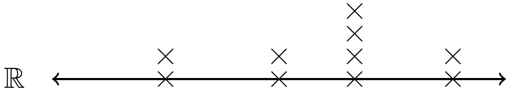

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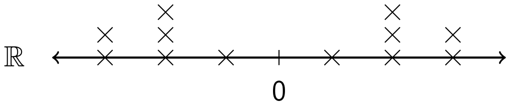

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4. $\dim \ker D\!\!\!/^{g} \geq c(M)$ (**Atiyah-Singer index theorem**)

$$c(M) = \begin{cases} |\hat{A}(M)|, & m \equiv 0, 4 \mod 8, \\ 1, & m \equiv 1 \mod 8 \text{ and } \alpha(M) \neq 0, \\ 2, & m \equiv 2 \mod 8 \text{ and } \alpha(M) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

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Apart from these, it seems that Dirac spectra can take any form.

Properties of Dirac Spectra

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Conjecture (Dahl '05)**:** For any spin manifold (M, Θ) and any $\lambda_1 \leq \ldots \leq \lambda_n \in]\Lambda_1, \Lambda_2[$ satisfying 2.-4. there exists a metric g on M such that

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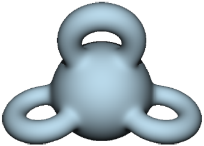
Can you go the other way around?



fix (M, Θ)

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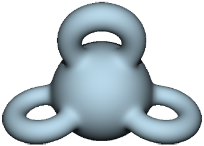

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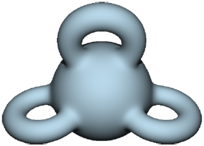

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Dirac spectrum

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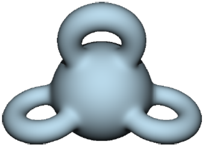

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$\lambda_j = \lambda_j(g)$ at least for finitely many j?

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A subtlety: $\lambda \in \mathbb{R}$ is **simple**, if

$$\dim_{\mathbb{K}} \ker(D\!\!\!/^g_{\mathbb{C}} - \lambda) = 1, \qquad \mathbb{K} = \begin{cases} \mathbb{H}, & m \equiv 2,3,4 \mod 8, \\ \mathbb{C}, & \text{otherwise} \end{cases}$$

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Main Theorem (Nowaczyk 2014): Let (M, Θ) be a closed spin manifold of dimension $m \equiv 0, 6, 7 \mod 8$.

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1. There exists a Riemannian metric g on M such that the Dirac operator $D\!\!\!/^g$ has at **least one eigenvalue of multiplicity at least two**.
2. The metric g can be chosen such that it agrees with an arbitrary metric $\tilde{g}$ outside an arbitrary small neighborhood on the manifold.

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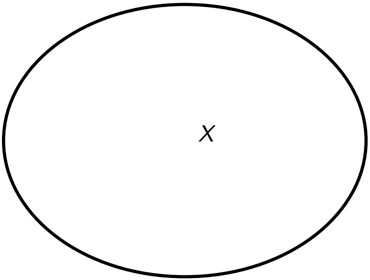
Lasso Lemma

Lemma: Let $X := (\mathcal{R}(M), \mathcal{C}^1)$ (simply connected)



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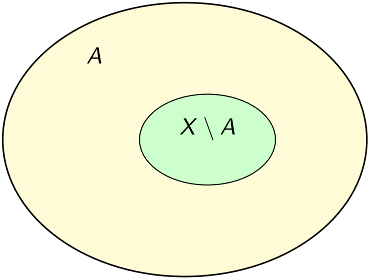
Lemma: Let $X := (\mathcal{R}(M), \mathcal{C}^1)$ (simply connected) and $\mathcal{R}_1(M) \subset A \subset X$ be any subspace.



$$\mathcal{R}_1(M) := \{g \in \mathcal{R}(M) \mid \text{all eigenvalues of } \not{D}^g \text{ are simple}\}$$

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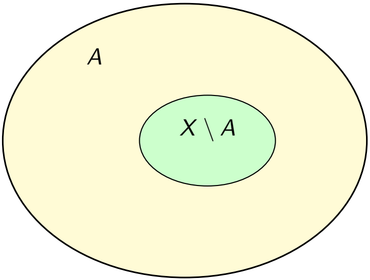
Want: Then $X \setminus A$ not empty.



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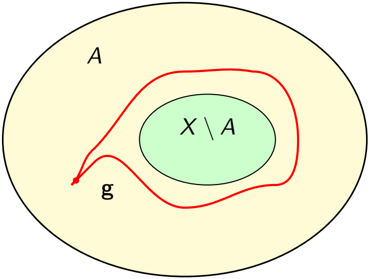
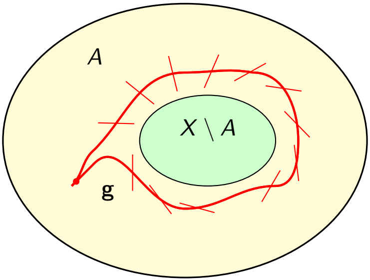
Lemma: Let $X := (\mathcal{R}(M), \mathcal{C}^1)$ (simply connected) and $\mathcal{R}_1(M) \subset A \subset X$ be any subspace. **Let $\mathbf{g} : S^1 \to A$ be not null-homotopic.** Then $X \setminus A \neq \emptyset$.



Lasso Lemma

Lemma: Let $X := (\mathcal{R}(M), \mathcal{C}^1)$ (simply connected) and $\mathcal{R}_1(M) \subset A \subset X$ be any subspace. **Let $E \to A$ be a vector bundle such that $\mathbf{g}^* E \to S^1$ is not trivial.** Then $X \setminus A \neq \emptyset$.



$$\begin{array}{ccc} \mathbf{g}^* E & \longrightarrow & E \\ \downarrow & & \downarrow \\ S^1 & \xrightarrow{\mathbf{g}} & A \end{array}$$

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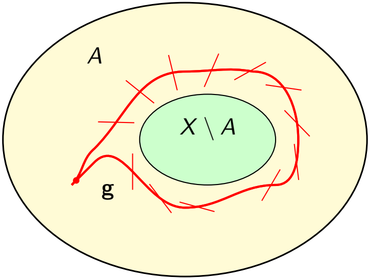

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Construction of A

Definition: For $n \in 2\mathbb{N}+1$ define

$$A := \{g \in \mathcal{R}(M) \mid \exists 1 \leq j \leq n : \mu_{\mathbb{R}}(\lambda_j(g)) \in 2\mathbb{N}+1, \\ \lambda_0(g) < \lambda_1(g), \lambda_n(g) < \lambda_{n+1}(g)\}$$

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Remark:

$$X \setminus A = \{g \in \mathcal{R}(M) \mid \forall 1 \leq j \leq n : \mu_{\mathbb{R}}(\lambda_j(g)) \in 2\,\mathbb{N} \text{ or } \\ \lambda_0(g) = \lambda_1(g) \text{ or } \lambda_n(g) = \lambda_{n+1}(g)\} \\ \subset \{g \in \mathcal{R}(M) \mid \exists 1 \leq j \leq n : \mu_{\mathbb{R}}(\lambda_j(g)) \in 2\,\mathbb{N}\}.$$

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Theorem (Bär 1996)**:** A bounded spectral interval of the Dirac operator can be described locally by continuous functions.

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Theorem (Bär 1996): Let $\Lambda > 0$ such that $\pm\Lambda \notin \operatorname{spec} \not{D}^g_{\mathbb{C}}$ and enumerate

$$\operatorname{spec} \not{D}^g_{\mathbb{C}} \cap]-\Lambda, \Lambda[= \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\}.$$

For any $\varepsilon > 0$, there exists a $\mathcal{C}^1$-neighborhood U of g such that for any $g' \in U$, we have

$$\operatorname{spec} \not{D}^{g'}_{\mathbb{C}} \cap]-\Lambda, \Lambda[= \{\lambda'_1 \leq \cdots \leq \lambda'_n\}$$

and

$$\forall 1 \leq i \leq n : |\lambda_i - \lambda'_i| < \varepsilon.$$

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2. and represents all the eigenvalues of $\not{D}^g$ (counted with multiplicities).
3. **In addition, the sequence $\{\operatorname{arsinh}(\lambda_j)\}_{j\in\mathbb{Z}}$ is equicontinuous.**

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Definition of E

Remark: Recall

$$A = \{g \in \mathcal{R}(M) \,|\exists 1 \leq j \leq n : \mu_{\mathbb{R}}(\lambda_j(g)) \in 2\,\mathbb{N}+1, \\ \lambda_0(g) < \lambda_1(g), \lambda_n(g) < \lambda_{n+1}(g)\}$$

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Definition:

$$E := \coprod_{g \in A} \sum_{1 \le j \le n} \ker(\not{D}_{\mathbb{R}}^{g} - \lambda_j(g)) \to A$$

Universal Spinor Field Bundle

Theorem (Bourguignon, Gauduchon 1992): For any $g, h \in \mathcal{R}(M)$, there exists an **identification isomorphism**

$$\bar{\beta}_{g,h} : L^2(\Sigma^g_{\mathbb{K}} M) \to L^2(\Sigma^h_{\mathbb{K}} M),$$

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Theorem: The **universal spinor field bundle**

$$\begin{array}{rcl} L^2(\Sigma_{\mathbb{K}} M) := \coprod_{g \in \mathcal{R}(M)} L^2(\Sigma^g_{\mathbb{K}} M) & \to & \mathcal{R}(M) \\ \psi \in L^2(\Sigma^g_{\mathbb{K}} M) & \mapsto & g \end{array}$$

carries a unique **Hilbert bundle** topology such that any identification isomorphism yields a global trivialization.

Outline

Loops induced by Diffeomorphisms

Definition: Let $F : S^1 \to \mathrm{Diff}(M)$ be a loop of diffeomorphisms and $g \in \mathcal{R}(M)$ be any metric. Then

$$\begin{aligned} \mathbf{g} : S^1 &\to \mathcal{R}(M) \\ \alpha &\mapsto g_\alpha := (F_\alpha^{-1})^* g \end{aligned}$$

is the associated loop of metrics.

Spin Diffeomorphisms

Definition: A diffeomorphism $f : M \to M$ is a **spin diffeomorphism**, if

$$\begin{array}{ccc}
\widetilde{\mathrm{GL}}^{+} M & \overset{\exists \tilde{F}}{\dashrightarrow} & \widetilde{\mathrm{GL}}^{+} M \\
{\scriptstyle 2:1}\downarrow{\scriptstyle \Theta} & & {\scriptstyle 2:1}\downarrow{\scriptstyle \Theta} \\
\mathrm{GL}^{+} M & \xrightarrow{f_*} & \mathrm{GL}^{+} M \\
\downarrow & & \downarrow \\
M & \xrightarrow{f} & M
\end{array}$$

commutes.

Odd and even Loops

Let M be connected.

$$\mathrm{Diff}^{\mathrm{spin}}(M) := \{f \in \mathrm{Diff}(M) \mid \exists \tilde{F} : \Theta \circ \tilde{F} = f_* \circ \Theta\}$$

$$\widetilde{\mathrm{Diff}}^{\mathrm{spin}}(M) := \{(f, \tilde{F}) \mid f \in \mathrm{Diff}^{\mathrm{spin}}(M)\}$$

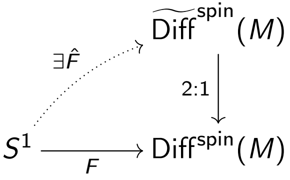
Odd and even Loops

Let M be connected.

$$\mathrm{Diff}^{\mathrm{spin}}(M) := \{f \in \mathrm{Diff}(M) \mid \exists \tilde{F} : \Theta \circ \tilde{F} = f_* \circ \Theta\}$$

$$\widetilde{\mathrm{Diff}}^{\mathrm{spin}}(M) := \{(f, \tilde{F}) \mid f \in \mathrm{Diff}^{\mathrm{spin}}(M)\}$$

Definition: A loop $F : S^1 \to \mathrm{Diff}^{\mathrm{spin}}(M)$ is **even**, if



and **odd** otherwise.

The sgn Invariant

Remark: Let $F : [0,1] \to \mathrm{Diff}^{\mathrm{spin}}(M)$ be a loop. Then the lift

$$\begin{array}{ccc} & \overset{\exists \hat{F}}{\nearrow} & \widetilde{\mathrm{Diff}}^{\mathrm{spin}}(M) \\ & & \downarrow {\scriptstyle 2:1} \\ [0,1] & \xrightarrow[F]{} & \mathrm{Diff}^{\mathrm{spin}}(M) \end{array}$$

exists always, but $\hat{F}(1) = \pm \hat{F}(0)$.

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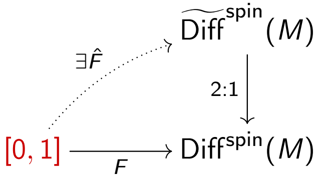

exists always, but $\hat{F}(1) = \pm \hat{F}(0)$. This defines an **invariant**

$$\begin{aligned} \mathrm{sgn} : \pi_1(\mathrm{Diff}^{\mathrm{spin}}(M), \mathrm{id}_M) &\to \mathbb{Z}_2 \\ F &\mapsto \begin{cases} +1, & F \text{ is even} \\ -1, & F \text{ is odd}. \end{cases} \end{aligned}$$

Non-Orientability

Theorem: Let $F : S^1 \to \mathrm{Diff}^{\mathrm{spin}}(M)$ and $g \in A$ ($\Longrightarrow \mathbf{g} : S^1 \to A$).

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Recall:

$$\begin{array}{ccc} \mathbf{g}^* E & \longrightarrow & E \\ \downarrow & & \downarrow \\ S^1 & \xrightarrow{\mathbf{g}} & A \end{array}$$

In that case, the hypothesis of the Lasso Lemma is satisfied and we can conclude $X \setminus A \neq \emptyset$.

Odd Loops of Metrics

Definition: A spin manifold (M, Θ) **admits an odd loop of metrics**, if there exists a continuous map

$$\mathbf{g} : (S^1, \tau_{S^1}) \to (A, \mathcal{C}^2)$$

such that the bundle E is odd dimensional and

$$\mathbf{g}^* E \to S^1$$

is non-orientable.

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Finding $F : S^1 \to \mathrm{Diff}^{\mathrm{spin}}(M)$ is more difficult than
finding $\mathbf{g} : S^1 \to A$ is more difficult than
finding $g \in \mathcal{R}(M)$ as required by the Main Theorem.

The Sphere

Theorem (odd loops on the sphere)**:** The sphere (S^m, Θ), $m \geq 3$, admits an odd loop of metrics.

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Lemma: The map $F : S^1 \to \mathrm{Diff}^{\mathrm{spin}}(S^m)$, $\alpha \mapsto R_{2\pi\alpha}|_{S^m}$, where

$$\forall \alpha \in \mathbb{R} : R_\alpha := \begin{pmatrix} I_{m-1} & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix} : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$$

is an odd loop of spin diffeomorphisms.

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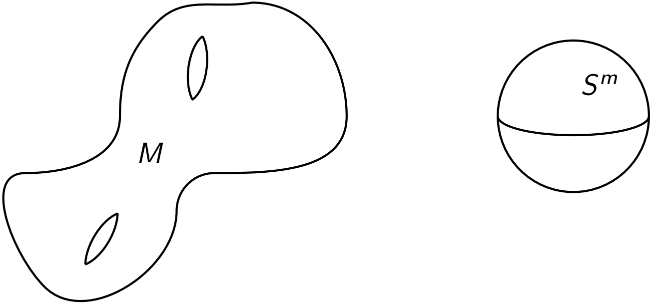
$n := \dim \ker(D\!\!\!/^g - \lambda)$.

Surgery

Idea: Transport odd loop of metrcis from S^m to M^m



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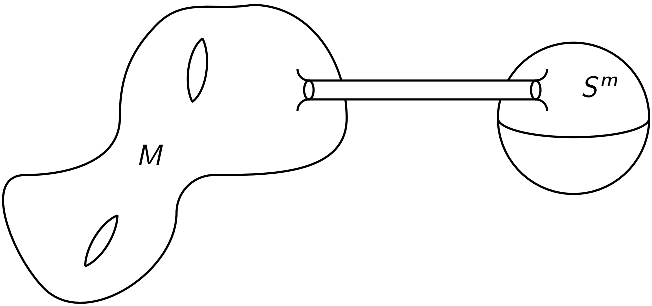

$$N := M \amalg S^m \rightsquigarrow \tilde{N} = M \sharp S^m \cong M.$$

Surgery and Dirac Spectra

Theorem (Bär, Dahl 2002): Let (N, g) be a closed Riemannian spin manifold. Let $S \subset N$ be an embedded sphere of codimension $m - k \geq 3$ and with trivialized tubular neighborhood. Let $\tilde{N}$ be obtained from N by surgery along S together with the resulting spin structure. Let $\varepsilon > 0$ and $\Lambda > 0$, $\pm\Lambda \notin \operatorname{spec} D\!\!\!/^{g}$. Then there exists a Riemannian metric $\tilde{g}$ on $\tilde{N}$ such that $D\!\!\!/^{g}$ and $D\!\!\!/^{\tilde{g}}$ are (Λ, ε)-spectral close.

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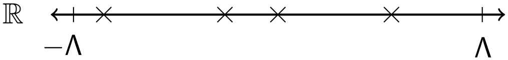

Dirac spectrum before surgery

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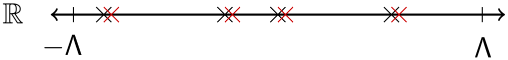
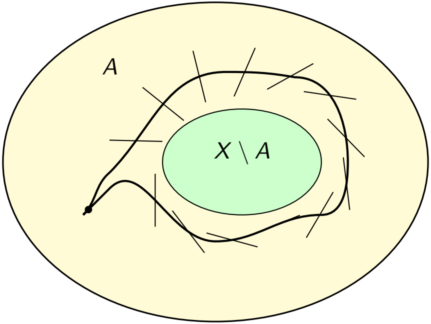


Dirac spectrum after surgery

Surgery Stability

Theorem (Nowaczyk 2014)**:** Odd loops of metrics are stable under surgeries.



$X = \mathcal{R}(N)$

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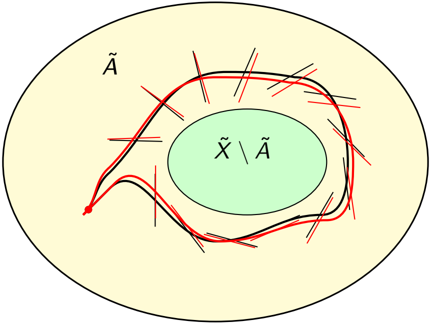

$$\tilde{X} = \mathcal{R}(\tilde{N})$$

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1. Construction of the set A ✓
   $\rightsquigarrow$ Continuity of Dirac Spectra
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   $\rightsquigarrow$ Universal Spinor Bundle
3. Construction of the Loop $\mathbf{g}$
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$\Longrightarrow$ **Existence of Dirac eigenvalues of higher multiplicity is not topologically obstructed in dimensions $m \equiv 0, 6, 7 \mod 8$.**

Comparison to other operators

- For the Laplace operator $g \mapsto \Delta^g$ on a compact manifold, it is kown that finitely many eigenvalues can be prescribed arbitrarily including multiplicity (by Verdiere 1986-1993, Jammes 2009-2012).
- For the Laplace operator $\Omega \mapsto \Delta_\Omega$ on a domain Ω with Dirichlet boundary conditions, the spectrum cannot be prescribed arbitrarily if the volume of Ω is fixed. (Henrot 2006)
- For the Sturm-Liouville operator,

  $$(p, q) \mapsto -\Big(\frac{d}{dx}\Big(p \cdot \frac{d}{dx}\Big) + q\Big),$$

  (with boundary conditions) higher multiplicities do not exist. (Hartmann 1964)

Thanks for your attention.

Open Problems

- Can one prescribe double eigenvalues?
- Is the set of metrics $\mathcal{R}_1(M)$ for which all Dirac eigenvalues are simple generic in $\mathcal{R}(M)$?
- Can one build a finite dimensional spinor bundle that is independent of the metric?
- Can one approach these problems by variational techniques?