Overview of Atiyah-Singer Index Theory

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Abstract. The aim of this text is to give an overview of the Index Theorems by Atiyah and Singer. Our primary motivation is to understand the formulation of the $C\ell_k$-linear Index Theorem. The primary reference for this is [LM89].

Contents

1. Reminder on K-Theory

[LM89, I.§9, 10]

Definition 1.1 ($K(X)$)**.** Let X be a compact space and let $V(X)$ be the isomorhpism classes of complex vector bundles over X. We define

$$K(X) := F(X)/E(X),$$

where $F(X)$ is the free abelian semi-group generated by elements of $V(X)$ and $E(X)$ is the subgroup in $F(X)$ generated by elements of the form $[V] + [W] - ([V] \oplus [W])$, where $+$ is addition in $F(X)$ and $\oplus$ is addition in $V(X)$. This is a ring with respect to

$$[u] \cdot [v] := \Delta^*[u \otimes v],$$

where $\Delta : X \to X \times X$ is the diagonal map. ♢

Definition 1.2 ($KO(X)$)**.** $KO(X)$ is defined exactly as $K(X)$, but with $V(X)$ replaced by $V_{\mathbb{R}}(X)$, the isomorphism classes of real vector bundles. ♢

Lemma 1.3 (Functoriality)**.** *K and KO are functors from TOP to RINGS. In particular, if $f : X \to Y$ is a map, we get an induced map $K(f) : K(Y) \to K(X)$ constructed using the pull-back $f^* : V(Y) \to V(X)$.* ♢

Definition 1.4 ($\tilde{K}(X)$)**.** Let $i : \{\mathrm{pt}\} \to X$ be the inclusion. Let $\tilde{K}(X)$ be the kernel of the induced map $K(i) : K(X) \to K(\mathrm{pt})$. We obtain a split exact sequence

$$0 \longrightarrow \tilde{K}(X) \longrightarrow K(X) \longrightarrow K(\mathrm{pt}) = \mathbb{Z} \longrightarrow 0.$$ ♢

Definition 1.5 ($K^{-i}(X)$)**.** For any space X, let $\Sigma(X) := S^1 \wedge X$ be the *reduced suspension of* X and $\Sigma^i(X) \approx S^i \wedge X$ be the i-fold suspension, $i \in \mathbb{N}$. We define for any $Y \subset X$:

$$\tilde{K}^{-i}(X) := \tilde{K}(\Sigma^i(X)), \qquad K^{-i}(X) := \tilde{K}^i(X/Y) := \tilde{K}(\Sigma^i(X/Y)).$$ ♢

Definition 1.6 (L-Theory)**.** Let $Y \subset X$ be a closed subspace. For each $n \geq 1$, let $\mathscr{L}_n(X, Y)$ be the space of tuples $\mathbf{V} = (V_0, \ldots, V_n; \sigma_1, \ldots, \sigma_n)$, where $V_0, \ldots V_n$ are vector bundles over X, $\sigma_i : V_{i-1} \to V_i$ are vector bundle morphisms such that

$$0 \longrightarrow V_0|_Y \xrightarrow{\sigma_1} V_1|_Y \xrightarrow{\sigma_2} \ldots \xrightarrow{\sigma_n} V_n|_Y \longrightarrow 0 \tag{1.1}$$

is an exact sequence. Two such elements $\mathbf{V}$ and $\mathbf{V}'$ are *isomorphic*, if there are bundle isomorphisms $\varphi_i : V_i \to V_i'$ such that

$$\begin{array}{ccc} V_{i-1}|_Y & \xrightarrow{\sigma_i} & V_i|_Y \\ \downarrow{\scriptstyle \varphi_{i-1}} & & \downarrow{\scriptstyle \varphi_i} \\ V_{i-1}' & \xrightarrow{\sigma_i'} & V_i'|_Y \end{array}$$

commutes, $i = 1, \ldots, n$. An element $\mathbf{V} = (V_0, \ldots, V_n; \sigma_1, \ldots, \sigma_n)$ is *elementary*, if there exists i such that

$$V_i = V_{i-1}, \sigma_i = \mathrm{id}, \qquad \forall j \neq i, i-1 : V_j = \{0\}.$$

We say $\mathbf{V}, \mathbf{V}'$ are *equivalent*, if there exist elementary elements $\mathbf{E}_1, \ldots, \mathbf{E}_k, \mathbf{F}_1, \ldots, \mathbf{F}_k \in \mathscr{L}_n(X, Y)$ and an isomorphism

$$\mathbf{V} \oplus \mathbf{E}_1 \oplus \ldots \oplus \mathbf{E}_k \cong \mathbf{V}' \oplus \mathbf{F}_1 \oplus \ldots \oplus \mathbf{F}_l.$$

Denote by $L_n(X, Y)$ the set of all equivalence classes. This is an abelian group under $\oplus$. We get a map $L_n(X, Y) \to L_{n+1}(X, Y)$ by extending as squence with the zero bundle and the zero morphism. We define

$$L(X, Y) := \varinjlim_{n} L_n(X, Y)$$

to be the L-theory of (X, Y). $\lozenge$

Theorem 1.7. There exists a unique equivalence $\chi : L(X, Y) \to K(X, Y)$ satisfying

$$\chi([V_0, \ldots, V_n]) = \sum_{k=0}^{n} (-1)^k [V_k],$$

when $Y = \emptyset$. $\lozenge$

Definition 1.8 (K-Theory with compact support)**.** Let X be locally compact. Then

$$K_{\mathrm{cpt}}(X) := \tilde{K}(X^+),$$

where $X^+ := X \cup \{\mathrm{pt}\}$ is the one point compactification of X. We also set

$$K_{\mathrm{cpt}}^{-i}(X) := K_{\mathrm{cpt}}(X \times \mathbb{R}^i). \qquad \lozenge$$

Remark 1.9. One can show that any element in $K_{\mathrm{cpt}}(X)$ can be represented as the formal difference of two vector bundles over X, which are trivialized outside a compact subset of X. $\lozenge$

Remark 1.10 (L_{cpt})**.** One can also define L_{cpt} in a similar fashion: One replaces the compact space X by a locally compact space X. We require that **??** is exact outside a compact set. We also get isomorphisms $L_1(X)_{\mathrm{cpt}} \to L_2(X)_{\mathrm{cpt}} \to \ldots K_{\mathrm{cpt}}(X)$. Consequently, any element in $L(X)_{\mathrm{cpt}}$ can be represented by a map $\sigma : V_0 \to V_1$ which is an isomorhpism outside a compact set. We denote this equivalence class by

$$[V_0, V_1; \sigma] \in L(X)_{\mathrm{cpt}} \cong K_{\mathrm{cpt}}(X). \tag{1.2}$$

Definition 1.11 (KR-Theory)**.** Consider the category of bundles $(V, c_V) \to (X, c_X)$, where $V \to X$ is a complex vector bundle $c_X : X \to X$ is an involution and c_V is a $\mathbb{C}$-antilinear lift of c_X. Let $VR(X, c_X)$ be the abelian semi-group of isomorphism classes of such bundles. The resulting Grothendieck group

$$KR(X, c_X)$$

is the KR-Theory of (X, c_X). $\lozenge$

Remark 1.12. One can also consider an LR-Theory and $KR_{\mathrm{cpt}}(X, Y)$ in an analogous fashion. $\lozenge$

2. $C\ell_k$-LINEARITY AND REAL DIRAC BUNDLES

Remark 2.1. In this section, all the bundles and operators are real. ◊

2.1. $C\ell_k$-linear Dirac Operators

[LM89, II.§7]

Definition 2.2 ($\not{S}(X)$)**.** Let (X, g) be a Riemannian spin manifold of dimension n and $\rho : \mathrm{Spin}_n \to \mathrm{Aut}(V)$ be a real spinor representation. Then

$$\not{S}(X) := P_{\mathrm{spin}}(X) \times_\rho V \to X$$

is the *spinor bundle of* X. ◊

Definition 2.3 ($C\ell(X)$)**.** Let (X, g) be a Riemannian spin manifold of dimension n. Then

$$C\ell(X) := \coprod_{x \in X} C\ell(T_x X, g_x) \to X$$

is the *Clifford-Algebra bundle of* X. ◊

Definition 2.4 (Spinor-Clifford bundle)**.** Let X be a spin manifold of dimension n, l : $\mathrm{Spin}_n \to \mathrm{Iso}(C\ell_n)$ be the left multiplication. We define

$$\not{\mathfrak{S}}(X) := P_{\mathrm{spin}}(X) \times_l C\ell_n \,.$$

This bundle carries

- A canonical connection ∇ just as $\not{S}(X)$.
- A canonical right multiplication $\not{\mathfrak{S}}(X) \times C\ell_n \to \not{\mathfrak{S}}(X)$ and therefore, the fibres are $C\ell_n$-modules of rank 1. This multiplication is parallel.
- A canonical left action of $C\ell(X)$ that commutes with the right multiplication.
- A $\mathbb{Z}_2$-grading $\not{\mathfrak{S}}(X) = \not{\mathfrak{S}}^0(X) \oplus \not{\mathfrak{S}}^1(X)$ over $C\ell(X)$ satisfying

  $$\forall i, j \in \mathbb{Z}_2 : \not{\mathfrak{S}}(X)^i \cdot C\ell_n^j \subseteq \not{\mathfrak{S}}^{i+j}(X). \tag{2.1}$$

  This splitting is induced from $C\ell_n = C\ell_n^0 \oplus C\ell_n^1$.
- A Dirac-Operator $\not{\mathfrak{D}} : \Gamma(\not{\mathfrak{S}}(X)) \to \Gamma(\not{\mathfrak{S}}(X))$, which is $C\ell_n$*-linear*, i.e. it commutes with the action of $C\ell_n$. With respect to the splitting, this operator is of course of the form

  $$\not{\mathfrak{D}} = \begin{pmatrix} 0 & \not{\mathfrak{D}}^1 \\ \not{\mathfrak{D}}^0 & 0. \end{pmatrix}$$ ◊

Lemma 2.5. The operator $\not{\mathfrak{D}}^0 : \Gamma(\not{\mathfrak{S}}^0(X)) \to \Gamma(\not{\mathfrak{S}}^1(X))$ is a real, elliptic first-order operator which commutes with the action of $C\ell_n^0 \cong C\ell_{n-1}$ on $\not{\mathfrak{S}}(X) = \not{\mathfrak{S}}^0(X) \oplus \not{\mathfrak{S}}^1(X)$. ◊

Definition 2.6 ($C\ell_k$-Dirac bundle). A $C\ell_k$-*Dirac bundle* over a Riemannian manifold X is a real Dirac bundle $\mathfrak{S} \to X$ together with a right action $C\ell_k \to \mathrm{Aut}(\mathfrak{S})$ which is parallel and commutes with multiplication by elements of $C\ell(X)$. Such a bundle is Z_2*-graded*, if it is $\mathbb{Z}_2$-graded as a Dirac bundle $\mathfrak{S} = \mathfrak{S}^0 \oplus \mathfrak{S}^1$ and the splitting is also a $\mathbb{Z}_2$-grading for the right action, i.e. (2.1) is satisfied. This also yields a Dirac operator $\mathfrak{D}$. ◊

Definition 2.7 (analytic index). Let X be compact and $\mathfrak{S} \to X$ be a $C\ell_k$-linear $\mathbb{Z}_2$-graded Clifford bundle with Dirac operator $\mathfrak{D}^0 : \Gamma(\mathfrak{S}^0) \to \Gamma(\mathfrak{S}^1)$. Then

$$\mathrm{ind}_k(\mathfrak{D}^0) := [\ker \not{D}^0] \in \mathfrak{M}_{k-1}/i^*\mathfrak{M}_k \cong KO^{-k}(\mathrm{pt}) \cong \begin{cases} \mathbb{Z}, & k \equiv 0 \mod 4, \\ \mathbb{Z}_2, & k \equiv 1,2 \mod 8, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Remark 2.8 (Explaination of (2.2)). Since $\mathfrak{D}$ commutes with $C\ell_k^0 \cong C\ell_{k-1}$, $\ker \mathfrak{D}^0$ is a finite-dimensional $C\ell_{k-1}$-module. Consequently, $\ker \not{D}^0$ determines an element in the Grothendieck group $\mathfrak{M}_{k-1}$ of isomorphism classes of $C\ell_{k-1}$-modules. Let $i : C\ell_{k-1} \to C\ell_k$ be induced by the canonical inclusion $\mathbb{R}^{k-1} \to \mathbb{R}^k$. Then $[\ker \not{D}^0]$ simply denotes the residue class. The isomorphism to $KO^{-k}(\mathrm{pt})$ is the *Atiyah-Bott-Shapiro-Isomorphism*, see [LM89, I.Prop. 9.27].◊

Remark 2.9 (Alternative Description of the Index). By [LM89, I. Prop. 5.20] there is an equivalce between the category of $\mathbb{Z}_2$-graded modules over $C\ell_n$ and the category of ungraded modules over $C\ell_{n-1}$ induced by projecting

$$C\ell_n = C\ell_n^0 \oplus C\ell_n^1 \mapsto C\ell_n^0 .$$

Consequently, if $\widehat{\mathfrak{M}}_k$ denotes the Grothendieck group of $\mathbb{Z}_2$-graded $C\ell_k$-Clifford modules. Clearly, $\ker \mathfrak{D}$ is a $\mathbb{Z}_2$-graded module and $(\ker \not{D})^0 = \ker \not{D}^0$. Consequently, we can also define

$$\mathrm{ind}_k(\mathfrak{D}) := [\ker \mathfrak{D}] \in \widehat{\mathfrak{M}}_k/i^*\widehat{\mathfrak{M}}_{k+1}.$$

This index agrees with (2.2) under the isomorphism $\widehat{\mathfrak{M}}_k \cong \mathfrak{M}_{k-1}$. ◊

Lemma 2.10. ind_k is a generalization of ind in the sense that

$$\mathrm{ind}_0(\mathfrak{D}) = \mathrm{ind}(\mathfrak{D}) = \dim_{\mathbb{R}} \ker \mathfrak{D}^0 - \dim_{\mathbb{R}} \mathrm{coker}\, \mathfrak{D}^0$$ ◊

Proof. First notice that $C\ell_0 = \mathbb{R}$ and $C\ell_1 = \mathbb{C}$. A $\mathbb{Z}_2$-graded $C\ell_0$-module is just a pair of real vector spaces $V = V^0 \oplus V^1$. Now

$$V \oplus 0 + 0 \oplus V = V \oplus V \cong V \otimes \mathbb{C}$$

is a graded $C\ell_1 = \mathbb{C}$-module, thus $[V \oplus 0] = -[0 \oplus V]$ and therefore

$$\begin{aligned} \mathrm{ind}_0(\mathfrak{D}) &= [\ker \mathfrak{D}] \\ &= [\ker \mathfrak{D}^0 \oplus \ker \mathfrak{D}^1] \\ &= [\ker \mathfrak{D}^0 \oplus 0] + [0 \oplus \ker \mathfrak{D}^1] \\ &= [\ker \mathfrak{D}^0 \oplus 0] - [\ker \mathfrak{D}^0 \oplus 0] \\ &\cong \dim_{\mathbb{R}} \ker \mathfrak{D}^0 - \dim_{\mathbb{R}} \mathrm{coker}\, \mathfrak{D}^0 \end{aligned}$$ □

2.2. Analytic Clifford Index

[LM89, III.§10]

Definition 2.11 (Cℓ_k-bundle)**.** A Cℓ_k-bundle on a space X is a bundle $E \to X$ of real right[1] Cℓ_k-modules, i.e. $E \to X$ is a real vector bundle together with a continuous map $\Psi : \mathrm{C}\ell_k \times E \to E$ such that $\Psi_\varphi : E \to E$ is a bundle endomorphism for all $\varphi \in \mathrm{C}\ell_k$ and the restriction $\mathrm{C}\ell_k \times E_x \to E_x$ makes the fibre into a Cℓ_k-module for each $x \in X$. ♢

Definition 2.12 (analytic Index)**.** Let X be compact, $E \to X$ be a Cℓ_k bundle with $\mathbb{Z}_2$-grading, P be an elliptic graded self-adjoint PDO. Then

$$\mathrm{ind}_k(P) := [\ker P] \in \widehat{\mathfrak{M}}_k / i^* \widehat{\mathfrak{M}}_{k+1} \cong KO^{-k}(\mathrm{pt})$$

is the *analytic index of* P. ♢

3. Overview of complex Index Theory

3.1. Analytic index of a PDO

[LM89, III. §1]

Definition 3.1 (PDO)**.** Let $E, F \to X$ be $\mathbb{C}$-vector bundles over manifold X. A linear map $P : \Gamma(E) \to \Gamma(F)$ is a PDO of order $m \in \mathbb{N}$, if locally

$$P = \sum_{|\alpha| \leq m} A^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}. \qquad ♢$$

Definition 3.2 (Symbol)**.** For any P as above, we obtain the *symbol of* P, $\sigma(P) \in \Gamma(\bigodot^m TX \otimes \mathrm{Hom}(E, F))$ defined loclly by

$$\forall x \in X : \forall \xi \in T_x^* X : \sigma_\xi(P) := \sum_{|\alpha| = m} i^m A^\alpha \xi_\alpha \in \mathrm{Hom}(E_x, F_x). \qquad ♢$$

Definition 3.3 (elliptic)**.** We say P is *elliptic*, if $\sigma_\xi(P)$ is an isomorphism for all $0 \neq \xi \in T^*X$. ♢

[LM89, III. §7]

Definition 3.4 (analytic index)**.** Let P be a PDO of order $m \in \mathbb{N}$ and consider any Fredholm extension $P : L^2_s(E) \to L^2_{s-m}(F)$. Then

$$\text{a-ind}(P) := \dim \ker P - \dim \operatorname{coker} P \in \mathbb{Z}$$

is the *analytic index of* P. ♢

[1] In [LM89], there is a *left* here. We use a *right* action here in order to make this definition more compatible with Definition 2.6. Of course this is just cosmetics.

3.2. Topological index of a PDO

[LM89, III. §13]

Again, let $E, F \to X$ be complex vector bundles and $P : \Gamma(E) \to \Gamma(F)$ be a PDO of order m.

Definition 3.5 (K-Theory-class of principal symbol)**.** Consider the pullback diagram

$$\begin{array}{ccc} \pi^*E, \pi^*F & \longrightarrow & E, F \\ \downarrow & & \downarrow \\ T^*X & \xrightarrow{\pi} & X. \end{array}$$

We define [LM89, III,(1.9),(13.1)]

$$i(P) := [\pi^*E, \pi^*F; \sigma(P)] \in K_{\mathrm{cpt}}(T^*X) \cong K_{\mathrm{cpt}}(TX),$$

see also **??**. ◊

Definition 3.6 (topological index of a PDO)**.** Let $f : X \hookrightarrow \mathbb{R}^N$ be a smooth embedding for N large enough. This induces an embedding

$$f_! : K_{\mathrm{cpt}}(TX) \to K_{\mathrm{cpt}}(T\,\mathbb{R}^N),$$

see [LM89, III.(12.7)]. Now, consider $T\,\mathbb{R}^N = \mathbb{R}^N \oplus \mathbb{R}^N = \mathbb{C}^N$ and think of $\mathbb{C}^N$ as a vector bundle $q : \mathbb{C}^N \to \mathrm{pt}$. Let $q_! : K_{\mathrm{cpt}}(\mathbb{C}^N) \to K_{\mathrm{cpt}}(\mathrm{pt}) = K(\mathrm{pt})$ be the inverse of the Thom-Isomorphism $i_!$, see below, and define

$$\text{top-ind}(P) := q_! f_! i(P) \in \mathbb{Z}\,. \qquad \Diamond$$

Theorem 3.7 (Atiyah-Sinder Index Theorem for an operator)**.** Let P be an elliptic operator on a compact manifold. Then

$$\text{a-ind}(P) = \text{top-ind}(P). \qquad \Diamond$$

Remark 3.8 (Thom-Isomorphism)**.** Let $E \to X$ be a complex vector bundle and $i : X \to E$ be the inclusion of X into X via the zero section. Then there exists an isomorphism

$$i_! : K_{\mathrm{cpt}}(X) \to K_{\mathrm{cpt}}(E),$$

called *Thom-Isomorphism*, see [LM89, III. §12]. ◊

Lemma 3.9. Let $f : X \to Y$ be a proper embedding. Assume that the normal bundle $N \to f(X)$ carries a complex structure. Then there exists a natural mapping

$$f_! : K_{\mathrm{cpt}}(X) \to K_!(Y).$$

In particular, if $f : X \to Y$ is a proper embedding of manifolds, there exists an associated map

$$f_! : K_{\mathrm{cpt}}(X) \to K_{\mathrm{cpt}}(Y). \qquad \Diamond$$

Proof. For the first claim, we just define the map $f_!$ to be the composition

$$K_{\text{cpt}}(X) \xrightarrow{i!} K_{\text{cpt}}(N) K_{\text{cpt}}(Y). \qquad \square$$

Here, $i!$ is the Thom-Isomorphism, and the second map is obtained by identifying N with a regular neighborhood of X in Y. For the second claim, notice that if $f : X \to Y$ is a proper smooth embedding of manifolds, $f_* : TX \to TY$ is a proper smooth embedding as well. $\square$

3.3. Analytic Index of a family

[LM89, III.§8]

Definition 3.10. Let $E, F \to X$ be smooth vector bundles.

- We denote by $\text{Diff}(E; X)$ the group of vector bundle automorphisms of $E \to X$ and by $\text{Diff}(X)$ the diffeomorphism group of X. We endow $\text{Diff}(X)$ and $\text{Diff}(E; X)$ with the $\mathcal{C}^\infty$-topology.
- There is a canonical homomorphism

  $$\beta : \text{Diff}(E; X) \to \text{Diff}(X)$$

  of topological groups.
- We define $\mathscr{D} := \text{Diff}(E, F; X)$ to be the subgroup of $\text{Diff}(E \oplus F; X)$, which maps E to E and F to F.
- Let $\text{Op}^m(E, F)$ the space of all PDOs $P : \Gamma(E) \to \Gamma(F)$ of order $\leq m$.
- We have a canonical group action

  $$\mathscr{D} \times \text{Op}^m(E, F) \to \text{Op}^m(E, F), \qquad (g = (g_E, g_F), P) \mapsto g_F \circ P \circ g_E^{-1}. \qquad \diamondsuit$$

Definition 3.11 (structure group)**.** Let $Z \to X$ be a smooth resp. continuous fibre bundle with fibre type Y. Then a subgroup G of $\text{Diff}(Y)$ resp. $\text{Homeo}(Y)$ is a *structure group of* $Z \to X$, if there exists an open cover of X such that all cocycles take values in G. $\diamondsuit$

Definition 3.12 (family of vector bundles)**.** Let A be a Hausdorff space. Then a *family of smooth vector bundles over X paramatrized by A* is a fibre bundle $\mathscr{E} \to A$ such that each fibre is a vector bundle $E \to X$ and the structure group of $\mathscr{E} \to A$ is $\text{Diff}(E; X)$. $\diamondsuit$

Remark 3.13. One should think about X as fixed only up to diffeomorphisms. For any $a \in A$, the fibre of the bundle $\mathscr{E} \to A$ over a is a vector bundle $E_a \to X_a$, isomorphic to $E \to X$. $\diamondsuit$

Remark 3.14. Let $\mathscr{E} \to A$ be a family of vector bundles and $\beta : \text{Diff}(E; X) \to \text{Diff}(X)$ as above. The associated bundle

$$\mathscr{X} := \mathscr{E} \times_\beta X \to A$$

is a bundle with structure group $\text{Diff}(X)$ and $\mathscr{E} \to \mathscr{X}$ is a vector bundle, i.e. we have a sequence

$$\mathscr{E} \to \mathscr{X} \to A$$

and over any $a \in A$ lies the manifold $\mathscr{X}_a$ and over $\mathscr{X}_a$ lies the vector bundle $\mathscr{E}_a \to \mathscr{X}_a$. $\diamondsuit$

Definition 3.15 (continuous pair)**.** A *continuous pair of vector bundles over X parametrized by A* is a bundle $\mathscr{E} \oplus \mathscr{F} \to A$ such that each fibre is a split bundle $E \oplus F \to X$ and whose structure group is $\mathscr{D} = \mathrm{Diff}(E, F; X)$. ♢

Definition 3.16 (operator bundle)**.** Let $\mathscr{E} \oplus \mathscr{F} \to A$ be a continuous pair. Then

$$\mathrm{Op}^m(\mathscr{E}, \mathscr{F}) := \mathscr{E} \oplus \mathscr{F} \times_{\mathscr{D}} \mathrm{Op}^m(E; F) \to A \qquad \diamondsuit$$

is the *operator bundle.* ♢

Definition 3.17 (family of elliptic operators)**.** A *family of elliptic operators* is a section P of the operator bundle $\mathscr{E} \oplus \mathscr{F} \to A$ such that for each $a \in A$, $P_a \in \mathrm{Op}^m(\mathscr{E}_a, \mathscr{F}_a)$ is an elliptic operator. ♢

Definition 3.18 (analytic index)**.** Let P be a family of elliptic operators as above. Then

$$\text{a-ind}(P) := [\ker P] - [\operatorname{coker} P] \in K(A)$$

is the *analytic index of P.* ♢

Remark 3.19. In general, neither $\ker P$ nor $\operatorname{coker} P$ are well-defined vector bundles over A, since their dimensions can jump. Nevertheless, one can show that their formal difference still gives a well-defined element in $K(A)$. ♢

3.4. Topological index of a family

[LM89, III. §15]

Definition 3.20 (topological index of a family)**.** Let $\mathscr{E} \oplus \mathscr{F} \to A$ be a continuous pair and P be a family of elliptic operators on the operator bundle $\mathrm{Op}^m(\mathscr{E}, \mathscr{F}) \to A$, where A is compact Hausdorff. Let $\pi : \mathscr{X} \to A$ again be the underlying family of manifolds. Define

$$T\mathscr{X} := \bigcup_{a \in A} T\mathscr{X}_a$$

to be the *vertical tangent bundle.* For N large enough, we can find a map $f : \mathscr{X} \to A \times \mathbb{R}^N$ such that for each $a \in A$, $f_a : \mathscr{X}_a \hookrightarrow \{a\} \times \mathbb{R}^N$ is an embedding. This induces a map $T\mathscr{X} \to A \times T\mathbb{R}^N$, which induces a map

$$f_! : K_{\mathrm{cpt}}(T\mathscr{X}) \to K_{\mathrm{cpt}}(A \times \mathbb{C}^N). \qquad \diamondsuit$$

Analogously, we get a map $q_! : K_{\mathrm{cpt}}(A \times \mathbb{C}^N) \to K_{\mathrm{cpt}}(A) = K(A)$. The composition

$$\text{top-ind}(P) := q_! f_! \sigma(P) \in K(A)$$

is the *topological index.* Here, $\sigma(P)$ is defined fibrewise as $\sigma(P_a)$. ♢

Theorem 3.21 (Atiyah-Singer Index Theorem for Families)**.** Let P be a family of elliptic operators as above. Then

$$\text{a-ind}(P) = \text{top-ind}(P). \qquad \diamondsuit$$

3.5. Index for $\mathrm{C}\ell_k$-family

[LM89, III. §16]

Remark 3.22. In this section, all the bundles and operators are real. ◊

Definition 3.23 (topological index)**.** Let $E, F \to X$ be real bundles. Consider $\pi : TX \to X$ as equipped with the involution $TX \to TX$, $v \mapsto -v$. Consider $\pi^*(E \otimes \mathbb{C}) \to TX$ as equipped with the complex conjugation. For any real elliptic operator $P : \Gamma(E) \to \Gamma(F)$, we obtain

$$\sigma(P) \in [\pi^*(E \otimes \mathbb{C}), \pi^*(F \otimes \mathbb{C}); \sigma(P)] \in KR_{\mathrm{cpt}}(TX).$$

Choose an embedding $f : X \hookrightarrow \mathbb{R}^N$ such that the associated embedding $TX \hookrightarrow T\mathbb{R}^N$ is compatible with the involutions. Using the Thom-Isomorphism in KR-Theory, we obtain a map

$$f_! : KR_{\mathrm{cpt}}(TX) \to KR_{\mathrm{cpt}}(T\mathbb{R}^N)$$

and compose with $KR_{\mathrm{cpt}}(T\mathbb{R}^N) \to KR_{\mathrm{cpt}}(\mathrm{pt})$. This gives top-ind$(P)$. ◊

Definition 3.24 (topological index of a family)**.** Let P be a family of elliptic operators on a real continuous pair $\mathscr{E} \oplus \mathscr{F} \to A$. Using local triviality, we get a map

$$f_! : KR_{\mathrm{cpt}}(T\mathscr{X}) \to KR_{\mathrm{cpt}}(A \times T\mathbb{R}^N) \cong KR_{\mathrm{cpt}}(A \times \mathbb{C}^N)$$

and there also is a Thom-Isomorphism

$$q_! : KR_{\mathrm{cpt}}(A \times \mathbb{C}^N) \to KR(A) \cong KO(A).$$ ◊

Theorem 3.25 (Atiyah-Singer)**.** Let P be a family of real elliptic operators on a compact manifold paramatrized by a compact Hausdorff space A. Let a-ind$(P) \in KO(A)$ be the analytic index of P (as defined for complex P by replacing complex with real objects). Then

$$\text{a-ind}(P) = \text{top-ind}(P).$$ ◊

REFERENCES

[LM89] H. B. Lawson Jr. and M.-L. Michelsohn. *Spin geometry.* Vol. 38. Princeton Mathematical Series. Princeton, NJ: Princeton University Press, 1989, pp. xii+427. ISBN: 0-691-08542-0.