# The Volume of $n$-dimensional Xmas Balls

Nikolai Nowaczyk

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## Contents



## 1 Eulers Gamma function

**1.1 Definition** (Gamma function)**.** The function $\Gamma : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$

$$x \mapsto \int_0^\infty t^{x-1} \exp(-t) dt$$

is *Eulers Gamma function*. The integral is to be interpreted as an improper Riemann integral.

**1.2 Theorem** (Properties of the Gamma function)**.**

(i) The $\Gamma$ function is well defined.

(ii) The $\Gamma$ function satisfies

$$\Gamma(1) = 1 \qquad\qquad \forall x \in \mathbb{R}_{>0} : x \cdot \Gamma(x) = \Gamma(x+1)$$

(iii) For $n \in \mathbb{N}$ we have

$$\Gamma(n+1) = n!$$

(iv) The $\Gamma$ function in strictly increasing, i.e.

$$\forall x, y \in \mathbb{R} : 1 < x < y \Rightarrow \Gamma(x) < \Gamma(y)$$

*Proof.*

(i) We have to show that for any $x \in \mathbb{R}_{>0}$ the integral

$$\int_0^\infty t^{x-1} \exp(-t) dt = \lim_{\varepsilon \searrow 0} \int_\varepsilon^1 t^{x-1} \exp(-t) dt + \lim_{R \nearrow \infty} \int_1^R t^{x-1} \exp(-t) dt$$

converges. On the one hand

$$t^{x-1} \exp(-t) \leq t^{x-1} \Rightarrow \int_0^1 t^{x-1} \exp(-t) dt \leq \int_0^1 t^{x-1} dt$$

The right hand integral converges if and only if

$$1 - x < 1 \Leftrightarrow 0 < x \Leftrightarrow x > 0$$

which holds by hypothesis. On the other hand

$$\lim_{t\to\infty} t^{x+1}\exp(-t) = 0$$

so there exists $t_0 \in \mathbb{R}$ such that for all $t \geq t_0$:

$$t^{x+1}\exp(-t) \leq 1 \Rightarrow t^{x-1}\exp -t \leq \frac{1}{t^2} = t^{-2} \Rightarrow \int_{t_0}^{\infty} t^{x-1}\exp(-t)dt \leq \int_{t_0}^{\infty} t^{-2}dt$$

and the last integral converges if and only if $2 > 1$ which is clearly true.

(ii) A direct calculation reveals

$$\Gamma(1) = \lim_{R\to\infty}\int_0^R \exp(-t)dt = \lim_{R\to\infty}[-\exp(-t)]_0^R = \lim_{R\to\infty} -\exp(-R) + \exp(0) = 1$$

By partial integration we obtain

$$\int_\varepsilon^R t^x e^{-t}dt = \int_\varepsilon^R t^x(-e^{-t})'dt = -t^x e^{-t}|_\varepsilon^R + x\int_\varepsilon^R t^{x-1}e^{-t}dt$$

By sending $\varepsilon \to 0$, $R \to \infty$ we obtain the desired result. (Notice that it does not matter which limit we take first).

(iii) This follows immediately from (ii).

(iv) We just calculate

$$1 < x < y \Rightarrow 0 < x-1 < y-1 \Rightarrow \forall t \in \mathbb{R}_{>0} : t^{x-1} < t^{y-1} \Rightarrow \int_0^\infty t^{x-1}e^{-t}dt < \int_0^\infty t^{y-1}e^{-t}dt$$

□

**1.3 Lemma** (Asymptotic Behaviour of the $\Gamma$ function)**.**

(i) $n! > \left(\frac{n}{3}\right)^n$

(ii) For any $C > 0$:

$$\lim_{x\to\infty}\frac{C^x}{\Gamma(x)} = 0$$

*Proof.*

(i) We use induction over $n$. Clearly the statement holds for $n = 1$. The induction step follows via

$$\left(\frac{n+1}{3}\right)^{n+1} = \left(\frac{n+1}{3}\right)^n \frac{n+1}{3} = \left(\frac{n}{3}\right)^n\left(1+\frac{1}{n}\right)^n\frac{n+1}{3} < n!(n+1)\frac{e}{3} < (n+1)!$$

(ii) First consider the subseries of even integers $n = 2m$. In that case

□

In other words

$$e^n \in o(\Gamma(n))$$

## 2 The $n$-dimensional unit ball

**2.1 Definition.** For any $n \in \mathbb{N}$ we denote by

$$B^n := \{x \in \mathbb{R}^n | ||x||_2 \leq 1\}$$

the euclidian unit ball and by

$$V^n := \mu(B^n)$$

its volume with respect to the standard $n$-dimensional Lebesgue measure.

**2.2 Theorem** (Integration of rotational symmetric functions)**.** Let $I \subseteq \mathbb{R}$ be an interval and denote by $K(I) := \{x \in \mathbb{R}^n : \inf I \leq ||x||_2 \leq \sup I\}$ the generated ball in $\mathbb{R}^n$. Let $F : K(I) \to \mathbb{R}$ be a rotational symmatric function, i.e. there exists a function $f : I \to \mathbb{R}$ such that $F(x) = f(||x||_2)$. Then $F$ is integrable over $K(I)$ if and only if $r \mapsto f(r)r^{n-1}$ is integrable over $I$ and

$$\int_{K(I)} F(x)dx = nV^n \int_I f(r)r^{n-1}dr$$

*Proof.* Employing polar coordinates (see Appendix) the transformation theorem implies

$$\int_{K(I)} F(x)dx = \int_{I\times\Pi} (F \circ P)(r,\varphi)|\det \nabla P((r,\varphi))|d((r,\varphi)) = \int_I f(r)r^{n-1}dr \int_\Pi C(\varphi)d\varphi \tag{1}$$

So especially

$$V^n = \int_{K(I)} 1dx = \int_{[0,1]} r^{n-1}dr \int_\Pi C(\varphi)d\varphi = \frac{1}{n}\int_\Pi C(\varphi)d\varphi \tag{2}$$

Multiplying by $n$ and substituting (2) into (1) we obtain

$$\int_{K(I)} F(x)dx = nV^n \int_I f(r)r^{n-1}dr \tag{3}$$

□

**2.3 Theorem** (Volume of $n$-balls)**.**

$$V^n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$$

*Proof.* On the one hand Fubinis theorem implies

$$\int_{\mathbb{R}^n} \exp(-||x||_2^2)dx = \int_{\mathbb{R}^n} \exp(\sum_{i=1}^n -x_i^2)dx = \int_{\mathbb{R}^n} \prod_{i=1}^n \exp(-x_i^2)dx$$
$$= \prod_{i=1}^n \int_\mathbb{R} \exp(-x_i^2)dx_i = \left(\int_\mathbb{R} \exp(-t^2)dt\right)^n$$

On the other hand applying theorem 2.2 we obtain

$$\int_{\mathbb{R}^n} \exp(-||x||_2^2)dx = nV^n \int_0^\infty e^{-r^2}r^{n-1}dx = \frac{n}{2}V^n \int_0^\infty e^{-(r^2)}r^{2(\frac{n}{2}-1)}2rdx$$
$$= \frac{n}{2}V^n \int_0^\infty e^{-t}t^{\frac{n}{2}-1}dt = V^n\frac{n}{2}\Gamma\left(\frac{n}{2}\right) = V^n\Gamma\left(\frac{n}{2}+1\right)$$

Combining both calculations for $n=2$ we obtain:

$$\left(\int_{\mathbb{R}} \exp(-t^2)dt\right)^2 = V^2\Gamma\left(\frac{2}{2}+1\right) = \pi\Gamma(2) = \pi\cdot 1\cdot\Gamma(1) = \pi$$

This implies the equation

$$\int_{\mathbb{R}} \exp(-t^2)dt = \sqrt{\pi}$$

which is quite famous by the way.
Alltogether we obtain

$$\pi^{\frac{n}{2}} = \left(\int_{\mathbb{R}} \exp(-t^2)dt\right)^n = \int_{\mathbb{R}^n} \exp(-||x||_2^2)dx = V^n\Gamma\left(\frac{n}{2}+1\right)$$

Rearranging we obtain the desired result. □

**2.4 Corrolary** (Asymptotics)**.**

$$\lim_{n\to\infty} V^n = 0$$

*Proof.* This follows more ore less directly from lemma 1.3,(ii). Alternatively we can use lemma 1.3,(i) by first considering the subseries of even integers $n=2m$:

$$V^n = V^{2m} = \frac{\pi^{\frac{2m}{2}}}{\Gamma(\frac{2m}{2}+1)} = \frac{\pi^m}{m!} < \frac{\pi^m}{\left(\frac{m}{3}\right)^m} = \left(\frac{3\pi}{m}\right)^m \xrightarrow{m\to\infty} 0$$

For the odd subseries write $n=2m+1$, use monotony of the $\Gamma$ function and then the sandwich lemma:

$$0 \xleftarrow{m\to\infty} \frac{\pi^m}{(m+1)!} = \frac{\pi^{\lfloor\frac{n}{2}\rfloor}}{\Gamma(\lceil\frac{n}{2}\rceil+1)} < \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} < \frac{\pi^{\lceil\frac{n}{2}\rceil}}{\Gamma(\lfloor\frac{n}{2}\rfloor+1)} = \frac{\pi^{m+1}}{m!} \xrightarrow{m\to\infty} 0$$

□

## 3 Generalization to $p$-Norm unit balls

**3.1 Definition.** For any $1 \le p \le \infty$, $R > 0$ we denote by

$$B_p^n(R) := \{x \in \mathbb{R}^n | ||x||_p \le R\}$$

the $n$-dimensional ball with radius $R$ respect to the $p$-norm. We denote by

$$B_p^n := B_p^n(1) \qquad\qquad V_p^n := \mu(B_p^n)$$

the Lebesgue measure of the unit ball.

**3.2 Lemma.** We can immediately establish the following relation: For any $R > 0$

$$B_p^n(R) = R^n \cdot B_p^n$$

*Proof.* Clearly the map $T : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Rx$ is a diffeomorphism with functional determinant $R^n$. Since $T(B_p^n) = R^n B_p^n(R)$ the transformation theorem yields

$$B_p^n(R) = \int_{B_p^n(R)} 1dx = \int_{T(B_p^n)} 1dx = \int_{B_p^n} R^n dx = R^n B_p^n$$

□

**3.3 Definition** (Beta function)**.** The function $B : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0}$

$$(x, y) \mapsto \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

is *Eulers Beta function.*

**3.4 Theorem.** The beta function satisfies

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

*Proof.* Applying the theorems of Tonelli and Fubini we obtain existence and equality of the following integrals:

$$\Gamma(x)\Gamma(y) = \int_0^\infty t^{x-1}e^{-t}dt \int_0^\infty s^{y-1}e^{-s}ds = \int_0^\infty \int_0^\infty t^{x-1}s^{y-1}e^{-(s+t)}dsdt$$

We apply the transformation theorem to the map $T : \mathbb{R}_+^2 \to \operatorname{im} T$, $(s, t) \mapsto (s + t, t) =: (\sigma, \tau)$ wich is a diffeomorphism with $\det \nabla T(t, s) = 1$. So

$$= \int_0^\infty \int_0^\infty t^{x-1}(s+t-t)^{y-1}e^{-(s+t}dtds = \int_0^\infty \int_0^\sigma \tau^{x-1}(\sigma-\tau)^{y-1}e^{-\sigma}d\tau d\sigma$$
$$= \int_0^\infty \int_0^\sigma \left(\frac{\tau}{\sigma}\right)^{x-1} \sigma^{y-1+x-1}(1-\frac{\tau}{\sigma})^{y-1}e^{-\sigma}d\tau d\sigma$$

Again we apply transformation theorem to $S : \operatorname{im} T \to \operatorname{im} S$, $(\sigma, \tau) \mapsto (\frac{\tau}{\sigma}, \sigma) =: (u, v)$ with $\det \nabla S = \sigma^{-1}$:

$$= \int_0^\infty \int_0^1 u^{x-1}v^{x+y-1}(1-u)^{y-1}e^{-v}dudv = \int_0^\infty v^{x+y-1}e^{-v}dv \int_0^1 u^{x-1}(1-u)^{y-1}du$$

and the right hand expression is by definition equal to $\Gamma(x + y)B(x, y)$. □

**3.5 Corrolary.**

$$\int_0^{\frac{\pi}{2}} \sin(t)^{2\alpha-1}\cos(t)^{2\beta-1}dt = \frac{1}{2}B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{2\Gamma(\alpha+\beta)}$$

*Proof.*

$$\int_0^{\frac{\pi}{2}} \sin(t)^{2\alpha-1}\cos(t)^{2\beta-1}dt = \int_0^{\frac{\pi}{2}} \sin(t)^{2\alpha-1}\cos(t)^{2\beta-2}\cos(t)dt$$
$$= \int_0^{\frac{\pi}{2}} \sin(t)^{2\alpha-1}(1-\sin(t)^2)^{\beta-1}\cos(t)dt \mid \sin(t) = s$$
$$= \int_0^1 s^{2\alpha-1}(1-s^2)^{\beta-1}ds = \frac{1}{2}\int_0^1 \left(s^2\right)^{\alpha-1}(1-s^2)^{\beta-1}2sds \mid s^2 = u$$
$$= \frac{1}{2}\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du = \frac{1}{2}B(\alpha, \beta)$$

□

**3.6 Theorem** (Main Theorem)**.**

$$V_p^n = \frac{\left(2\Gamma(\frac{1}{p}+1)\right)^n}{\Gamma(\frac{n}{p}+1)}$$

*Proof.* This is a rather complex, but direct calculation. Applying Fubinis theorem and lemma 3.2 we obtain

$$\begin{aligned} V_p^n &= \mu(B_p^n) = \int_{B_p^n} 1dx = \int_{\sum_{i=1}^n |x_i|^p \le 1} 1d(x_1 \dots x_n) = \int_{-1}^1 \int_{\sum_{i=1}^{n-1} |x_i|^p \le 1-|x_n|^p} d(x_1 \dots dx_{n-1})dx_n \\ &= \int_{-1}^1 \int_{B_{n-1}^p((1-|x_n|^p)^{\frac{1}{p}})} d(x_1 \dots dx_{n-1})dx_n \overset{3.2}{=} \int_{-1}^1 (1-|x_n|^p)^{\frac{n-1}{p}} dx_n \cdot V_p^{n-1} \\ &\overset{(*)}{=} \frac{4}{p} \int_0^{\frac{\pi}{2}} \cos(u)^{2\frac{n-1}{p}+1} \sin(u)^{\frac{2}{p}-1} du \cdot V_p^{n-1} \overset{3.5}{=} \frac{4}{p} \cdot \frac{\Gamma(\frac{n+p-1}{p})\Gamma(\frac{1}{p})}{2\Gamma(\frac{n}{p}+1)} \cdot V_p^{n-1} \end{aligned}$$

The step (*) follows from:

$$\begin{aligned} I &:= \frac{4}{p} \int_0^{\frac{\pi}{2}} \cos(u)^{2\frac{n-1}{p}+1} \sin(u)^{\frac{2}{p}-1} du = \frac{4}{p} \int_0^{\frac{\pi}{2}} \cos(u)^{2\frac{n-1}{p}} \sin(u)^{\frac{2}{p}-1} \cos(u) du \\ &= \frac{4}{p} \int_0^{\frac{\pi}{2}} (1-\sin(u)^2)^{\frac{n-1}{p}} \sin(u)^{\frac{2}{p}-1} \cos(u) du \mid \sin(u) = s \\ &= \frac{4}{p} \int_0^1 \left(1-s^2\right)^{\frac{n-1}{p}} s^{\frac{2}{p}-1} ds = \frac{2}{p} \int_0^1 \left(1-s^2\right)^{\frac{n-1}{p}} (s^2)^{\frac{1}{p}-1} 2s ds \mid s^2 = t \\ &= \frac{2}{p} \int_0^1 (1-t)^{\frac{n-1}{p}} t^{\frac{1}{p}-1} dt = 2 \int_0^1 (1-t)^{\frac{n-1}{p}} \left(\frac{1}{p} t^{\frac{1}{p}-1}\right) dt = 2 \int_0^1 \left(1-\left(t^{\frac{1}{p}}\right)^p\right)^{\frac{n-1}{p}} \left(t^{\frac{1}{p}}\right)' ds \mid y = s^{\frac{1}{p}} \\ &= 2 \int_0^1 (1-y^p)^{\frac{n-1}{p}} dy = \int_{-1}^1 (1-|y|^p)^{\frac{n-1}{p}} dy \end{aligned}$$

Alltogether we obtain the following recursion formula

$$V_p^n = \frac{2}{p} \cdot \frac{\Gamma(\frac{n+p-1}{p})\Gamma(\frac{1}{p})}{\Gamma(\frac{n}{p}+1)} \cdot V_p^{n-1}$$

This equation can now be employed to obtain the final result using telescope products (Notice that $V_1^p = 2$ for any $p$. ):

$$\begin{aligned} V_p^n &= \prod_{i=2}^n \frac{V_p^n}{V_p^{n-1}} V_p^1 = 2 \prod_{i=2}^n \frac{2}{p} \cdot \frac{\Gamma\left(\frac{i+p-1}{p}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{i}{p}+1\right)} = \frac{2^n}{p^{n-1}} \Gamma\left(\frac{1}{p}\right)^{n-1} \prod_{i=2}^n \cdot \frac{\Gamma\left(\frac{i-1}{p}+1\right)}{\Gamma\left(\frac{i}{p}+1\right)} \\ &= \frac{2^n}{p^{n-1}} \Gamma\left(\frac{1}{p}\right)^{n-1} \cdot \frac{\Gamma\left(\frac{1}{p}+1\right)}{\Gamma\left(\frac{n}{p}+1\right)} = \frac{\frac{2^n}{p^{n-1}} \Gamma\left(\frac{1}{p}\right)^{n-1} \frac{1}{p} \Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{n}{p}+1\right)} \\ &= \frac{\left(2\frac{1}{p}\Gamma\left(\frac{1}{p}\right)\right)^n}{\Gamma\left(\frac{n}{p}+1\right)} = \frac{\left(2\Gamma\left(\frac{1}{p}+1\right)\right)^n}{\Gamma\left(\frac{n}{p}+1\right)} \end{aligned}$$

□

**3.7 Corrolary** (Asymptotics)**.** For any $1 \le p < \infty$

$$\lim_{n\to\infty} V_p^n = 0$$

*Proof.* This follows from lemma 1.3 or as in the proof of 2.4. □

The case $p = \infty$ is much easier: In that case $B_\infty^n = [-1, 1]^n$, so by definition of the lebesgue measure

$$V_\infty^n = 2^n \xrightarrow{n\to\infty} \infty$$

## 4 Appendix: Polar Coordinates

**4.1 Definition** (Polar Coodinate Map)**.** For each integer $n \geq 2$ the map $P_n : \mathbb{R}^n \to \mathbb{R}^n$ which is recursively defined as follows, is the *polar coordinate map*:

$$P_2(r,\varphi) := \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \end{pmatrix} \qquad P_{n+1}(r,\varphi_1,\dots,\varphi_{n-1}) := \begin{pmatrix} P_n(r,\varphi_1,\dots,\varphi_n)\cos(\varphi_n) \\ r\sin(\varphi_n) \end{pmatrix}$$

**4.2 Theorem** (Functional Determinant)**.** For each $n$ $P_n \in \mathscr{C}^\infty(\mathbb{R}^n,\mathbb{R}^n)$ and

$$\det(\nabla P_n(r,\varphi_1,\dots,\varphi_{n-1}) = r^{n-1}\prod_{k=2}^{n-1}\cos(\varphi_k)^{k-1}$$

*Proof.* We proof this theorem by induction. The smoothness claim follows directly from the definition. In the following we drop the arguments of $P_n$ in notation, i.e. we write $P_n = P_n(r,\varphi_1,\dots,\varphi_{n-1})$.
For $n = 2$ we have

$$\nabla P_2(r,\varphi) = \begin{pmatrix} \cos(\varphi) & -r\sin(\varphi) \\ \sin(\varphi) & r\cos(\varphi) \end{pmatrix} \Rightarrow \det(\nabla P_2(r,\varphi)) = r\cos(\varphi)^2 + r\sin(\varphi)^2 = r$$

For $n \to n+1$ we denote the jacobian of $P_{n+1}$ as a system of columns

$$\nabla P_{n+1} = (\partial_r P_{n+1}, \dots \partial_{\varphi_\nu} P_{n+1} \dots, \partial_{\varphi_n} P_{n+1})$$

which can we expressed as follows:

$$\partial_r P_{n+1} = \begin{pmatrix} \partial_r P_n \cdot \cos(\varphi_n) \\ \sin(\varphi_n) \end{pmatrix}$$

For $1 \leq \nu \leq n-1$:

$$\partial_{\varphi_\nu} P_{n+1} = \begin{pmatrix} \partial_{\varphi_\nu} P_n \cdot \cos(\varphi_n) \\ 0 \end{pmatrix}$$

and

$$\partial_{\varphi_n} P_{n+1} = \begin{pmatrix} -P_n \cdot \sin(\varphi_n) \\ r\cos(\varphi_n) \end{pmatrix}$$

Alltogether we obtain

$$\det(\nabla P_{n+1}) = \begin{vmatrix} \partial_r P_n \cos(\varphi_n) & \dots \partial_{\varphi_\nu} P_n \cos(\varphi_n) \dots & -P_n \sin(\varphi_n) \\ \sin(\varphi_n) & 0 & r\cos(\varphi_n) \end{vmatrix}$$

First we want to proof the following recursion formula

$$\det \nabla P_{n+1} = r\cos(\varphi_n)^{n-1}\det(\nabla P_n)$$

By the representation above clearly this holds, if $\cos(\varphi_n) = 0$. If $\cos(\varphi_n) \neq 0$, we add $(r\sin(\varphi_n)\cos(\varphi_n)^{-1})$ times the first column to the third and obtain at the top rows:

$$-P_n\sin(\varphi_n) + r\sin(\varphi_n)\cos(\varphi_n)^{-1}\partial_r P_n\cos(\varphi_n) = -P_n\sin(\varphi_n) + \sin(\varphi_n) r\partial_r P_n = 0$$

The last equality follows since $r\partial_r P_n = P_n$. This can be seen by induction directly from the definitions.
In the last row we have

$$r\cos(\varphi_n) + r\sin(\varphi_n)\cos(\varphi_n)^{-1}\sin(\varphi_n) = r\cos(\varphi_n)^{-1}(\cos(\varphi_n)^2 + \sin(\varphi_n)^2) = r\cos(\varphi_n)^{-1}$$

So alltogether

$$\det(\nabla P_{n+1}) = \begin{vmatrix} \partial_r P_n \cos(\varphi_n) & \ldots \partial_{\varphi_\nu} P_n \cos(\varphi_n) \ldots & 0 \\ \sin(\varphi_n) & 0 & r\cos(\varphi_n)^{-1} \end{vmatrix}$$

$$= r\cos(\varphi_n)^{-1}\cos(\varphi_n)^n \det(\nabla P_n)$$

where we have expanded the determinant into the last row. This proofs the recursion formula which implies the explicit formula. □

Certainly the map $P_n : \mathbb{R}^n \to \mathbb{R}^n$ is not a global diffeomorphism. We have to restrict properly in order to use it for transformation theorems.

**4.3 Theorem.** For any intervall $I \subseteq [0, \infty[$ and any integer $n$ we define

$$K(I) := \{x \in \mathbb{R}^n | ||x||_2 \in I\}$$

Define furthermoore

$\Pi := ]-\pi, \pi[ \times ]-\pi/2, \pi/2[^{n-2}$ $\quad S := \{(x_1, 0) | x_1 \in \mathbb{R}, x_1 \leq 0\} \subset \mathbb{R}^2$ $\quad \mathbb{R}_0^{n-2} := \{(x_1, \ldots, x_{n-2}, 0, 0)\} \subset \mathbb{R}^n$

Then

$$P_n : I \times \Pi \to K(I) \setminus (S \times \mathbb{R}_0^{n-2})$$

is a diffeomorphism. For $n = 2$ this is to be interpreted as $\Pi = ]-\pi, \pi[$, $S \times \mathbb{R}_0^{n-2} = S$.

## References


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