# Vector Bundles and Pullbacks

Nikolai Nowaczyk

November 2008

## Contents



The main result of this article is theorem 2.9 stating that the pullbacks of a vector bundle over paracompact base spaces via two homotopic maps are isomorphic. So in section 1 we repeat some basic facts about vector bundles mainly to introduce notation. We also repeat the notion of paracompactness, which we will need for the proof of the Homotopy Invariance Theorem. Readers who are already familiar whith these concepts can skip this paragraph entirely.
In section 2 we define pullbacks of vector bundles and proof some algebraic properties. Then we are able to tackle the proof of the Homotopy Invariance Theorem.
Throughout the article, the word *map* will always mean a continuous map.

## 1 Basics

**1.1 Definition** (Vector Bundle)**.** Let $B, E$ be topological spaces and $p : E \to B$. Then $(E, B, p)$ is a *real vector bundle*, if:

(i) The map $p$ is continuous and surjective.

(ii) There exists an $n \in \mathbb{N}$ such that for any $b \in B$ the *fibre* $E_b := p^{-1}(b)$ is an $n$-dimensional vector space. The number $n$ is the *rank* of the vector bundle.

(iii) For any $b \in B$ there exists an open neighourhood $U$ of $b$ and a homeomorphism $h : p^{-1}(U) \to U \times \mathbb{R}^n$ such that $\pi \circ h = p$ where $\pi : U \times \mathbb{R}^n \to U$, $(u, x) \mapsto u$ is the projection onto the first factor and for any $c \in U$ $h : E_c \to \{c\} \times \mathbb{R}^n \cong \mathbb{R}^n$ is a vector space isomorphism. Such an $h$ is a *local trivialization.*

Complex vector bundles are defined entirely analogously. We only demand that for each $b \in B$, the fibre $E_b$ is a complex $n$-dimensional vector space. The space $B$ is called *base space* and $E$ is the *total space.* We also say $p : E \to B$ is a vector bundle, if $(E, B, p)$ is a vector bundle. If the base space is clear from the context we will also shorten this further by saying $E$ is a vector bundle.

**1.2 Definition** (Isomorphism)**.** Let $(E, B, p)$, $(F, B, q)$ be two vector bundles over the same base space. A mapping $\Psi : E \to F$ is a *bundle isomorphism*, if it is a homeomorphism and restricts to a vector space isomorphism $\Psi : E_b \to F_b$ on each fibre. We denote the isomorphism class of a bundle $(E, B, p)$ by $[E, B, p]$.

**1.3 Definition** (Vect)**.** Let $B$ be a topological space. We define

$$\mathrm{Vect}^n(B) := \mathrm{Vect}^n_{\mathbb{R}}(B) := \{[E,B,p] | (E,B,p) \text{ is a real vector bundle of rank } n \text{ over B}\}$$

The set of isomorphism classes of complex vector bundles of rank $n$ over $B$ is denoted by

$$\mathrm{Vect}^n_{\mathbb{C}}(B)$$

**1.4 Lemma.** Let $(E_1, B, p)$, $(E_2, B, p_2)$ be vector bundles over the same base space. Let $\varphi : E_1 \to E_2$ be a continuous map, such that for each $b \in B$ $\varphi : p_1^{-1}(b) \to p_2^{-1}(b)$ is a vector space isomorphism. Then $\varphi$ is a bundle isomorphism.

*Proof.* By hypothesis $\varphi$ is bijective and continuous, so it remains only to show, that $\varphi^{-1}$ is continuous as well. This is a local question, so we chose $U \subset B$ such that $E_1, E_2$ are both trivial over $U$ via the local trivializations $h_1 : p_1^{-1}(U) \to U \times \mathbb{R}^n$, $h_2 : p_2^{-1}(U) \to U \times \mathbb{R}^n$. Then $\varphi$ is an isomorphism if and only if $\psi : U \times \mathbb{R}^n \to U \times \mathbb{R}^n$

$$\psi := h_2 \circ \varphi \circ h_1^{-1}$$

is an isomorphism. By construction there exists a continuous function $g : U \to GL(n)$ such that

$$\psi(u, v) = (u, g(u)v)$$

Since $g$ depends continuously on $u$ and the inverse $g(u)^{-1}$ depends continuously on $g(u)$ (one can see this via Cramers Rule for example) the inverse of $\psi$ wich is

$$\psi^{-1}(u, v) = (u, g(u)^{-1}v)$$

is also continuous. □

## 1.1 Paracompactness

**1.5 Definition** (Paracompactness)**.** A Hausdorff space $X$ is *paracompact*, if for each open cover $\{U_\alpha\}_{\alpha \in A}$ there exists a refinement $\{V_\beta\}_{\beta \in B}$ of $\{U_\alpha\}_{\alpha \in A}$, that is locally finite.

**1.6 Definition** (Partition of Unity)**.** Let $X$ be a topological space and $\mathfrak{U} := \{U_\alpha\}_{\alpha \in A}$ be an open cover of $X$. Then a *partition of unity subordinate to* $\mathfrak{U}$ is and index set $B$ together with a system of maps $\varphi_\beta : X \to I$, $\beta \in B$, such that:

(i) $\forall \beta \in B \exists \alpha \in A : \operatorname{supp} \varphi_\beta \subset U_\alpha$

(ii) For any $x \in X$ there exists an open neighbourhood $U$ of $x$ such that $\operatorname{supp} \varphi_\beta \cap U \neq \emptyset$ for only finitely many $\beta \in B$.

(iii) For any $x \in X$: $\sum_{\beta \in B} \varphi_\beta(x) = 1$.

**1.7 Theorem.** A topological space $X$ is paracompact if and only if for each open cover $\mathfrak{U}$ there exists a partition of unity subordinate to this cover.

**1.8 Theorem.** Every compact Hausdorff space $X$ is paracompact.

**1.9 Lemma.** Let $X$ be a paracompact space.

(i) Let $\mathfrak{U} := \{U_\alpha\}_{\alpha \in A}$ be an open cover of a $X$. Then there is a countable open cover $\mathfrak{V} := \{V_k\}_{k \in \mathbb{N}}$ of $X$ such that each $V_k$ is the disjoint union of open sets each contained in some $U_\alpha$.

(ii) There is a partition of unity $\{\varphi_k\}_{k \in \mathbb{N}}$ such that $\forall k \in \mathbb{N} : \operatorname{supp} \varphi_k \subset V_k$.

*Proof.* If the index set $A$ is countable istelf, we are done. So let $A$ be uncountable. By theorem 1.7 there exists a parition of unity $\{\varphi_\beta\}_{\beta\in B}$ subordinate to $\mathfrak{U}$. For any finite subset $S \subset B$ define

$$V_S := \{x \in X | \forall\beta \in S : \forall\beta' \in B \setminus S : \varphi_\beta(x) > \varphi_{\beta'}(x)\}$$

We claim that $V_S$ is open: For any $x \in V_S \subset X$ there exists a finite set $T \subset B$ and an open neighbourhood $U \subset X$ of $x$ such that

$$\forall\beta \in B : \varphi_\beta|_U = 0 \Leftrightarrow \beta \in T$$

It follows, that

$$\begin{aligned} V_S \cap U &= \{x \in U | \forall\beta \in S : \forall\beta' \in (B \setminus S) \cap T : \varphi_\beta(x) > \varphi_{\beta'}(x)\} \\ &= \bigcap_{\beta\in S} \bigcap_{\beta'\in(B\setminus S)\cap T} \{x \in U | \varphi_\beta(x) > \varphi_{\beta'}(x)\} \\ &= \bigcap_{\beta\in S} \bigcap_{\beta'\in(B\setminus S)\cap T} (\varphi_\beta - \varphi_{\beta'})^{-1}(]0, 1]) \end{aligned}$$

is a finite intersection of open sets and hence open.
If $S = (\beta_1, \dots, \beta_n)$ then there exists $\alpha_1 \in A$ such that $\operatorname{supp}\varphi_1 \subset U_{\alpha_1}$. We claim that $V_S \subset U_{\alpha_1}$. Suppose there is an $x \in V_S$ such that $x \notin U_{\alpha_1}$. This implies $x \notin \operatorname{supp}\varphi_{\beta_1} \Rightarrow \varphi_{\beta_1}(x) = 0$. On the other hand: Since $S$ is finite and $B$ is not, there exists $\beta \in B \setminus S$. This implies $0 = \varphi_{\beta_1}(x) > \varphi_\beta(x) \geq 0$. Contradiction.
We define for any $k \in \mathbb{N}$

$$V_k := \bigcup_{S\subset B, \#S=k} V_S$$

and claim, that this is a disjoint union. To see this let $S_1, S_2 \subset B$, $S_1 \neq S_2$, $\#S_1 = \#S_2$. This implies

$$\exists\beta_1 \in S_1 : \beta_1 \notin S_2 \qquad\qquad \exists\beta_2 \in S_2 : \beta_2 \notin S_1$$

If there existed an $x \in V_{S_1} \cap V_{S_2}$ it would follow

$$x \in V_{S_1} \Rightarrow \varphi_{\beta_1}(x) > \varphi_{\beta_2}(x) \qquad\qquad x \in V_{S_2} \Rightarrow \varphi_{\beta_2}(x) > \varphi_{\beta_1}(x)$$

which clearly is a contradiction. So the union is disjoint.
Finally $\mathfrak{B} := \{V_k | k \in \mathbb{N}\}$ is a cover of $X$ since for any $x \in X$ the set $S := \{\beta \in B | \varphi_\beta(x) > 0\}$ is finite and $x \in V_S$.
For the second statement let $\{\psi_\gamma\}_{\gamma\in\Gamma}$ be a partition of unity subordinate to $\mathfrak{B}$. For each $k \in \mathbb{N}$ define

$$\Gamma_k := \{\gamma \in \Gamma | \operatorname{supp}\psi_\gamma \subset V_k, \forall j < k : \operatorname{supp}\psi_\gamma \subseteq X \setminus V_j\}$$

and $\varphi_k := \sum_{\gamma\in\Gamma_k} \psi_\gamma$. Then obviously $\operatorname{supp}\varphi_k \subset V_k$ and additionally

$$\sum_{k\in\mathbb{N}} \varphi_k = \sum_{k\in\mathbb{N}} \sum_{\gamma\in\Gamma_k} \psi_\gamma = \sum_{\gamma\in\Gamma} \psi_\gamma = 1$$

So this of course is still a partition of unity. □

# 2 Pullbacks

## 2.1 Definition and Algebraic Properties

**2.1 Theorem** (Pullback)**.** *Let $A, B$ be topological spaces, $f : A \to B$ be a map and $(E, B, p)$ be a vector bundle over $B$.*

(i) Then there exists a vector bundle $(F, A, q)$ and a map $g : F \to E$ which restricts to a vector space isomorphism $g : F_a \to E_{f(a)}$ for any $a \in A$. We call $(F, A, q)$ a *pullback bundle of* $(E, B, p)$.

(ii) The bundle $(F, A, q)$ is determined uniquely by this property up to isomorphism.

(iii) Let $(E', B, p') \cong (E, B, p)$, let $(F', A, q')$ be the pullback of $(E', B, p')$ and $(F, A, q)$ be the pullbacks of $(E, B, p)$ then $(F', A, q') \cong (F, A, q)$.

*Proof.*

(i) We define

$$F := \{(a, v) \in A \times E | f(a) = p(v)\}$$

$q : F \to A$, $(a, v) \mapsto a$ and $g : F \to E$, $(a, v) \mapsto v$. We have to show, that $(A, F, q)$ is a vector bundle. In order to do so, we will first of all show, that the following diagram commutes

$$\begin{array}{ccccc} & & F & \xrightarrow{g} & E \\ & \swarrow^{q_1} & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ \Gamma_f & \xrightarrow[q_2]{} & A & \xrightarrow[f]{} & B \end{array}$$

The definition implies the relation $f \circ q = p \circ g$, because for any $(a, v) \in F$:

$$f(q(a, v)) = f(a) = p(v) = p(g(a, v))$$

Let $\Gamma_f := \{(a, f(a))|a \in A\} \subset A \times B$ denote the graph of $f$ and let $q_1 : F \to \Gamma_f$, $(a, v) \mapsto (a, p(v)) = (a, f(a))$, $q_2 : \Gamma_f \to A$, $(a, f(a)) \mapsto a$. Then $q = q_2 \circ q_1$ because

$$q_2(q_1(a, v)) = q_2((a, f(a))) = a = q(a, v)$$

The map $q_2$ is a homeomorphism since it has the continuous inverse $q_2^{-1} : A \to \Gamma_f$, $a \mapsto (a, f(a))$. We claim that $q_1 : F \to \Gamma_f$ is a vector bundle. To see this consider the map $\mathrm{id} \times p : A \times E \to A \times B$, $(a, v) \mapsto (a, p(v))$. It is continuous and surjective, because $p$ is. For any $(a, b) \in A \times B$ $(\mathrm{id} \times p)^{-1}(a, b) = \{a\} \times p^{-1}(b) \cong p^{-1}(b)$ wich is a vector space by hypothesis. If $U \subset B$ is an open neighbourhood of $b$, any local trivialization $h : p^{-1}(U) \to U \times \mathbb{R}^n$ extends to a local trivialization $\mathrm{id} \times h : (\mathrm{id} \times h)^{-1}(A \times U) \to (A \times U) \times \mathbb{R}^n$. So $\mathrm{id} \times p : A \times E \to A \times B$ is a vector bundle. Restricting yields

$$\mathrm{id} \times p|_F(a, v) = (a, p(v)) = (a, f(a)) = q_2(a, v)$$

As a restriction of the vector bundle $\mathrm{id} \times p$ $q_2$ is a vector bundle as well. It follows that $q = q_2 \circ q_1$ is a vector bundle as well.

It remains to show that $g$ restricts to a vector space isomorphism $g : F_a \to E_{f(a)}$. But by construction

$$F_a = \{v \in E|p(v) = f(a)\} = p^{-1}(f(a)) = E_f(a)$$

Since $g(a, v) = v$ it follows that it is an isomorphism.

(ii) Let $(F', A, q')$ be a vector bundle and $g' : F' \to F$ a map satisfying property (i).

$$\begin{array}{ccc} F & \xrightarrow{g} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & B \\ {\scriptstyle q'}\uparrow & & \uparrow{\scriptstyle p} \\ F' & \xrightarrow{g'} & E \end{array}$$

Define a map $\varphi : F' \to F$, $v' \mapsto (q'(v'), g'(v'))$. The image really is $F$, since for any $v' \in q'^{-1}(a)$ we have $g'(v') \in E_{f(a)} = p^{-1}(f(a))$, which implies

$$p(g'(v')) = f(a) = f(q'(v')) \Leftrightarrow (g'(v'), q'(v')) \in F \subset A \times E$$

Obviously $\varphi$ is continuous and restricts to an isomorphism on each fibre by hypothesis. Thus it is an isomorphism by Lemma 1.4.

(iii) Let's denote $\Psi : (E', B, p') \to (E, B, p)$ the given isomorphism.

$$\begin{array}{ccccc} F' & \xrightarrow{g'} & E' & \xrightarrow{\Psi} & E \\ \downarrow{\scriptstyle q'} & & {\scriptstyle p'}\downarrow & \swarrow{\scriptstyle p} & \\ A & \xrightarrow{f} & B & & \end{array}$$

By definition $g' : F' \to E'$ is continuous and restricts to an isomorphism on each fibre of $F'$. Thus $\Psi \circ g' : F' \to E$ also restricts to an isomorphism on each fibre by definition of a vector bundle isomorphism. Thus the statement follows from (ii).

□

**2.2 Definition** (Pullback)**.** Let $A, B$ be topological spaces and $f : A \to B$ be a map. Then we denote by $f^* : \mathrm{Vect}(B) \to \mathrm{Vect}(A)$ the *pullback of* $f$ maping each isomorphism class of vector bundles in $\mathrm{Vect}(B)$ to the isomorphism class of its corresponding pullback bundle. Theorem 2.1 above ensures that this is well defined.

**2.3 Example.**

(i) Let $(E, B, p)$ be a vector bundle and $\iota : A \hookrightarrow B$ be the inclusion of a subspace. Then clearly $\iota^*(E) = (p^{-1}(A), B, p|_A)$.

(ii) Let $\pi : \mathbb{S}^n \to \mathbb{R}P^n$ be the canonical projection and $(T\mathbb{R}P^n, \mathbb{R}P^n, p)$ be the tangential bundle of $\mathbb{R}P^n$. Then $\pi^*(T\mathbb{R}P^n) = T\mathbb{S}^n$. We define a map

$$\begin{array}{ccc} T\mathbb{S}^n & \xrightarrow{f} & T\mathbb{R}P^n \\ {\scriptstyle q}\downarrow & & {\scriptstyle p}\downarrow \\ \mathbb{S}^n & \xrightarrow{\pi} & \mathbb{R}P^n \end{array}$$

$f : T\mathbb{S}^n \to T\mathbb{R}P^n$ by simply sending a tangential vector $(p, v)$ to its equivalence class $[p, v]$ in $T\mathbb{R}P^n$.

(iii) Define a map $f : \mathbb{S}^1 \to \mathbb{S}^1$, $z \mapsto z^2$. We regard the Möbius bundle to be $E := \mathbb{S}^1 \times \mathbb{R}/\sim$, where $(z, t)\ (-z, -t)$, together with the projection map $p : E \to \mathbb{S}^1$, $[z, t] \mapsto z^2$. The we obtain

$$\begin{array}{ccc} \mathbb{S}^1 \times \mathbb{R} & \xrightarrow{f'} & E = \mathbb{S}^1 \times \mathbb{R}/\sim \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ \mathbb{S}^1 & \xrightarrow[f]{} & \mathbb{S}^1 \end{array}$$

So the pullback of the Möbius bundle via $z \mapsto z^2$ is the trivial bundle $\mathbb{S}^1 \times \mathbb{R}$. Informally spoken: The Möbius bundle has one "half-twist" and if we pull it back via $f$ it gets another "half-twist" and thus is "twist free".

**2.4 Theorem** (Algebraic Properties of Pullbacks)**.**

(i) Let $f : A \to B$, $g : B \to C$ be maps and $(E, C, p)$ be a vector bundle over $C$. Then

$$(g \circ f)^*([E]) = (f^* \circ g^*)([E])$$

(ii) For any vector bundle $(E, B, p)$

$$\mathrm{id}_B^*([E]) = \mathrm{id}_{\mathrm{Vect}(B)}([E]) = [E]$$

(iii) Let $(E_1, B, p_1)$, $(E_2, B, p_2)$ be vector bundles over $B$ and $f : A \to B$ be a map. Then

$$f^*([E_1 \oplus E_2]) = f^*([E_1]) \oplus f^*([E_2])$$

(iv) Analogously:

$$f^*([E_1 \otimes E_2]) = f^*([E_1]) \otimes f^*([E_2])$$

*Proof.* In all four cases we will show that the bundle on the right side satisfys the characteristic property stated in Theorem 2.1,(i). The statement is then an immediate consequence of the uniqueness property stated in Theorem 2.1,(ii).

(i) By hypothesis there exists a pullback $(F := g^*(E), B, q)$ of $(E, C, p)$ via $g$ and a map $g' : F \to E$. By repeating this procedure we obtain a pullback $(G := f^*(F) := f^*(g^*(E)), A, r)$ of $(F, B, q)$ and a map $f' : G \to F$. We obtain the commutative diagram:

$$\begin{array}{ccccc} G & \xrightarrow{f'} & F & \xrightarrow{g'} & E \\ \downarrow r & & \downarrow q & & \downarrow p \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

We define a map $h' : G \to E$, $(a, v) \mapsto g'(f'(a, v))$. It is continuous by construction and for any $a \in A$ it takes the fibre $G_a$ isomorphically to the fibre $E_{f(g(a))}$, because first $f'$ takes $G_a$ isomorphically to $F_{f(a)}$ by construction and then $g'$ maps $F_{f(a)}$ isomorphically to $G_{g(f(a))}$ also by construction.

(ii) We just have to show, that in the following commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\mathrm{id}_E} & E \\ p\downarrow & & p\downarrow \\ B & \xrightarrow{\mathrm{id}_B} & B \end{array}$$

the identity $\mathrm{id}_E : E \to E$ maps each fibre of $E_b$ isomorphically to the corresponding fibre of $E_{\mathrm{id}_B(b)}$ for all $b \in B$. This is obviously the case.

(iii) Let $(E_1', A, p_1')$ and $(E_2', A, p_2')$ be pullbacks of $(E_1, B, p_1)$ and $(E_2, B, p_2)$. By construction there exist maps $f_1' : E_1' \to E$ and $f_2' : E_2' \to E_2$, which restrict to vector space isomorphisms on their fibres and make the following diagram commutative:

$$\begin{array}{ccccc} E_2' & \xrightarrow{p_2'} & A & \xleftarrow{p_1'} & E_1' \\ f_2'\downarrow & & f\downarrow & & \downarrow f_1' \\ E_2 & \xrightarrow[p_2]{} & B & \xleftarrow[p_2']{} & E_1 \end{array}$$

It follows that the map $f_1' \oplus f_2' : E_1' \oplus E_2' \to E_1 \oplus E_2$ also maps each fibre of $E_1' \oplus E_2'$ isomorphically to the corresponding fibre of $E_1 \oplus E_2$.

(iv) Analogous to (iii) there is a canonical map $f_1' \otimes f_2' : E_1' \otimes E_2' \to E_1 \otimes E_2$ taking the fibres isomorphically to one another.

□

**2.5 Remark.** For each $n \in \mathbb{N}$ we can think of $\text{Vect}^n$ as a contravariant functor $\text{Vect}^n : \text{TOP} \to \text{SETS}$ sending each topological space $X$ to $\text{Vect}^n(X)$ and any morphism $f : X \to Y$ to the pullback $f^* : \text{Vect}^n(Y) \to \text{Vect}^n(X)$. Properties (i),(ii) of the preceeding theorem 2.4 together with theorem 2.1 ensure that this really is a functor.

## 2.2 Homotopy Invariance

**2.6 Lemma.** Let $a, b \in \mathbb{R}$. If there exists a $c \in ]a, b[$ such that the restrictions of the vector bundle $(E, X \times [a, b], p)$ to $X \times [a, c]$ and $X \times [c, b]$ are both trivial, then $(E, X \times [a, b], p)$ is trivial itself.

*Proof.* Let $F := p^{-1}(X \times [a, c])$ and $G := p^{-1}(X \times [c, b])$ be the restrictions and $h = (h_1, h_2, h_3) : G \to (X \times [a, c]) \times \mathbb{R}^n$, $g = (g_1, g_2, g_3) : F \to (X \times [c, b]) \times \mathbb{R}^n$ be the trivializing isomorphisms. If $h|_{(p^{-1}(X \times \{c\}))} = g|_{(p^{-1}(X \times \{c\}))}$ we can patch these isomorphisms together and optain a global isomorphism $E \to (X \times [a, b]) \times \mathbb{R}^n$.
If this is not the case, we have to replace e.g. $g$ by an isomorphism $g'$ such that $h|_{F \cap G} = g'|_{F \cap G}$. In order to do so, define $\Psi : X \times [c, b] \times \mathbb{R}^n \to X \times [c, b] \times \mathbb{R}^n$, $(x, t, v) \mapsto (x, t, h_3(g^{-1}(x, c, v)))$. Then $\Psi$ is a vector bundle isomorphism and thus $g' := \Psi \circ g$ is an isomorphism as well. Remember that $g, h$ are trivializations, so if $(x, c) \in X \times \{c\}$ and $e \in p^{-1}(x, c)$ we have:

$$g'(e) = \Psi(g(e)) = (g_1(e), g_2(e), h_3(g^{-1}(g(e)))) = (x, c, h_3(e)) = (h_1(e), h_2(e), h_3(e)) = h(e)$$

□

**2.7 Lemma.** Let $I := [0, 1] \subset \mathbb{R}$ and $(E, X \times I, p)$ be a vector bundle. Then there exists an open cover $\{U_\alpha\}_{\alpha \in A}$ such that each restriction $p^{-1}(U_\alpha \times I) \to U_\alpha \times I$ is trivial.

*Proof.* First fix an arbitrary $x \in X$. By definition of a vector bundle for any $(x, t) \in X \times I$ we can find an open neighbourhood $U_x \times U_t$ such that $p^{-1}(U_x \times U_t) \to U_x \times U_t$ is trivial. It follows that $\bigcup_{t \in I} U_t$ is an open cover of $I$. Since $I$ is compact there exists a finite partition $0 = t_0 < t_1 < \ldots < t_k = 1$ of $I$ and corresponding neighbourhoods $U_{x,1}, \ldots, U_{x,k} \subset X$ such that E is trivial over each $U_i := U_{x,i} \times [t_{i-1}, t_i]$. Define $V_x := \bigcap_{i=1}^k U_i$. Then iterative application of Lemma 2.6 implies that the bundle is trivial over $V_x \times I$. Since this can be done with any $x \in X$ we obtain an open cover of the entire space $X \times I$. □

**2.8 Theorem.** Let $X$ be a topological space and $(X \times I, E, p)$ be a vector bundle.

(i) If $X$ is compact Hausdorff, then its restrictions over $X \times \{0\}$ and $X \times \{1\}$ are isomorphic.

(ii) This still holds, if $X$ is only paracompact.

*Proof.* By Lemma 2.7 we can chose an open cover $\{U_\alpha\}_{\alpha \in A}$ of $X$ such that $E$ is trivial over each $U_\alpha \times I$.

(i) Since $X$ is compact, there exists a finite subcover of $X$ which we can relabel to $\{U_i\}_{i=1}^m$. There exists a subordinate partition of unity $\{\varphi_i\}_{i=1}^m$, so $\operatorname{supp} \varphi_i \subset U_i$. Define $\psi_i : X \to I$, $\psi_i := \sum_{\nu=1}^i \varphi_\nu$, $X_i := \Gamma_{\Psi_i} \subset X \times I$, $p_i := p|_{X_i}$, $E_i := p_i^{-1}(X_i)$, $\pi_i : X_i \to X_{i-1}$, $(x, \psi_i(x)) \mapsto (x, \psi_{i-1}(x))$. Then $\pi_i$ is a homeomorphism since it has the continuous inverse $\pi_i^{-1} : X_{i-1} \to X_i$, $(x, \psi_{i-1}(x)) \mapsto (x, \psi_i(x))$.
We claim, that for each $1 \leq i \leq m$ there exists an isomorphism $h_i : E_i \to E_{i-1}$. Let $1 \leq i \leq m$ be arbitrary. Since $\operatorname{supp} \varphi_i \subset U_i$ we have $\psi_i|_{X \setminus U_i} = \psi_{i-1}|_{X \setminus U_i}$ and thus $p^{-1}(\Gamma_{\psi_i|_{X \setminus U_i}}) = p^{-1}(\Gamma_{\psi_{i-1}|_{X \setminus U_i}})$. So the homeomorphism is given by the identity on there. By construction the

bundle is trivial over $U_i \times I$. Since $X_i \subset U_i \times I$ it is also trivial there and we obtain the following chain of homeomorphisms

$$p^{-1}\left(\Gamma_{\Psi_i|_{U_i}}\right) \approx \Gamma_{\Psi_i|_{U_i}} \times \mathbb{R}^n \approx \Gamma_{\Psi_{i-1}|_{U_i}} \times \mathbb{R}^n \approx p^{-1}\left(\Gamma_{\Psi_{i-1}|_{U_i}}\right)$$

where the homeomorphisms right and left are induced by the local trivializations and the homeomorphism in the middle is induced by the $\pi_i$. Patching together we obtain a bundle isomorphism $h_i : E_i \to E_{i-1}$. Because $\psi_0 = 0$ and $\psi_m = 1$, $X_0 = X \times \{0\}$, $X_m = X \times \{1\}$, $E_0 = p^{-1}(X \times \{0\})$, $E_m = p^{-1}(X \times \{1\})$. So the composition $h := h_1 \circ h_2 \circ \ldots \circ h_m : E_m \to E_0$ is the isomorphism satisfying the disired propertys.

(ii) By Lemma 1.9,(i) there is a countable open cover $\{V_i\}_{i \in \mathbb{N}}$ such that each $V_i$ is the disjoint union of sets each contained in some $U_\alpha$. This implies that $E$ is trivial over every $V_i \times I$. By Lemma 1.9,(ii) there exists a partition of unity $\{\varphi_i\}_{i \in \mathbb{N}}$ subordinate to $\{V_i\}$. Define again $\psi_i := \sum_{\nu=1}^{i} \varphi_\nu$ and $X_i$, $E_i$ as in (i). We can construct homeomorphisms $h : E_i \to E_{i-1}$ entirely analogous since the bundle is trivial over every $V_i \times I$. Again $\psi_0 = 0$ and $\psi := \sum_{\nu=1}^{\infty} \varphi_\nu = 1$. So the infinite composition $h := h_1 \circ h_2 \circ \ldots$ is an ismorphism $p^{-1}(X \times \{0\}) \to p^{-1}(X \times \{1\})$ provided that it is well defined. But this is the case: For any $x \in X$ there exists an open neighbourhood $U$ such that $\varphi_i|_U = 0$ for all but finitely many $i$. So for all but finitely many $i$ we have $h_i|_U = \mathrm{id}_U$.

□

**2.9 Theorem** (Homotopy Invariance Theorem)**.** Let $f_0, f_1 : A \to B$ be homotopic maps. Let $(E, B, p)$ be a vector bundle over $B$ and let $A$ be paracompact. Then

$$f_0^*([E]) = f_1^*([E])$$

This is especially true, if $A$ is compact Haudorff.

*Proof.* Let $F : A \times I \to B$ be a homotopy from $f_0$ to $f_1$. Then we obtain a pullback

$$\begin{array}{ccc} F^*(E) & \longrightarrow & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A \times I & \xrightarrow[F]{} & B \end{array}$$

By definition $F|_{A \times \{0\}} = f_0$, $F|_{A \times \{1\}} = f_1$ and thus

$$[f_0^*([E])] = [q^{-1}(A \times \{0\})] \overset{2.8}{=} [q^{-1}(A \times \{1\})] = [f_1^*([E])]$$

□

**2.10 Corrolary.** A homotopy equivalence $f : A \to B$ of paracompact spaces induces a bijection $f^* : \mathrm{Vect}^n(B) \to \mathrm{Vect}^n(A)$. In particular, every vector bundle over a contractible paracompact base is trivial.

*Proof.* Let $g : B \to A$ be a homotopy inverse to $f$. Then

$$f^* \circ g^* \overset{2.4,(i)}{=} (g \circ f)^* = \mathrm{id}_B^* \overset{2.4,(ii)}{=} \mathrm{id}_{\mathrm{Vect}(B)}$$
$$g^* \circ f^* \overset{2.4,(i)}{=} (f \circ g)^* = \mathrm{id}_A^* \overset{2.4,(ii)}{=} \mathrm{id}_{\mathrm{Vect}(A)}$$

□

**2.11 Remark.** Theorem 2.9 holds for fibre bundles as well with the same proof.

## References

[1] Hatcher, Allen, Vector Bundles and K-Theory