Introduction to Partial Differential Operators

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This article is designed to give a short introduction to the theory of partial differential operators ("PDO"). It is though of as an analouge to the "Introduction to Pseudo-Differential Operators" by Wong and in fact contains the solution of some prerequisites and exercises from this book. The treatment of Pseudo-Differential Operators ("ΨDO") is even more technical than the one of partial differential operators. So in order to understand the ideas behind the ΨDO theory, the PDOs are helpful and of course also useful and nice for themselves.

Contents

1 Local Theory

1.1 Notation and Basic Definitions

Global notation Conventions

- n denotes the Dimension of the space in which we are operating
- $U \subset \mathbb{R}^n$ is an open set
- ∂_j denotes the partial differentitation with respect to the j-th coordinate direction e_j
- $D_j := -i\partial_j$, where i is the imaginary unit
- For any multi-index $\alpha \in \mathbb{N}^n$ and any $i \in \mathbb{N}$, $0 \leq i \leq n$, we denote by

$$\alpha + i := (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \ldots, \alpha_n),$$

  which will be very helpful for inductions.

(to be extended...)

1.1.1 Definition (Differential Operator)**.** Let $U \subset \mathbb{R}^n$ be open. A *complex (partial) differential operator on* U *of order* k, $k \in \mathbb{N}$, (a "PDO") is a $\mathbb{C}$-linear map $P : \mathcal{C}^\infty(U, \mathbb{C}^r) \to \mathcal{C}^\infty(U, \mathbb{C}^s)$ such that for every $\alpha \in \mathbb{N}^n$, $|\alpha| \leq k$, there exists $P_\alpha \in \mathcal{C}^\infty(U, \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^r, \mathbb{C}^s))$ such that

$$P = \sum_{|\alpha| \leq k} P_\alpha D^\alpha$$

A real differential operator is defined analogously. The set of all such operators is denoted by

$$\mathrm{Diff}^k_{\mathbb{C}}(U, \mathbb{C}^r, \mathbb{C}^s)$$

1.1.2 Remark. Of course we assume that $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^r, \mathbb{C}^s)$ is given the smooth structure obtained by identifying with $\mathbb{C}^{s\times r}$. The set $\mathrm{Diff}^k_{\mathbb{C}}(U, \mathbb{C}^r, \mathbb{C}^s)$ itself is a module over $\mathcal{C}^\infty(U)$ and a $\mathbb{C}$-vector space. Chosing a bases $\{E_\mu\}$ of $\mathbb{C}^r$ and $\{F_\nu\}$ of $\mathbb{C}^s$, we can fully expand P in coordinates as

$$Ps = \sum_{|\alpha| \leq k} \sum_{\mu=1}^{r} \sum_{\nu=1}^{s} (P^\alpha)^\nu_\mu D^\alpha(s^\mu) e_\nu.$$

1.1.3 Definition (Symbol)**.** Define the map $\sigma : \mathrm{Diff}^k_{\mathbb{C}}(U, \mathbb{C}^r, \mathbb{C}^s) \to \mathcal{C}^\infty(U, \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^r, \mathbb{C}^s))[\xi_1, \ldots, \xi_n]$ by

$$P = \sum_{|\alpha| \leq k} P_\alpha D^\alpha \mapsto \sum_{|\alpha| \leq k} P_\alpha \xi^\alpha$$

where $\mathcal{C}^\infty(U, \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^r, \mathbb{C}^s))[\xi_1, \ldots, \xi_n]$ is the "polynomial ring" in the n variables $\xi = (\xi_1, \ldots, \xi_n)$ over $\mathcal{C}^\infty(U, \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^r, \mathbb{C}^s))$. We call $\sigma(P)$ the *symbol of* P.

It is clear, that we may identify differential operators with their symbols.

1.1.4 Lemma. The map σ is bijective. Its inverse is given by the substitution homomorphism induced by

$$\xi_j \mapsto D_j$$

1.2 Leibniz Formulae

Before we can start with the analysis of the PDO Algebra itself, we first have to investigate the application of a PDO to a function especially to the product of two functions. We assume the reader to be very familiar with the product rule from basic calulus, i.e. if $f, g \in \mathcal{C}^1(U)$, then

$$\forall x \in U : \partial_i(fg)(x) = (\partial_i f)(x)g(x) + f(x)(\partial_i g(x))$$

This can be generalized considerably and will be done in this section.

1.2.1 Theorem (Leibniz Rule)**.** Let $U \subset \mathbb{R}^n$ be open, $f, g \in \mathcal{C}^k(U)$, $\alpha \in \mathbb{N}^n$, $|\alpha| = k$. Then

$$\partial^\alpha(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} (\partial^\beta f)(\partial^\gamma g)$$

As a didactial motivation we will proof the very important special case $n = 1$ seperately. Logically it is not needed in the proof of the general case and thus may be skipped.

Proof. [for $n = 1$] In that case, the statement is

$$(f \cdot g)^{(k)} = \sum_{\nu=0}^{k} \binom{k}{\nu} f^{(\nu)} \cdot g^{(k-\nu)}$$

We will proof this via induction over k. For $k = 1$ this is the ordinary product rule:

$$(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$$

For the induction step $k \to k+1$ consider

$$\begin{aligned}
(f \cdot g)^{(k+1)} &= \left((f \cdot g)^{(k)}\right)' = \left(\sum_{\nu=0}^{k} \binom{k}{\nu} f^{(\nu)} \cdot g^{(k-\nu)}\right)' \\
&= \sum_{\nu=0}^{k} \binom{k}{\nu} f^{(\nu+1)} \cdot g^{(k-\nu)} + f^{(\nu)} \cdot g^{(k+1-\nu)} \\
&\overset{(1)}{=} \sum_{\nu=1}^{k+1} \binom{k}{\nu-1} f^{(\nu)} \cdot g^{(k-(\nu-1))} + \sum_{\nu=0}^{k} \binom{k}{\nu} f^{(\nu)} \cdot g^{(k+1-\nu)} \\
&\overset{(2)}{=} \sum_{\nu=0}^{k+1} \binom{k}{\nu-1} f^{(\nu)} \cdot g^{(k+1-\nu)} + \sum_{\nu=0}^{k+1} \binom{k}{\nu} f^{(\nu)} \cdot g^{(k+1-\nu)} \\
&\overset{(3)}{=} \sum_{\nu=0}^{k+1} \binom{k+1}{\nu} f^{(\nu)} \cdot g^{(k+1-\nu)}
\end{aligned}$$

(1): Here we splitted up the sum and shifted the first one up by one.
(2): Here we added the zero summands

$$\binom{k}{-1} = 0 \qquad \text{and} \qquad \binom{k}{k+1} = 0$$

(3): This uses the addition theorem for binomials:

$$\binom{k}{\nu-1} + \binom{k}{\nu} = \binom{k+1}{\nu}$$

□

Proof. [General Case] We will proof this statment as well by induction over $k = |\alpha|$. If $k = 1$, the statement is just the ordinary product rule. So by induction assume the statement is valid for k. If $|\alpha| = k+1$, there exists $1 \leq j \leq n$ such that α can be written as $\alpha = \tilde{\alpha} + (0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 is at position j and $|\tilde{\alpha}| = k$. So

$$\forall 1 \leq i \neq j \leq n : \tilde{\alpha}_i = \alpha_i \qquad \text{and} \qquad \tilde{\alpha}_j + 1 = \alpha_j$$

Using induction hypothesis we calculate:

$$
\begin{aligned}
\partial^\alpha(fg) &= \partial_j \partial^{\tilde{\alpha}}(fg) = \partial_j \sum_{\beta \le \tilde{\alpha}} \binom{\tilde{\alpha}}{\beta} (\partial^\beta f)(\partial^{\tilde{\alpha}-\beta} g \\
&= \sum_{0 \le \beta \le \tilde{\alpha}} \binom{\tilde{\alpha}}{\beta} (\partial_j \partial^\beta f)(\partial^{\tilde{\alpha}-\beta} g) + (\partial^\beta f)(\partial_j \partial^{\tilde{\alpha}-\beta} g) \\
&\overset{(1)}{=} \sum_{i \ne j : 0 \le \beta_i \le \tilde{\alpha}_i, 0 \le \beta_j \le \tilde{\alpha}_j} \prod_{i=1, i \ne j}^{n} \binom{\tilde{\alpha}_i}{\beta_i} \binom{\tilde{\alpha}_j}{\beta_j} (\partial_j \partial^\beta f)(\partial^{\tilde{\alpha}-\beta} g) + \sum_{0 \le \beta \le \alpha} \binom{\tilde{\alpha}}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g) \\
&\overset{(2)}{=} \sum_{i \ne j : 0 \le \beta_i \le \alpha_i, 1 \le \beta_j \le \alpha_j} \prod_{i=1, i \ne j}^{n} \binom{\alpha_i}{\beta_i} \binom{\alpha_j - 1}{\beta_j - 1} (\partial^\beta f)(\partial^{\alpha-\beta} g) + \sum_{0 \le \beta \le \alpha} \binom{\tilde{\alpha}}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g) \\
&\overset{(3)}{=} \sum_{0 \le \beta \le \alpha} \prod_{i=1, i \ne j}^{n} \binom{\alpha_i}{\beta_i} \binom{\alpha_j - 1}{\beta_j - 1} (\partial^\beta f)(\partial^{\alpha-\beta} g) + \sum_{0 \le \beta \le \alpha} \prod_{i=1, i \ne j}^{n} \binom{\alpha_i}{\beta_i} \binom{\alpha_j - 1}{\beta_j} (\partial^\beta f)(\partial^{\alpha-\beta} g) \\
&= \sum_{0 \le \beta \le \alpha} \left(\binom{\alpha_j - 1}{\beta_j - 1} + \binom{\alpha_j - 1}{\beta_j} \right) \prod_{i=1, i \ne j}^{n} \binom{\alpha_i}{\beta_i} (\partial^\beta f)(\partial^{\alpha-\beta g}) \\
&\overset{(4)}{=} \sum_{0 \le \beta \le \alpha} \binom{\alpha_j}{\beta_j} \prod_{i=1, i \ne j}^{n} \binom{\alpha_i}{\beta_i} (\partial^\beta f)(\partial^{\alpha-\beta} g) \\
&= \sum_{0 \le \beta \le \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g)
\end{aligned}
$$

Where we have used the following facts:

(1) In the first sum, we just wrote down the index set and the expression more complicated. In the second sum we only added summands with multi-indices β, such that

$$\binom{\tilde{\alpha}}{\beta} = \prod_{1 \le i \ne j \le n} \binom{\alpha_i}{\beta_i} \cdot \binom{\alpha_j}{\alpha_j + 1}$$

(2) In the first sum this is an index shift

$$(0 \le \beta_j \le \tilde{\alpha}_j = \alpha_j - 1) \mapsto (1 \le \beta_j \le \alpha_j)$$

and the plugging in of the definition of $\tilde{\alpha}$.

(3) In the first sum, we just added summands where $\beta_j = 0$ and thus

$$\binom{\alpha_j - 1}{\beta_j - 1} = 0$$

In the second sum we plugged in the definitions.

(4) This is the addition law for binomial coefficients.

□

1.2.2 Remark. This statement is also valid for D^α instead of ∂^α (by just multiplying the equation with $(-i)^\alpha$).

1.2.3 Corollary (Vector valued Leibniz Formula)**.** Let $U \subset \mathbb{R}^n$ be open and $F \in \mathcal{C}^k(U, \mathbb{C}^{s \times r})$, $g \in \mathcal{C}^k(U, \mathbb{C}^r)$ and $\alpha \in \mathbb{N}^n$ such that $|\alpha| \le k$. Then

$$\partial^\alpha(Fg)(x) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (\partial^\beta F)(x)(\partial^{\alpha-\beta} g)(x)$$

Proof. We calculate

$$\partial^\alpha(Fg)(x) = \partial^\alpha\Big(\sum_{i=1}^{s}\sum_{j=1}^{r} F_j^i g^j e_i\Big)(x) = \sum_{i=1}^{s}\sum_{j=1}^{r} \partial^\alpha(F_j^i g^j e_i)(x)$$

$$\overset{1.2.1}{=} \sum_{i=1}^{s}\sum_{j=1}^{r}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(\partial^\beta F_j^i)(x)(\partial^{\alpha-\beta}g^j)(x)e_i = \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\sum_{i=1}^{s}\sum_{j=1}^{r}(\partial^\beta F_j^i)(x)(\partial^{\alpha-\beta}g^j)(x)e_i$$

$$= \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(\partial^\beta F)(x)(\partial^{\alpha-\beta}g)(x)$$

□

One specific product is of particular importance.

1.2.4 Lemma. Let $\alpha, \delta \in \mathbb{N}^n$ be any two multi-indices. Then for all $\xi \in \mathbb{R}^n$

$$\partial_\xi^\delta \xi^\alpha = \begin{cases} \delta!\binom{\alpha}{\delta} & ,\delta\leq\alpha \\ 0 & ,\text{otherwise} \end{cases}$$

Proof. By induction over n.
$n = 1$: Then there exist $i, j \in \mathbb{N}$ such that $\partial_\xi^\delta = \partial_i$ and $\xi^\alpha = \xi_j$. If $\delta \leq \alpha$ then $i = j$ and thus

$$\partial_\xi^\delta \xi^\alpha = \partial_i\xi_i = 1 = \delta!\binom{\alpha}{\delta}\xi^{\alpha-\delta}$$

Otherwise $\partial_\xi^\delta\xi^\alpha = \partial_i\xi_j = 0$.
$n \to n+1$: Suppose the formula is valid for multi-indices of length n and $\alpha, \delta \in \mathbb{N}^{n+1}$. Define

$$\tilde\alpha := (\alpha_1, \ldots, \alpha_n), \tilde\delta := (\delta_1, \ldots, \delta_n) \in \mathbb{N}^n$$

We regard

$$\alpha_{n+1} \triangleq (\alpha_{n+1}, 0, \ldots, 0) \in \mathbb{N}^n \qquad \delta_{n+1} \triangleq (\delta_{n+1}, 0, \ldots, 0) \in \mathbb{N}^n$$

Thus if $\delta_{n+1} \leq \alpha_{n+1}$

$$\partial_\xi^\delta\xi^\alpha = \partial_\xi^{\delta_{n+1}}(\partial_\xi^{\tilde\delta}(\xi^{\tilde\alpha}\xi_{n+1}^{\alpha_{n+1}})) = \partial_\xi^{\delta_{n+1}}\xi_{n+1}^{\alpha_{n+1}}\cdot\tilde\delta!\binom{\tilde\alpha}{\tilde\delta}\xi^{\tilde\alpha-\tilde\delta} = \prod_{i=0}^{\delta_{n+1}-1}(\alpha_{n+1}-i)\xi_{n+1}^{\alpha_{n+1}-\delta_{n+1}}\cdot\tilde\delta!(\binom{\tilde\alpha}{\tilde\delta}\xi^{\tilde\alpha-\tilde\delta}$$

$$= \prod_{i=1}^{\delta_{n+1}}(\alpha_{n+1}-i+1)\xi_{n+1}^{\alpha_{n+1}-\delta_{n+1}}\cdot\tilde\delta!\frac{\tilde\alpha!}{\tilde\delta!(\tilde\alpha-\tilde\delta)!}\xi^{\tilde\alpha-\tilde\delta} = \frac{\alpha_{n+1}!}{(\alpha_{n+1}-\delta_{n+1})!}\xi_{n+1}^{\alpha_{n+1}-\delta_{n+1}}\cdot\tilde\delta!\frac{\tilde\alpha!}{\tilde\delta!(\tilde\alpha-\tilde\delta)!}\xi^{\tilde\alpha-\tilde\delta}$$

$$= \delta!\frac{\alpha!}{\delta!(\tilde\alpha-\tilde\delta)!(\alpha_{n+1}-\delta_{n+1})!}\xi^{\alpha-\delta} = \delta!\binom{\alpha}{\delta}\xi^{\alpha-\delta}$$

Otherwise $\partial_\xi^{\delta_{n+1}}\xi_{n+1}^{\alpha_{n+1}} = 0$ and thus the entire expression would be zero. □

This allows us to generalize Leibniz formula for PDOs.

1.2.5 Theorem (Leibniz Formula for PDO)**.** Let $U \subset \mathbb{R}^n$ be open $F \in \mathcal{C}^k(U, \mathbb{C}^{r\times s})$, $g \in \mathcal{C}^k(U, \mathbb{C}^r)$ open and

$$P(x,D) = \sum_{|\alpha|\leq k} p_\alpha(x) D^\alpha \in \operatorname{Diff}^k_{\mathbb{C}}(U,s,t)$$

be a PDO with symbol $p(x,\xi)$. Then

$$P(D)(Fg) = \sum_{|\mu|\leq k} \frac{1}{\mu!} P^{(\mu)}(D)(F) D^\mu g$$

where $P^{(\mu)}(D) \in \operatorname{Diff}^{k-|\mu|}_{\mathbb{C}}(U,s,t)$ is the PDO with symbol

$$p^{(\mu)}(x,\xi) = \partial^\mu_\xi p(x,\xi)$$

Proof. By the Leibniz formula 1.2.3 above

$$P(x,D)(Fg) = \sum_{|\alpha|\leq k} p_\alpha(x) D^\alpha(Fg) = \sum_{|\alpha|\leq k} p_\alpha(x) \sum_{\mu\leq\alpha} \binom{\alpha}{\mu} D^{\alpha-\mu}(F) D^\mu(g)$$

Furthermore

$$p^{(\mu)}(x,\xi) = \sum_{|\alpha|\leq k} p_\alpha(x) \partial^\mu_\xi \xi^\alpha \overset{1.2.4}{=} \sum_{|\alpha|\leq k} p_\alpha(x) \mu! \binom{\alpha}{\mu} \xi^{\alpha-\mu}$$

Consequently

$$\begin{aligned} P(x,D)(Fg) &= \sum_{|\alpha|\leq k} \sum_{\mu\leq\alpha} p_\alpha(x) \binom{\alpha}{\mu} D^{\alpha-\mu}(F) D^\mu g \overset{1.2.4}{=} \sum_{|\mu|\leq k} \frac{1}{\mu!} \sum_{|\alpha|\leq k} p_\alpha(x) \mu! \binom{\alpha}{\mu} D^{\alpha-\mu}(F) D^\mu g \\ &= \sum_{|\mu|\leq k} \frac{1}{\mu!} P^{(\mu)}(D)(F) D^\mu g \end{aligned}$$

□

We would like to generalize generalize these results to products with finitely many factors.

1.2.6 Theorem (Binomial Theorem)**.** For any $x, y \in \mathbb{C}$, $n \in \mathbb{N}$

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

In this context $0^0 := 1$.

1.2.7 Theorem (Multinomial Theorem)**.** Let $z_1, \ldots, z_k \in \mathbb{C}^n$, $\alpha \in \mathbb{N}^n$. Then

$$\left(\sum_{i=1}^k z_i \right)^\alpha = \sum_{B \in (\mathbb{N}^n)^k, |B| = \alpha} \binom{\alpha}{B} Z^B$$

where $B = (B_1, \ldots, B_k)$ is a tuple of multi-indices, $Z^B := z_1^{B_1} \ldots z_k^{B_k}$ and

$$|B| := \sum_{i=1}^k B_i \qquad\qquad \binom{\alpha}{B} := \frac{\alpha!}{B_1! \ldots B_k!}$$

Proof. We use induction over k. For $k=1$, the statement is clear since both side equal z_1^α. For $k=2$

$$\left(\sum_{i=1}^k z_i\right)^\alpha = \prod_{i=1}^n (z_1^i + z_2^i)^{\alpha_i} \overset{1.2.6}{=} \prod_{i=1}^n \sum_{\gamma_i=1}^{\alpha_i} \binom{\alpha_i}{\gamma_i} (z_1^i)^{\gamma_i} (z_2^i)^{\alpha_i-\gamma_i}$$
$$= \sum_{\gamma_1=1}^{\alpha_1} \cdots \sum_{\gamma_n=1}^{\alpha_n} \prod_{i=1}^n \binom{\alpha_i}{\gamma_i} z_1^{\gamma_i} z_2^{\alpha_i-\gamma_i} = \sum_{\gamma\le\alpha} \frac{\alpha!}{\gamma!(\alpha-\gamma)!} z_1^\gamma z_2^{\alpha-\gamma} = \sum_{B\in(\mathbb{N}^n)^2, |B|=\alpha} \binom{\alpha}{B} Z^B$$

For the induction step assume $k \ge 3$ and that the statement holds for k. Define $y := \sum_{i=1}^k z_k$, $Y = (z_1, \ldots, z_k) \in (\mathbb{C}^n)^k$ Using the induction start for $k = 1, 2$ and the induction hypothesis, we obtain:

$$\left(\sum_{i=1}^{k+1} z_i\right)^\alpha = \left(\sum_{i=1}^k z_i + z_{k+1}\right)^\alpha = (y + z_{k+1})^\alpha = \sum_{\gamma+\delta=\alpha} \binom{\alpha}{(\gamma,\delta)} (y, z_{k+1})^{(\gamma,\delta)}$$
$$= \sum_{\gamma+\delta=\alpha} \frac{\alpha!}{\gamma!\delta!} z_{k+1}^\delta \left(\sum_{i=1}^k z_i\right)^\gamma = \sum_{\gamma+\delta=\alpha} \frac{\alpha!}{\gamma!\delta!} z_{k+1}^\delta \left(\sum_{C\in(\mathbb{N}^n)^k, |C|=\gamma} \binom{\gamma}{C} Y^C\right)$$
$$= \sum_{\gamma+\delta=\alpha} \sum_{C\in(\mathbb{N}^n)^k, |C|=\gamma} \frac{\alpha!}{\gamma!\delta!} \frac{\gamma!}{C_1!\ldots C_k!} z_1^{C_1} \ldots z_k^{C_k} z_{k+1}^\delta = \sum_{B\in(\mathbb{N}^n)^{k+1}, |B|=\alpha} \binom{\alpha}{B} Z^B$$

□

1.2.8 Theorem (Leibniz rule for multiple factors)**.** Let $f_1, \ldots, f_k \in \mathcal{C}^\infty(U, \mathbb{C})$, $U \subset \mathbb{R}^n$, $\alpha \in \mathbb{N}^n$. Then

$$\partial^\alpha \left(\prod_{i=1}^k f_i\right) = \sum_{B\in(\mathbb{N}^n)^k, |B|=\alpha} \binom{\alpha}{B} \partial^B F$$

where $F = (f_1, \ldots, f_k) : U \to \mathbb{C}^k$, $B = (B_1, \ldots, B_k)$ is a tuple of multi-indices, $\partial^B F := (\partial^{B_1} f_1) \ldots (\partial^{B_k} f_k)$ and

$$|B| := \sum_{i=1}^k B_i \qquad\qquad \binom{\alpha}{B} := \frac{\alpha!}{B_1! \ldots B_k!}$$

Proof. We use induction over k. For $k=1$, the statement is clear since both side equal $\partial^\alpha f_1$. For $k=2$ this has already been proven as 1.2.1. For the induction step $k \to (k+1)$ consider:

$$\partial^\alpha \left(\prod_{i=1}^{k+1} f_i\right) = \partial^\alpha \left(\prod_{i=1}^k f_i f_{k+1}\right) = \sum_{\gamma+\delta=\alpha} \binom{\alpha}{(\gamma,\delta)} \partial^\gamma \left(\prod_{i=1}^k f_i f_{k+1}\right) f_{k+1}^\delta$$
$$= \sum_{\gamma+\delta=\alpha} \frac{\alpha!}{\gamma!\delta!} z_{k+1}^\delta \left(\sum_{C\in(\mathbb{N}^n)^k, |C|=\gamma} \binom{\gamma}{C} \partial^C (f_1, \ldots, f_k)\right)$$
$$= \sum_{\gamma+\delta=\alpha} \sum_{C\in(\mathbb{N}^n)^k, |C|=\gamma} \frac{\alpha!}{\gamma!\delta!} \frac{\gamma!}{C_1!\ldots C_k!} (\partial^{C_1} f_1) \ldots (\partial^{C_k} f_k)(\partial_{k+1}^\delta f) = \sum_{B\in(\mathbb{N}^n)^{k+1}, |B|=\alpha} \binom{\alpha}{B} \partial^B F$$

□

1.3 The Composition of two PDO

1.3.1 Theorem (Composition Symbol)**.** Let $U \subset \mathbb{R}^n$ be open and

$$P = \sum_{|\alpha|\leq k} p_\alpha D^\alpha \in \mathrm{Diff}^k_{\mathbb{C}}(U,r,s) \qquad \text{and} \qquad Q = \sum_{|\beta|\leq l} q_\beta D^\beta \in \mathrm{Diff}^k_{\mathbb{C}}(U,s,t)$$

be two PDO with symbols

$$\sigma(P)(x,\xi) = p(x,\xi) = \sum_{|\alpha|\leq k} p_\alpha(x)\xi^\alpha \qquad \text{and} \qquad \sigma(Q)(x,\xi) = q(x,\xi) = \sum_{|\beta|\leq l} q_\beta(x)\xi^\beta$$

Then the $P \circ Q \in \mathrm{Diff}^{k+l}_{\mathbb{C}}(U,r,t)$ is a PDO with symbol

$$\sigma(P\circ Q)(x,\xi) = \sum_{|\gamma|\leq l} \frac{(-i)^\gamma}{\gamma!}(\partial_\xi^\gamma p)(x,\xi)(\partial_x^\gamma q)(x,\xi)$$

Proof. For any $f \in \mathcal{C}^\infty(U, \mathbb{C}^r)$

$$\begin{aligned}(P\circ Q)(f)(x) &= \sum_{|\alpha|\leq k} p_\alpha(x) D_x^\alpha \left(\sum_{|\beta|\leq l} (q_\beta D_x^\beta f)(x)\right) = \sum_{|\alpha|\leq k}\sum_{|\beta|\leq l} p_\alpha(x) D_x^\alpha (q_\beta D_x^\beta f)(x) \\ &\overset{1.2.3}{=} \sum_{|\alpha|\leq k}\sum_{|\beta|\leq l} p_\alpha(x) \sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}(D_x^\gamma q_\beta)(x)(D_x^{\beta+\alpha-\gamma}f)(x) \\ &= \sum_{|\alpha|\leq k}\sum_{|\beta|\leq l}\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma} p_\alpha(x)(D_x^\gamma q_\beta)(x)(D_x^{\beta+\alpha-\gamma}f)(x)\end{aligned}$$

Consequently

$$\begin{aligned}\sigma(P\circ Q)(x,\xi) &= \sum_{|\alpha|\leq k}\sum_{|\beta|\leq l}\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma} p_\alpha(x)(D_x^\gamma q_\beta)(x)\xi^{\beta+\alpha-\gamma} \\ &= \sum_{|\alpha|\leq k}\sum_{|\gamma|\leq\alpha}\sum_{|\beta|\leq l}\binom{\alpha}{\gamma} p_\alpha(x)(D_x^\gamma q_\beta)(x)\xi^{\beta+\alpha-\gamma} = \sum_{|\alpha|\leq k}\sum_{|\gamma|\leq\alpha}\binom{\alpha}{\gamma}p_\alpha(x)\xi^{\alpha-\gamma}D_x^\gamma\Big(\sum_{|\beta|\leq l} q_\beta(x)\xi^\beta\Big) \\ &= \sum_{|\alpha|\leq k}\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\xi^{\alpha-\gamma}p_\alpha(x)D_x^\gamma q(x,\xi) \overset{(1)}{=} \sum_{|\alpha|\leq k}\sum_{\gamma\leq\alpha}\frac{1}{\gamma!}(\partial_\xi^\gamma\xi^\alpha)p_\alpha(x)D_x^\gamma q(x,\xi) \\ &\overset{(2)}{=} \sum_{|\alpha|\leq k}\sum_{|\gamma|\leq k}\frac{1}{\gamma!}(\partial_\xi^\gamma\xi^\alpha)p_\alpha(x)D_x^\gamma q(x,\xi) = \sum_{|\gamma|\leq k}\Big(\frac{1}{\gamma!}\partial_\xi^\gamma\big(\sum_{|\alpha|\leq k} p_\alpha(x)\xi^\alpha\big)D_x^\gamma q(x,\xi)\Big) \\ &= \sum_{|\gamma|\leq k}\frac{1}{\gamma!}(\partial_\xi^\gamma p)(x,\xi)D_x^\gamma q(x,\xi) = \sum_{|\gamma|\leq k}\frac{(-i)^\gamma}{\gamma!}(\partial_\xi^\gamma p)(x,\xi)\partial_x^\gamma q(x,\xi)\end{aligned}$$

Remember from Lemma 1.2.4, that for any two multi-indices α,γ, we have

$$\partial_\xi^\gamma \xi^\alpha = \begin{cases}\gamma!\binom{\alpha}{\gamma} & ,\gamma\leq\alpha \\ 0 & ,\text{otherwise}\end{cases}$$

This is the justification for (1) and also for (2) since we only added zero summands! □

1.4 The Adjoint of a PDO

1.4.1 Definition (L^2 scalar product)**.** We define the space

$$L^2(U,\mathbb{C}^r) := \{f : U \to \mathbb{C}^r \mid \|f\|^2_{L^2(U,\mathbb{C}^r)} := \sum_{i=1}^r ||f_i||_{L_2(U)} < \infty\}$$

and call $\|\_\|_{L^2(U,\mathbb{C}^r)}$ the L^2*-norm.* This norm is induced by the L^2*-scalar product*

$$\langle f,g\rangle_{L^2(U,\mathbb{C}^r)} := \sum_{i=1}^r \langle f,g\rangle_{L^2(U)} := \sum_{i=1}^r \int_U f_i(x)\bar{g}_i(x)dx$$

on $L^2(U,\mathbb{C}^r)$.

1.4.2 Definition (Vector valued Schwarz space)**.** The space

1.4.3 Theorem (Adjoint Symbol)**.** Let $P(x,D) = \sum_{|\alpha|\le m} p_\alpha(x)D_x^\alpha$ be a PDO with symbol

$$\sigma(P)(x,\xi) = p(x,\xi) = \sum_{|\alpha|\le m} p_\alpha(x)\xi^\alpha$$

Then the adjoint Operator P^* has the symbol

$$\sigma(P^*) = \sum_{|\gamma|\le m} \frac{(-i)^\gamma}{\gamma!}\partial_x^\gamma\partial_\xi^\gamma\bar{p}(x,\xi)$$

Proof. Take $\varphi,\psi \in \mathcal{S}$ and consider

$$\langle P(x,D)\varphi,\psi\rangle_{L^2} = \int_{\mathbb{R}^n}\sum_{|\alpha|\le m} p_\alpha(x)D_x^\alpha\varphi(x)\cdot\bar{\psi}(x)dx = \sum_{|\alpha|\le m}(-i)^\alpha\int_{\mathbb{R}^n}(\partial_x^\alpha\varphi)(x)p_\alpha(x)\cdot\bar{\psi}(x)dx$$

$$= \sum_{|\alpha|\le m} i^\alpha\int_{\mathbb{R}^n}\varphi(x)\partial_x^\alpha(p_\alpha(x)\cdot\bar{\psi}(x))dx = \int_{\mathbb{R}^n}\varphi(x)\overline{\sum_{|\alpha|\le m}(-i)^\alpha\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}\partial_x^\gamma(\bar{p}_\alpha(x))(\partial_x^{\alpha-\gamma}\psi)(x)}dx$$

$$= \left\langle\varphi, \sum_{|\alpha|\le m}(-i)^\alpha\sum_{\gamma\le\alpha}(\binom{\alpha}{\gamma}\partial_x^\gamma(\bar{p}_\alpha(x))(\partial_x^{\alpha-\gamma}\psi)(x)\right\rangle$$

$$= \left\langle\varphi, \sum_{|\alpha|\le m}(-i)^\alpha\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}\partial_x^\gamma(\bar{p}_\alpha(x))i^{\alpha-\gamma}(D_x^{\alpha-\gamma}\psi)(x)\right\rangle$$

Consequently

$$\sigma(P^*)(x,\xi) = \sum_{|\alpha|\le m}\sum_{\gamma\le a}(-i)^\alpha i^{\alpha-\gamma}\partial_x^\gamma(\bar{p}_\alpha(x))\binom{\alpha}{\gamma}\xi^{\alpha-\gamma} \overset{(1)}{=} \sum_{|\alpha|\le m}\sum_{\gamma\le\alpha}(-i)^\alpha i^{\alpha-\gamma}\partial_x^\gamma(\bar{p}_\alpha(x))\frac{1}{\gamma!}\partial_\xi^\gamma\xi^\alpha$$

$$\overset{(2)}{=} \sum_{|\alpha|\le m}\sum_{|\gamma|\le m}\frac{(-i)^\gamma}{\gamma!}\partial_x^\gamma(\bar{p}_\alpha(x))\partial_\xi^\gamma\xi^\alpha = \sum_{|\gamma|\le m}\frac{(-i)^\gamma}{\gamma!}\partial_x^\gamma\partial_\xi^\gamma(\sum_{|\alpha|\le m}\bar{p}_\alpha(x)\xi^\alpha)$$

$$= \sum_{|\gamma|\le m}\frac{(-i)^\gamma}{\gamma!}\partial_x^\gamma\partial_\xi^\gamma\bar{p}(x,\xi)$$

(1),(2): Remember from Lemma 1.2.4, that

$$\partial_\xi^\gamma\xi^\alpha = \begin{cases}\gamma!\binom{\alpha}{\gamma} & ,\gamma\le\alpha\\ 0 & ,\text{otherwise}\end{cases}$$

This justifies (1) and it also justifies (2), since we only added zero summands. $\square$

2 PDO between vector bundles

Let M be a smooth manifold of dimension m.

2.1 Basic Definitions and Properties

2.1.1 Definition (Complex vector bundle)**.** A map $\pi : E \to M$ is a *smooth complex vector bundle of rank* r if the following conditions are satisfied:

(i) E is a smooth manifold.

(ii) The map π is smooth and surjective.

(iii) For all $p \in M$ *fibre over* p $E_p := \pi^{-1}(p)$ is endowed with a complex vector space structure of dimension k.

(iv) For every $p \in M$ there exists an open neighbourhood $U \subset M$ of p and a *local trivialization*, i.e. a diffeomorphism $\Phi : E_U := \pi^{-1}(U) \to U \times \mathbb{C}^r$ such that $\mathrm{pr} \circ \Phi = \mathrm{id}_U$, where $\mathrm{pr} : U \times \mathbb{C}^r \to U$ is the canonical projection, and for every $q \in U$ the restriction $\Phi : E_q \to q \times \mathbb{C}^r \cong \mathbb{C}^r$ is a complex vector space isomorphism.

2.1.2 Definition (Section)**.** If $\pi : E \to M$ is a complex vector bundle, a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$ is a *section in* E *over* M. The space of all such sections is denoted by $\Gamma(M, E)$.

Before we are able to define PDO on vector bundles, one additional local property is required, we have not yet established.

2.1.3 Definition (Push-forward of operators)**.** Let $V, \tilde{V} \subset \mathbb{R}^n$ be open, $P \in \mathrm{Diff}^k(V, \mathbb{C}^r, \mathbb{C}^s)$ be a PDO and $F : V \to \tilde{V}$ be a smooth diffeomorphism. Then the map $\tilde{P} := F_* P : \mathcal{C}^\infty(\tilde{V}, \mathbb{C}^r) \to \mathcal{C}^\infty(\tilde{V}, \mathbb{C}^s)$ defined by

$$\tilde{s} \mapsto P(\tilde{s} \circ F) \circ F^{-1}$$

is the *push-forward of* P *along* F.

We would like to show, that PDOs are in some sense invariant under push-forwards.

2.1.4 Lemma (Diffeomorphism invariance)**.** With the notation of Definition 2.1.3 above: Let $\alpha \in \mathbb{N}^n$, $|\alpha| = k \geq 1$, be a multi-index and $P := D^\alpha \in \mathrm{Diff}^k(V, \mathbb{C}^r, \mathbb{C}^r)$. Then $\tilde{P} := F_*(D^\alpha) \in \mathrm{Diff}^k(\tilde{V}, \mathbb{C}^r, \mathbb{C}^r)$, thus there exist $\tilde{P}^\alpha \in \mathcal{C}^\infty(\tilde{V}, \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^r))$ such that

$$\forall \tilde{s} \in \mathcal{C}^\infty(\tilde{V}, \mathbb{C}^r, \mathbb{C}^r) = \tilde{P}(\tilde{s}) = F_*(D^\alpha)(\tilde{s}) = D^\alpha(\tilde{s} \circ F) \circ F^{-1} = \sum_{|\beta| \leq k} \tilde{P}^\beta D^\beta(\tilde{s}).$$

Moreover the symbols satisfy

$$\sigma_{D^\alpha}(x, \xi) = I_r \xi^\alpha \qquad \sigma_{F_*(D^\alpha)}(\tilde{x}, \xi) = \sum_{|\beta| = k} \tilde{P}^\beta(\tilde{x}) \xi^\beta = I_r (A^t(\tilde{x})\xi)^\alpha,$$

where $I_r \in \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^r)$ is the identity and $A := \nabla F \circ F^{-1}$.

Proof. We will show the statement by induction over k.
STEP 1 ($k = 1$): This implies, that $\alpha = e_j$ for some $1 \leq j \leq n$. The chain rule for total derivatives states

$$\nabla(\tilde{s} \circ F) = \nabla \tilde{s} \circ F \cdot \nabla F,$$

which implies

$$\partial_j(\tilde{s} \circ F) = \nabla \tilde{s} \circ F \cdot \partial_j F.$$

Consequently by definition

$$F_*(\partial^\alpha) = F_*(\partial_j) = \partial_j(\tilde{s} \circ F) \circ F^{-1} = \nabla\tilde{s} \cdot \partial_j F \circ F^{-1} = \sum_{i=1}^n \partial_j F^i \circ F^{-1} \partial_i \tilde{s}.$$

By multiplying with $-i$, this shows $F_*(D^\alpha) \in \mathrm{Diff}^1(\tilde{V}, \mathbb{C}^r, \mathbb{C}^r)$. The symbols are given by

$$\sigma_{\partial_j}(x,\xi) = \xi_j \qquad\qquad \sigma_{F_*(\partial_j)}(x,\xi) = \sum_{i=1}^n I_r(\partial_j F^i \circ F^{-1})(x)\xi_i = I_r(A^t\xi)_j.$$

Step 2 ($k \to k+1$): There exists $\hat{\alpha} \in \mathbb{N}^n$, $|\hat{\alpha}| = k$, and $1 \le j \le n$, such that $\alpha = \hat{\alpha} + e_j$. We calculate

$$\begin{aligned}
F_*(\partial^\alpha)(\tilde{s}) &= \partial^\alpha(\tilde{s} \circ F) \circ F^{-1} = \partial^{\hat{\alpha}}\partial_j(\tilde{s} \circ F) \circ F^{-1} = \partial^{\hat{\alpha}}\Big(\sum_{i=1}^n \partial_j F^i \cdot \partial_i \tilde{s} \circ F\Big) \circ F^{-1} \\
&= \Big(\sum_{\beta \le \hat{\alpha}} \sum_{i=1}^n \binom{\hat{\alpha}}{\beta} \partial^{\hat{\alpha}-\beta}\partial_j F^i \cdot \partial^\beta(\partial_i \tilde{s} \circ F)\Big) \circ F^{-1} \\
&= \sum_{\beta \le \hat{\alpha}} \sum_{i=1}^n \binom{\hat{\alpha}}{\beta} \partial^{\hat{\alpha}-\beta}\partial_j F^i \circ F^{-1} \cdot F_*(\partial^\beta)(\partial_i \tilde{s}) \\
&= \sum_{\beta \le \hat{\alpha}} \sum_{i=1}^n \sum_{|\gamma| \le |\beta|} P_\beta^\gamma \binom{\hat{\alpha}}{\beta} \partial^{\hat{\alpha}-\beta}\partial_j F^i \circ F^{-1} \partial^\gamma \partial_i(\tilde{s}).
\end{aligned}$$

By multiplying with $(-i)^{k-1}$, this shows $F_*(D^\alpha) \in \mathrm{Diff}^{k+1}(\tilde{V}, \mathbb{C}^r, \mathbb{C}^r)$. We analyse the highest order terms. These occure precisely, if $|\gamma + e_i| = k+1 \Leftrightarrow |\gamma| = k$. Since $|\gamma| \le |\beta|$ and $\beta \le \hat{\alpha}$, this can only happen if $\beta = \hat{\alpha}$ and $|\gamma| = k$. Obviously $\sigma_{\partial^\alpha}(x,\xi) = I_r\xi^\alpha$ and

$$\begin{aligned}
\sigma_{F_*(D^\alpha)}(x,\xi) &= \sum_{|\gamma|=k} P_{\hat{\alpha}}^\gamma(x)\Big(\sum_{i=1}^n (\partial_j F^i \circ F^{-1})(x)\xi_i\Big)\xi^\gamma = \sum_{|\gamma|=k} P_{\hat{\alpha}}^\gamma(x)(A^t(x)\xi)_j\xi^\gamma \\
&= (A^t(x)\xi)_j \sum_{|\gamma|=k} P_{\hat{\alpha}}^\gamma(x)\xi^\gamma = I_r(A^t(x)\xi)_j(A^t(x)\xi)^{\hat{\alpha}} = I_r(A^t(x)\xi)^\alpha.
\end{aligned}$$

□

2.1.5 Theorem (Diffeomorphism Invariance)**.** With the notation of Definition 2.1.3 we claim: $\tilde{P} = F_*(P) \in \mathrm{Diff}^k(\tilde{V}, \mathbb{C}^r, \mathbb{C}^s)$, i.e. there exist $\tilde{P}^\alpha$ such that

$$\forall \tilde{s} \in \mathcal{C}^\infty(\tilde{V}, \mathbb{C}^r, \mathbb{C}^s) : \tilde{P}(\tilde{s}) = F_*(P)(\tilde{s}) = \sum_{|\alpha| \le k} \tilde{P}^\alpha D^\alpha.$$

Moreover the symbol has a representation

$$\sigma_{\tilde{P}}(\tilde{x},\xi) = \sum_{|\alpha|=k} \tilde{P}^\alpha(\tilde{x})\xi_\alpha = \sum_{|\alpha|=k} (P^\alpha \circ F^{-1})(\tilde{x})(A^t(\tilde{x})\xi)^\alpha = \sigma_P(F^{-1}(\tilde{x}), A^t(\tilde{x})\xi),$$

where $A := \nabla F \circ F^{-1}$.

Proof. By definition we obtain

$$\tilde{P}(\tilde{s}) = F_*(P)(\tilde{s}) = \Big(\sum_{|\alpha| \le k} P^\alpha D^\alpha(s \circ F)\Big) \circ F^{-1} = \sum_{|\alpha| \le k} P^\alpha \circ F^{-1} F_*(D^\alpha)(\tilde{s}).$$

Applying the first part of Lemma 2.1.4, we conclude $\tilde{P} \in \mathrm{Diff}^k(\tilde{V}, \mathbb{C}^r, \mathbb{C}^s)$. Applying the second part and analyzing the highest order terms, we conclude, that the symbol satisfies:

$$\sigma_{\tilde{P}}(\tilde{x}, \xi) = \sum_{|\alpha|=k} (P^\alpha \circ F^{-1})(\tilde{x}) \sigma_{F_*(D^\alpha)}(\tilde{x}, \xi) = \sum_{|\alpha|=k} (P^\alpha \circ F^{-1})(\tilde{x}) \sum_{|\beta|=k} Q_\alpha^\beta(\tilde{x}) \xi^\beta$$
$$= \sum_{|\alpha|=k} (P^\alpha \circ F^{-1})(\tilde{x}) (A^t(\tilde{x})\xi)^\alpha.$$

□

2.1.6 Definition (Associated Pushforwards)**.** Let $\pi : E \to M$ be a complex vector bundle of rank r and $\Phi : E_U \to U \times \mathbb{C}^r$ be a local trivialization. Denote by $\mathrm{pr}_2 : U \times \mathbb{C}^r \to \mathbb{C}^r$ the canonical projection. We obtain the *pushforward* $\Phi_* : \Gamma(U, E) \to \mathcal{C}^\infty(U, \mathbb{C}^r)$ defined by

$$s \mapsto \mathrm{pr}_2 \circ \Phi \circ s = \Phi_2 \circ s$$

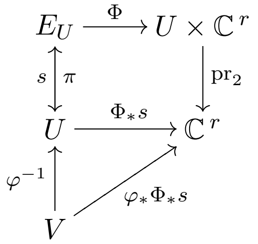
and for any chart $\varphi : U \to V$ of M the *pushforward* $\varphi_* : \mathcal{C}^\infty(U, \mathbb{C}^r) \to \mathcal{C}^\infty(V, \mathbb{C}^r)$

$$f \mapsto f \circ \varphi^{-1}.$$

By composing we obtain a map $\varphi_* \circ \Phi_* : \Gamma(U, E) \to \mathcal{C}^\infty(V, \mathbb{C}^r)$.

2.1.7 Lemma.

(i) By construction the following diagram commutes:



(ii) The map φ_* is bijective with inverse $(\varphi_*)^{-1} : \mathcal{C}^\infty(V, \mathbb{C}^r) \to \mathcal{C}^\infty(U, \mathbb{C}^r)$ given by $(\varphi^{-1})_*$.

(iii) The map Φ_* is bijective with inverse $\Phi_* : \mathcal{C}^\infty(U, \mathbb{C}^r) \to \Gamma(U, E)$ given by $f \mapsto \Phi^{-1} \circ \mathrm{id}_U \times f$.

Proof. The first two statements are clear. To see the third one, remember that any local trivialization can be written as $\Phi = (\Phi_1, \Phi_2) = (\mathrm{id}_U, \Phi_2) = \mathrm{id}_U \times \Phi_2$. Therefore we obtain

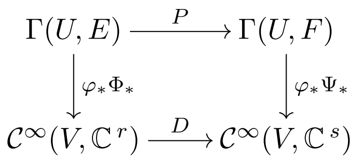
$$\forall s \in \Gamma(U, E) : (\Phi_*^{-1} \circ \Phi_*)(s) = \Phi_*^{-1}(\Phi_2 \circ s) = \Phi^{-1} \circ \mathrm{id}_U \times (\Phi_2 \circ s) = \Phi^{-1}((\Phi_1, \Phi_2)(s)) = s$$

and

$$\forall f \in \mathcal{C}^\infty(U, \mathbb{C}^r) : (\Phi_* \circ \Phi_*^{-1})(f) = \Phi_*(\Phi^{-1} \circ \mathrm{id}_U \times f) = \mathrm{pr}_2 \circ \Phi \circ \Phi^{-1} \circ \mathrm{id}_U \times f = f.$$

□

2.1.8 Definition (Differential operators between vector bundles)**.** Let E, F be smooth complex vector bundles over M of rank r and s. A linear map $P : \Gamma(M, E) \to \Gamma(M, F)$ is a *differential operator of rank* k, if for any chart $\varphi : U \to V$ and local trivializations $\Phi : E_U \to U \times \mathbb{C}^r$ and $\Psi : F_U \to U \times \mathbb{C}^s$, there exists $D \in \mathrm{Diff}^k(V; \mathbb{C}^r, \mathbb{C}^s)$, called a *local representation of* P, such that



commutes, i.e. $\varphi_*\Psi_* \circ P \circ (\varphi_*\Phi_*)^{-1} = D$. The set of all differential operators of order k between E and F is denoted by

$$\mathrm{Diff}^k(M; E, F).$$

2.1.9 Theorem (Local independence)**.** Let $P : \Gamma(M, E) \to \Gamma(M, F)$ be a linear map, let $\varphi : U \to V$, $\psi : U \to \tilde{V}$ be any charts and $\Phi, \tilde{\Phi} : E_U \to U \times \mathbb{C}^r$, $\Psi, \tilde{\Psi} : F_U \to U \times \mathbb{C}^s$ be local trivializations.

(i) Then

$$D := \varphi_*\Psi_* \circ P \circ (\varphi_*\Phi_*)^{-1} \in \mathrm{Diff}^k(V, \mathbb{C}^r, \mathbb{C}^s) \Longrightarrow \tilde{D} := \tilde{\psi}_*\tilde{\Psi}_* \circ P \circ (\psi_*\tilde{\Phi}_*)^{-1} \in \mathrm{Diff}^k(\tilde{V}, \mathbb{C}^r, \mathbb{C}^s).$$

So the local property of beeing a differential operator does not depend on the choice of charts or trivializations, but only on the smooth structures of M, E and F.

(ii) Denote by $F := \psi \circ \varphi^{-1} : V \to \tilde{V}$ the transition map between the charts, $A := \nabla F \circ F^{-1}$, and by g_E and g_F the transition functions between the local trivializations (see equation (2.1)) and let $D = \sum_{|\alpha| \leq k} P^\alpha D^\alpha$. Then the symbol satisfies

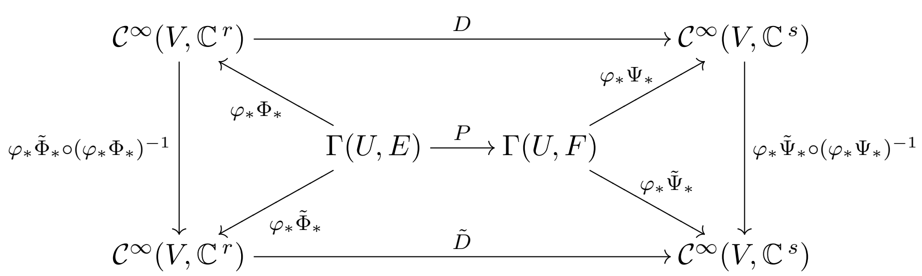
$$\forall \tilde{x} \in \tilde{V} : \forall \xi \in \mathbb{R}^n : \sigma_{\tilde{D}}(\tilde{x}, \xi) = \sum_{|\alpha|=k} (g_F P^\alpha g_E^{-1})(F^{-1}(\tilde{x}))(A^t(\tilde{x})\xi)^\alpha.$$

Proof.

Step 1 (Independence of trivializations): First we fix the chart φ and consider different trivializations. There exist functions (c.f. [2, 5.4])) $g_E \in \mathcal{C}^\infty(V, GL(r, \mathbb{C}))$, $g_F \in \mathcal{C}^\infty(V, GL(s, \mathbb{C}))$ such that

$$\begin{aligned} &\forall x \in V : \forall \xi \in \mathbb{C}^r : (\tilde{\Phi} \circ \Phi^{-1})(\varphi^{-1}(x), \xi) = (\varphi^{-1}(x), g_E(x)\xi) \\ &\forall x \in V : \forall \xi \in \mathbb{C}^s : (\tilde{\Psi} \circ \Psi^{-1})(\varphi^{-1}(x), \xi) = (\varphi^{-1}(x), g_F(x)\xi). \end{aligned} \tag{2.1}$$

We redefine $\tilde{D} := \varphi_*\tilde{\Psi}_* \circ P \circ (\varphi_*\tilde{\Phi}_*)^{-1}$ (valid for this step of the proof) and remark that the following diagram commutes:



We calculate

$$\begin{aligned} \tilde{D} &= \varphi_*\tilde{\Psi}_* \circ P \circ (\varphi_*\tilde{\Phi}_*)^{-1} = \varphi_*\tilde{\Psi}_* \circ (\varphi_*\Psi_*)^{-1} \circ D \circ \varphi_*\Phi_* \circ (\varphi_*\tilde{\Phi}_*)^{-1} \\ &= \varphi_* \circ \tilde{\Psi}_* \circ \Psi_*^{-1} \circ \varphi_*^{-1} \circ D \circ \varphi_* \circ \Phi_* \circ \tilde{\Phi}_*^{-1} \circ \varphi_*^{-1}. \end{aligned}$$

The map $\varphi_* \circ \tilde{\Psi}_* \circ \Psi_*^{-1} \circ \varphi_*^{-1} : \mathcal{C}^\infty(V, \mathbb{C}^r) \to \mathcal{C}^\infty(V, \mathbb{C}^r)$ can be simplified drastically. By construction for any $f \in \mathcal{C}^\infty(V, \mathbb{C}^r)$

$$\begin{aligned} &\varphi_*(\tilde{\Psi}_*(\Psi_*^{-1}(\varphi_*^{-1}(f)))) \overset{2.1.7,(iii)}{=} \varphi_*(\tilde{\Psi}_*(\Psi_*^{-1}(f \circ \varphi))) = \varphi_*(\tilde{\Psi}_*(\Psi^{-1} \circ \mathrm{id}_U \times (f \circ \varphi))) \\ &= \varphi_*(\mathrm{pr}_2 \circ \tilde{\Psi} \circ \Psi^{-1} \circ \mathrm{id}_U \times (f \circ \varphi)) = g_F (f \circ \varphi) \circ \varphi^{-1} = g_F f \end{aligned}$$

and analogously

$$(\varphi_* \circ \Phi_* \circ \tilde{\Phi}_*^{-1} \circ \varphi_*^{-1})(f) = g_E^{-1} f.$$

Since $D \in \mathrm{Diff}^k(M; E, F)$ by hypothesis, there exist $P_\alpha \in \mathcal{C}^\infty(V, \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s))$ such that

$$D = \sum_{|\alpha| \leq k} P_\alpha D^\alpha \in \mathrm{Diff}^k_{\mathbb{C}}(V, r, s).$$

Alltogether we obtain

$$\begin{aligned}\tilde{D}f &= g_F\Big(\sum_{|\alpha|\leq k} P_\alpha D^\alpha\Big)(g_E^{-1} f) = \sum_{|\alpha|\leq k} g_F P_\alpha D^\alpha (g_E^{-1} f) \overset{1.2.3}{=} \sum_{|\alpha|\leq k} g_F P_\alpha \Big(\sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (D^{\alpha-\beta} g_E^{-1}) D^\beta f\Big) \\ &= \sum_{|\alpha|\leq k} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} g_F P_\alpha (D^{\alpha-\beta} g_E^{-1}) D^\beta f,\end{aligned}$$

which shows $\tilde{D} \in \mathrm{Diff}^k(V, \mathbb{C}^r, \mathbb{C}^s)$.
We analyze the highest order terms: These occur precisely if $|\beta| = k$. But since $\beta \leq \alpha$ this happens if and only if $\beta = \alpha$. So the symbol is given by

$$\sigma_{\tilde{D}}(x, \xi) = \sum_{|\alpha|=k} g_F(x) P_\alpha(x) g_E^{-1}(x) \xi^\alpha.$$

Step 2 (Independence of the chart): Now fix the trivialization Φ and consider the two different charts φ, ψ. Analogously we redefine $\tilde{D} := \psi_* \Psi_* \circ P \circ (\psi_* \Phi_*)^{-1}$ (valid for this step of the proof) and calculate

$$\begin{aligned}\tilde{D} &= \psi_* \Psi_* \circ P \circ (\psi_* \Phi_*)^{-1} = \psi_* \Psi_* \circ (\varphi_* \Psi_*)^{-1} \circ D \circ \varphi_* \Phi_* \circ (\psi_* \Phi_*)^{-1} \\ &= \psi_* \circ \Psi_* \circ \Psi_*^{-1} \circ \varphi_*^{-1} \circ D \circ \varphi_* \circ \Phi_* \circ \Phi_*^{-1} \circ \psi_*^{-1} = \psi_* \circ \varphi_*^{-1} \circ D \circ \varphi_* \circ \psi_*^{-1}.\end{aligned}$$

Thus for any $\tilde{f} \in \mathcal{C}^\infty(\tilde{V}, \mathbb{C}^r, \mathbb{C}^s)$, we obtain

$$\tilde{D}(\tilde{f}) = D(\tilde{f} \circ F) \circ F^{-1} = F_*(D)(f),$$

which implies $\tilde{D} \in \mathrm{Diff}^k(\tilde{V}, \mathbb{C}^r, \mathbb{C}^s)$ by Theorem 2.1.5. It was already shown there, that the symbol is given by

$$\sigma_{\tilde{D}}(\tilde{x}, \xi) = \sum_{|\alpha|=k} P_\alpha(F^{-1}(\tilde{x}))(A^t(\tilde{x})\xi)^\alpha.$$

Redefining $\tilde{D} := \tilde{\psi}_* \tilde{\Psi}_* \circ P \circ (\psi_* \tilde{\Phi}_*)^{-1}$ as in the statement of the theorem and combining both steps, we obtain both claims. □

2.1.10 Definition (Symbol)**.** Let $P \in \mathrm{Diff}^k(M; E, F)$ be a PDO. For any $p \in M$ and any $\xi \in T_p^*M$ define $\sigma_P(p, \xi) \in \mathrm{Hom}(E_p, F_p)$ to be the homomorphism given as follows: Choose a chart $\varphi : U \to V$ near p and local trivializations $\Phi : E_U \to U \times \mathbb{C}^r$, $\Psi : F_U \to U \times \mathbb{C}^s$. Let D be the local coordinate representation of P with respect to this chart and these trivializations and define

$$\forall v \in E_p : \sigma_P(p, \xi)(v) := \Psi_2^{-1}(\sigma_D(\varphi(p), \xi_i e^i)(\Phi_2(e))).$$

We call σ_P the *symbol of* P.

2.1.11 Remark. This definition produces two problems: First of all, the homomorphism $\sigma_P(p, \xi)$ is defined in terms of various non canonical choices, so we have to show, that it is well-defined. Secondly, we would like to state more precisely, what σ_P is. Therefore denote by $\pi : T^*M \to M$ the cotangant bundle. Notice, that for any $\xi \in T^*M$ we could define a homomorphism $\sigma_P(\xi) := \sigma_P(\pi(\xi), \xi) \in \mathrm{Hom}(E_{\pi(\xi)}, F_{\pi(\xi)})$, so the base point $p \in M$ is somewhat superflous. But if we drop it, we can no longer think of $\mathrm{Hom}(E, F)$ as a bundle over M. But if we think of $\mathrm{Hom}(E, F)$ as a bundle over T^*M, then we may think of an element in $\mathrm{Hom}(E_{\pi(\xi)}, F_{\pi(\xi)})$ as beeing attatched to ξ. These notions are made precise in the following Lemma.

2.1.12 Lemma. The symbol is a well-defined section

$$\sigma_P \in \Gamma(T^*M, \mathrm{Hom}(\pi^*E, \pi^*F)),$$

i.e.: Let $\tilde{\varphi} : \tilde{U} \to \tilde{V}$ be another chart, $\tilde{\Phi}$, $\tilde{\Psi}$ be other local trivializations for E and F and let $\tilde{\sigma}_P$ be the symbol defined in terms of this chart and these local trivializations. Then

$$\forall p \in U \cap \tilde{U} : \forall \xi \in T_p^*M : \forall e \in E_p : \sigma_P(p, \xi)(e) = \tilde{\sigma}_P(p, \xi)(e).$$

Proof. By shrinking the coordinate neighbourhoods if necessary, we may assume that $U = \tilde{U}$, and calculate there. As usual, we define $F := \tilde{\varphi} \circ \varphi^{-1}$, $A := \nabla F \circ F^{-1}$. Denote by $\Xi := (\xi_1, \ldots, \xi_n)$ the coordinate vector function of ξ seen as a column vector in $\mathbb{R}^n$ (define $\tilde{\Xi}$ analogously). The transformation law for the cotangent bundle states that

$$\xi = \xi_i d\varphi^i = \tilde{\xi}_i d\tilde{\varphi}^i,$$

where $\Xi = \nabla F^t \circ \varphi \cdot \tilde{\Xi}$. This implies $\Xi = A^t \circ \tilde{\varphi} \cdot \tilde{\Xi}$, which is equivalent to

$$\tilde{\Xi} = (A^t)^{-1} \circ \tilde{\varphi} \cdot \Xi \tag{2.2}$$

Remember the defining equations (2.1) for the transition functions. Define $\tau : U \times \mathbb{C}^r \to U \times \mathbb{C}^r$, $(p, v) \mapsto (p, g_E(\varphi(p))v)$. Then we can reformulate

$$\tilde{\Phi} \circ \Phi = \tau \iff \Phi = \tilde{\Phi} \circ \tau^{-1},$$

which implies in particular

$$\tilde{\Phi}(e) = (\tau(\Phi(e))) = \tau(\Phi_1(e), \Phi_2(e)) = (p, g_E(\varphi(p))\Phi_2(e)) \tag{2.3}$$

and analogously for $\tilde{\Psi}$. Alltogether we obtain

$$\begin{aligned}
&\tilde{\sigma}_P(p, \xi)(e) = \tilde{\Psi}_2^{-1}(\sigma_{\tilde{D}}(\tilde{\varphi}(p), \tilde{\Xi}(p))(\tilde{\Phi}_2(e))) \\
&= \tilde{\Psi}_2^{-1}\Big(\sum_{|\alpha|=k} (g_F P^\alpha g_E^{-1})(F^{-1}(\tilde{\varphi}(p)))(A^t(\tilde{\varphi}(p))\tilde{\Xi}(p))^\alpha\Big)(\tilde{\Phi}_2(e))) \\
&\overset{(2.2)}{=} \tilde{\Psi}_2^{-1}\Big(\sum_{|\alpha|=k} (g_F P^\alpha g_E^{-1})(\varphi(p))(A^t(\tilde{\varphi}(p))(A^t)^{-1}(\tilde{\varphi}(p))\Xi(p)^\alpha\Big)(\tilde{\Phi}_2(e))) \\
&\overset{(2.3)}{=} \Psi_2^{-1} g_F^{-1}(\varphi(p))\Big(\sum_{|\alpha|=k} (g_F P^\alpha g_E^{-1})(\varphi(p))\Xi(p)^\alpha\Big)(g_E(\varphi(p)\Phi_2(e))) \\
&= \Psi_2^{-1}(\sigma_D(\varphi(p), \Xi(p))(\Phi_2(e))) = \sigma_P(p, \xi)(v).
\end{aligned}$$

□

2.1.13 Definition (Elliptic PDO)**.** An operator $P \in \mathrm{Diff}^k(M; E, F)$ is called *elliptic*, if its symbol is invertible outside the zero section, i.e.

$$\forall p \in M : \forall 0 \neq \xi \in T_p^* M : \sigma_P(p, \xi) \in \mathrm{Iso}(E_p, F_p).$$

References

[1] Wong, Introduction to Pseudo-Differential Operators

[2] Lee, Introduction to smooth Manifolds