The de Rham Isomorphism and the L_p-Cohomology of non-compact Riemannian Manifolds

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04.04.2011

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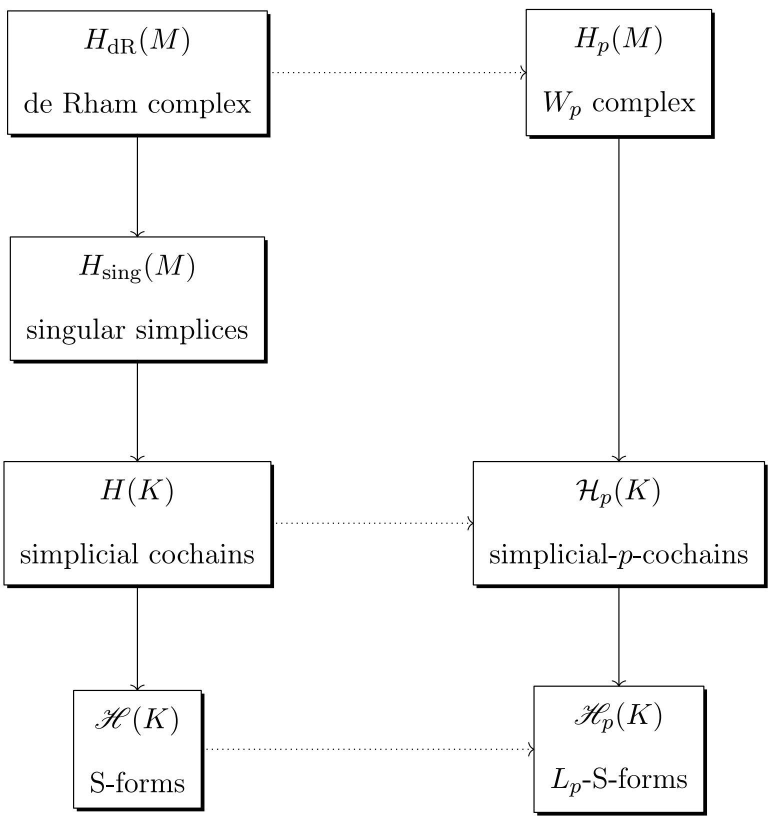
1 Overview and Main Result

Global Setup: Let M be a smooth oriented Riemannian m- manifold without boundary. Let K be a simplicial complex in some $\mathbb{R}^n$ with star-bound N (c.f. 2.12). Let $h : |K| \to M$ be a smooth triangulation of M, and $1 \leq p \leq \infty$. In the following graph we will add a node for every cohomology theory of interest and an edge, if under certain reasonable assumptions there exists an isomorphism between them.



1.1 Main Theorem. If $h : |K| \to M$ is GKS (c.f. 3.1), then there exists a commutative diagram of isomorphisms

$$\begin{array}{ccc} & H_p(M) & \\ \nearrow & & \nwarrow \\ \mathcal{H}_p(K) & \longrightarrow & \mathscr{H}_p(K). \end{array}$$

Therefore all L_p-cohomolgies of M are mutually isomorphic.

2 Definition of L_p-spaces

2.1 Differential Forms

2.1 Convention (exterior direct sums). We will frequently define $\mathbb{Z}$-indexed systems of vector spaces $(V^k)_{k\in\mathbb{Z}}$. In that case, the space V is understood to be

$$V := \bigoplus_{k\in\mathbb{Z}} V^k.$$

2.2 Definition (L_p-spaces). Let

$$L^k(M) := \{\omega : M \to \Lambda^k TM \mid \omega \text{ is a measurable section}\}/\sim,$$

where $\omega \sim \omega'$ if and only if ω equals ω' outside a set of measure zero. For any $1 \leq p < \infty$, define

$$L_p^k(M) := \{\omega \in L^k(M) \mid \|\omega\|^p_{L_p^k(M)} := \int_M |\omega|^p d_gV < \infty\}$$

the *p-integrable forms*. In case $p = \infty$ the norm is replaced by

$$\|\omega\|_{L^k_\infty(M)} := \text{ess sup}_{x\in M} |\omega(x)|.$$

Sometimes it is nice to have the following local version of L_p-spaces:

$$L^k_{p,\text{loc}}(M) := \{\omega \in L^k(M) \mid \forall K \subset M \text{ compact } : \omega \in L^k_p(K)\}.$$

The *modulus* $|\omega|$ of a differential form is defined below.

2.3 Definition (modulus). For any Riemannian metric g on M there exists exactly one fibre metric $\tilde{g}$ on $\Lambda^k TM$ such that for any g-ONB $B = (b_1, \ldots, b_k)$ of any T_pM the set

$$\Lambda^k B := \{b_{i_1} \wedge \ldots \wedge b_{i_k} \mid 0 \leq i_1 < \ldots < i_k \leq m\}$$

is a $\tilde{g}$-ONB for $\Lambda^k T_pM$. Denote by $|\_|$ the norm generated by $\tilde{g}$. Then $|\omega| : M \to \mathbb{R}$ is the *modulus* of ω.

2.4 Theorem (completeness of L_p-spaces). For every $1 \leq p \leq \infty$ and every $0 \leq k \leq m$ the space $L^k_p(M)$ is a Banach space.

2.5 Definition (weak differential). Let $\omega \in L^k_{p,\text{loc}}(M)$ and $\omega' \in L^{k+1}_{p,\text{loc}}(M)$. Then ω' is a *weak differential* of ω if

$$\forall \eta \in \Omega_c^{m-k-1}(M) : \int_M \omega' \wedge \eta = (-1)^{k+1} \int_M \omega \wedge d\eta.$$

In that case we denote $d\omega := \omega'$. The space of all those forms is denoted by $W^k_{p,\text{loc}}(M)$.

2.6 Lemma. The weak differential of a form $\omega \in L^k_{p,\text{loc}}(M)$ is uniquely determined (if it exists). If ω is smooth, the weak differential equals the exterior differential.

Proof. The first claim follows from the Fundamental Lemma of the calculus of variations. For the second, notice that by Stokes' theorem and Leibniz rule

$$\forall \eta \in \Omega_c^{m-k-1}(M) : 0 = \int_{\partial M} \omega \wedge \eta = \int_M d(\omega \wedge \eta) = \int_M d\omega \wedge \eta + (-1)^k \int_M \omega \wedge d\eta.$$

□

2.7 Definition (exterior Sobolev spaces). Employing the notation

$$\|\omega\|^p_{W^k_p(M)} := \|\omega\|^p_{L^k_p(M)} + \|d\omega\|^p_{L^{k+1}_p(M)}, \qquad \|\omega\|_{W^k_\infty(M)} := \max\{\|\omega\|_{L^k_\infty(M)}, \|\omega\|_{L^{k+1}_\infty(M)}\},$$

we define the *(exterior) Sobolev spaces*

$$W^k_p(M) := \{\omega \in W^k_{p,\mathrm{loc}}(M) \mid \|\omega\|_{W^k_p(M)} < \infty\}.$$

2.8 Lemma (completeness of exterior Sobolev spaces). For every $1 \leq p \leq \infty$ and every $0 \leq k \leq m$ the map $d : W^k_p(M) \to W^{k+1}_p(M)$ is a bounded linear operator between Banach spaces.

2.9 Theorem (Hölder Inequality and Leibniz rule). Let $1 \leq p, q \leq \infty$, $\omega \in L^k_p(M)$, $\eta \in L^l_q(M)$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

(i) $\omega \wedge \eta \in L^{k+l}_r(M)$,

(ii) $\|\omega \wedge \eta\|_{L_r(M)} \leq \|\omega\|_{L_r(M)} \|\eta\|_{L_r(M)}$,

(iii) if $\omega \in W^k_p(M)$, $\eta \in W^l_q(M)$, then $\omega \wedge \eta \in W^{k+l}_r(M)$ and

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

2.10 Definition (L_p-cohomology). The cohomology of the exterior Sobolev spaces called the *L_p-complex of M.* Its cohomology groups

$$H^k_p(M) := \frac{Z^k_p(M)}{B^k_p(M)} := \frac{\ker\left(d^k : W^k_p(M) \to W^{k+1}_p(M)\right)}{\mathrm{im}\left(d^k : W^{k-1}_p(M) \to W^k_p(M)\right)} = H^k((W_p(M), d))$$

are called *L_p-cohomology of M.* This is a $\mathbb{Z}$-indexed system of vector spaces endowed with the ordinary quotient semi-norm $\|\_\|_{H^k_p}$ induced by $\|\_\|_{W^k_p}$. The spaces

$$\bar{H}^k_p(M) := \frac{Z^k_p(M)}{\overline{B^k_p(M)}} \cong \frac{H^k_p(M)}{\{x \in H^k_p(M) \mid \|x\|_{H^k_p} = 0\}}$$

are called *reduced L_p-cohomology of M.* The space

$$T^k_p(M) := \frac{\overline{B^k_p(M)}}{B^k_p(M)}$$

is the *Torsion of M.*

2.11 Example (L_1-cohomology of the half-line). The following example illustrates that L_p-cohomology can be very different from the classical de Rham cohomology. For simplicity let $p = 1$, define $M :=]1, \infty[\subset \mathbb{R}$ and remember that

$$t \mapsto t^{-s} \in L^0_1(M) \Leftrightarrow s > 1.$$

Similiarly, for an antiderivative of this function, we obtain

$$t \mapsto \frac{1}{-s+1} t^{-(s-1)} \in L^0_1(M) \Leftrightarrow s > 2.$$

(i) Define

$$f : M \to \mathbb{R},\ f(t) := t^{-2},\ \omega := f\,dt \in L_1^1(M).$$

Since $\dim M = 1$, $d\omega = 0$, thus ω is closed and $\omega \in W_1^1(M)$. An antiderivative is easily seen to be $F : M \to \mathbb{R}$, $t \mapsto -t^{-1}$. So in the classical de Rham cohomology we would conclude that $\omega = dF$ is exact. Since every smooth function on M has an antiderivative, we obtain $H^1_{\mathrm{dR}}(M) = 0$. The crucial observation here is that

$$\forall c \in \mathbb{R} : F + c \notin L_1^0(M),$$

i.e. no antiderivative of ω is integrable. Thus $0 \neq [\omega] \in H_1^1(M)$.

(ii) The sequence

$$g_n := t^{-(2+\frac{1}{n})} \in W_1^1(M), \quad G_n := \frac{-1}{1+\frac{1}{n}} t^{-(1+\frac{1}{n})} \in L_1^0(M), \quad \omega_n := g_n dt = dG_n \in W_1^1(M)$$

satisfies

$$\forall n \in \mathbb{N} : 0 = [\omega_n] \in H_1^1(M), \qquad \omega_n \xrightarrow[W_1^1(M)]{} \omega\ , \qquad [\omega] \neq 0 \in H_1^1(M).$$

So this is an example of a non-exact W_1-form ω, which is a W_1^1-limit of exact forms ω_n. In particular $d : W_1^0(M) \to W_1^1(M)$ is not a closed operator.

(iii) It is also remarkable that $M =]1, \infty[$ is homotopy equivalent to the one point space $\{*\}$. Clearly $H_1^1(\{*\}) = 0$, so L_p-cohomology is not homotopy invariant (and therefore does not satisfy the Eilenberg-Steenrod axioms of a (co-)homology theory). One can show that $H_1^1(M)$ is not even finitely generated. For each $0 < \varepsilon < 1$ the form $\omega_\varepsilon := t^{-(1+\varepsilon)}dt \in W_1^1(M)$ is L_1-closed, but not exact, since its antiderivative $F_\varepsilon := -\frac{1}{\varepsilon}t^{-\varepsilon} \notin L_1^0(M)$.

2.2 Simplicial L_p-cohomology

We assume the audience to be familiar with the basic notions about simplices and simplicial complexes. Therefore we will briefly recall some selected definitions of particular importance.

2.12 Definition (star-bounded). Let K be a simplicial complex and $S \subset K$ be an arbitrary subset. The *star* of S in K is the set

$$\mathrm{st}(S) := \mathrm{st}_K(S) := \{\sigma \in K \mid \exists \tau \in S : \tau \leq \sigma\}.$$

A simplicial complex K is *star-bounded* N, if the stars of all the simplices in K contain no more than N simplices, i.e.

$$\exists N \in \mathbb{N} : \forall \sigma \in K : \sharp\, \mathrm{st}_K(\sigma) \leq N.$$

2.13 Definition (simplicial homology). Let K be a simplicial complex. Define

$$C_k(K) := \mathbb{R}\langle\{\text{oriented } k\text{-simplices } [\sigma] = [x_0, \ldots, x_k]\}\rangle/\sim,$$

where $[\sigma] \sim -[\sigma]^{-1}$. In other words: We take all the topological simplices in K, choose both possible orientations, take all these oriented simplices, form the free module and then identify. The module $C_k(K)$ is the *k-th simplicial chain group of K*.
The map $d : C_k(K) \to C_{k-1}(K)$ is defined as the linear extension of

$$[\sigma] = [x_{i_0}, \ldots, x_{i_k}] \mapsto \sum_{\nu=0}^{k} (-1)^\nu [x_{i_0}, \ldots, \hat{x}_{i_\nu}, \ldots, x_{i_k}].$$

These groups and maps assemble to a chain complex of R-modules (C_*, d_*). The homology groups

$$H_*(K) := H_*(C_*(K))$$

are the *simplicial homology groups.*

2.14 Definition (simplicial cohomology). Let K be a simplicial complex. We call

$$(C^k(K), d^k) := (\mathrm{Hom}_{\mathbb{R}}(C_k(K), \mathbb{R}), \mathrm{Hom}_{\mathbb{R}}(d_k, \mathbb{R}))$$

the *simplicial cochains of* K. The cohomology groups

$$H^*(K) := H^*(C^*(K))$$

are the *simplicial cohomology groups.*

2.15 Definition (simplicial L_p-cohomology). Let K be a simplicial complex. We call

$$\forall 1 \leq p < \infty : C_p^k(K) := \left\{ c \in C^k(K) \,\middle|\, \|c\|_{C_p^k(K)} := \Big(\sum_{\sigma \in K^{(k)}} |c([\sigma])|^p \Big)^{\frac{1}{p}} < \infty \right\},$$
$$C_\infty^k(K) := \Big(c \in C^k(K) \mid \|c\|_{C_\infty^k(K)} := \sup_{\sigma \in K^{(k)}} |c([\sigma])| < \infty \Big),$$

the *k-th simplicial L_p-cochain group of* K. Let K be a star-bounded simplicial complex. Then we call

$$\mathcal{H}_p^*(K) := H^*(C_p^*(K), d^*)$$

the *simplicial L_p-cohomology of* K. We denote its closed and exact forms by $\mathcal{Z}_p(K)$ respectively $\mathcal{B}_p(K)$. The norm on $C_p^k(K)$ induces a semi-norm on $\mathcal{H}_p^k(K)$. We call

$$\bar{\mathcal{H}}_p^*(K) := \frac{\mathcal{H}_p^*(K)}{\left\{ x \in H_p^*(K) \,\middle|\, \|x\|_{H_p^*(K)} = 0 \right\}}$$

the *reduced simplicial L_p-cohomology.*

2.3 S-Forms

2.16 Definition (S-form). Let K be a star-bounded simplicial complex. For any two simplices $\tau, \sigma \in K$, $\tau \leq \sigma$ consider the inclusion map $j_{\tau,\sigma} : \tau \hookrightarrow \sigma$. A collection of forms

$$\theta := \{\theta(\sigma) \in W_\infty^k(\sigma) \mid \sigma \in K\}$$

such that

$$\forall \tau \leq \sigma \in K : j_{\tau,\sigma}^*(\theta(\sigma)) = \theta(\tau),$$

is a *simplicial differential form of degree k* or just an "*S-form*". The space of all these S-forms of degree k on K is denoted by $S^k(K)$. For any S-form $\theta := \{\theta(\sigma)\}_{\sigma \in K}$ of degree k, the collection $d\theta := \{d\theta(\sigma)\}_{\sigma \in K}$ is an S-form of degree $k+1$. Thus the S-forms assemble to a cochain complex $(S^*(K), d^*)$, the *cochain complex of S-forms on K.*

2.17 Definition (L_p-cohomology of S-forms). Define

$$\forall 1 \le p < \infty : S_p^k(K) := \{\theta \in S_p^k(K) \mid \|\theta\|_{S_p^k(K)}^p := \sum_{\sigma \in K} \|\theta(\sigma)\|_{W_\infty^k(\sigma)}^p < \infty\}$$

$$S_\infty^k(K) := \{\theta \in S_p^k(K) \mid \|\theta\|_{S_\infty^k(K)} := \sup_{\sigma \in K} \|\theta(\sigma)\|_{W_\infty^k(\sigma)} < \infty\}.$$

The spaces

$$\mathscr{H}_p^k(K) := H^k(S_p(K), d)$$

are called *L_p-cohomology of S-forms on K*. The corresponding closed and exact forms are denoted by $\mathscr{Z}_p(K)$ and $\mathscr{B}_p(K)$.

2.18 Lemma. For every $1 \le p \le \infty$ the $S_p^*(K)$ assemble to a cochain complex of Banach spaces.

2.19 Definition (simplicial Riemannian Metric). Let K be a simplicial complex. For any $\sigma \in K$ we think of σ as a smooth manifold with corners. A system of Riemannian metrics

$$g = \{g(\sigma) \in \mathcal{T}^2(\sigma) \mid \sigma \in K\}$$

is a *simplicial Riemannian metric* or just an "*S-metric*", if whenever $\tau \le \sigma$ and $j_{\tau,\sigma} : \tau \hookrightarrow \sigma$ is the inclusion, then $j_{\tau,\sigma}^* g(\sigma) = g(\tau)$.

2.20 Lemma (canonical S-metric). There exists a canonical S-metric g_S on K such that every k-simplex $\sigma \in K$ is isometric to the standard simplex Δ_k. From now on, we will assume that K is endowed with this S-metric.

3 Isomorphism Theorem (Main Result)

3.1 Definition (GKS condition). A smooth triangulation $h : |K| \to M$ *satisfies the Gol'dshtein-Kuz'minov-Shvedov condition* (or just "*is GKS*"), if

(i) The simplicial complex K is star-bounded with star-bound N.

(ii) There are constants $C_1, C_2 > 0$ such that for every simplex $\sigma \in K$ the push-forward of the map $h : |\sigma| \to M$, seen as a smooth map between manifolds with corners, satisfies

$$\sup_{x \in \sigma} \|h_*|_x\| \le C_1. \qquad \sup_{x \in \sigma} \|h_*^{-1}|_{h(x)}\| \le C_2.$$

Here $\|\_\|$ denotes the operator norm, which is induced by the Riemannian metric on M and the S-metric on K (c.f. 2.20).

A Riemannian manifold M *is GKS* if there exists a triangulation $h : |K| \to M$ that is GKS.

Now let us rephrase the Main Theorem from 1.1:

3.2 Main Theorem. Assume that

(i) (M, g) is a smooth oriented Riemannian m-manifold without boundary,

(ii) K is a star-bounded (c.f. 2.12) simplicial complex with S-metric g_S (c.f. 2.20),

(iii) $h : |K| \to M$ is a smooth triangulation that is GKS (c.f. 3.1).

Then there exists a commutative diagram of isomorphisms

$$\begin{array}{ccc} & H_p(M) & \\ \nearrow & & \nwarrow \\ \mathcal{H}_p(K) & \longrightarrow & \mathscr{H}_p(K). \end{array}$$

Therefore all L_p-cohomolgies of M are mutually isomorphic.

4 Applications and Recent Developments

We give a short overview of more recent results concerning L_p-spaces and L_p-cohomology on manifolds.

4.1 Theorem (Kopylov, 2009, [3, Theorem 3.3, 3.4]). Suppose that $1 \leq p, q < \infty$, $\frac{1}{q} - \frac{1}{p} < \frac{1}{n+1}$ and let $M := M_f$ be a surface of revolution as above.

(i) If f is unbounded, then $T^j_{p,q}(M) \neq 0$ for any $1 \leq j \leq n+1$.

(ii) If $T^j_{p,q}(M) = 0$ for any $1 \leq j \leq n+1$, then

$$\lim_{x \to \infty} f(x) = 0 \qquad \text{and} \qquad \operatorname{vol}(M) < \infty.$$

In particular f is bounded.

4.2 Definition (Hodge Laplacian). Let (M, g) be a complete Riemannian manifold with exterior differential d. Denote by d^* the formal L^2-adjoint of d. The operator

$$\Delta := d \circ d^* + d^* \circ d$$

is called *Hodge Laplacian*. In case L_2-norms are taken with respect to some weight function $\sigma = e^{-\phi}$, we denote the corresponding operator by Δ_ϕ. Denote by

$$H_{k,p}(M, \sigma) := \ker \Delta \cap L^k_p(M, \sigma),$$

where $L^k_p(M, \sigma)$ is the L_p*-space with weight function* $\sigma : M \to \mathbb{R}$.

4.3 Theorem (Hodge decomposition, Kodaira, 1949, [2]). The L_2-space over M admits the following orthogonal direct sum decomposition

$$L^k_2(M) = H_{k,2}(M) \oplus \overline{d\Omega^k_c(M)} \oplus \overline{d^*\Omega^k_c(M)}.$$

4.4 Theorem (Hodge decomposition, non-compact case, Li, 2009, [4, Theorem 2.1]). Let (M, g) be a complete Riemannian manifold, $\phi \in \mathcal{C}^2(M)$, $\sigma := e^{-\phi}$, $p > 1$, $q := \frac{p}{p-1}$. Suppose that the *Riesz transforms* $d(\Delta^k_\phi)^{-\frac{1}{2}}$, $d^*(\Delta^k_\phi)^{-\frac{1}{2}}$ are bounded in L_p and L_q and the *Riesz potential* $(\Delta^k_\phi)^{-\frac{1}{2}}$ is bounded in L_p. Then the Strong L^p-Hodge direct sum decomposition holds:

$$L^k_p(M, \sigma) = H_{k,p}(M, \sigma) \oplus d\mathcal{W}^{k-1}_p(M, \sigma) \oplus d^*_\phi \mathcal{W}^{k+1}_p(M, \sigma)$$

(Warning: The definition of $\mathcal{W}^k_p(M, \sigma)$ is slightly different than $W^k_p(M)$.)

4.5 Theorem (analytic Poincaré duality). Let M be a smooth compact oriented manifold of dimension m. The bilinear pairing $\beta : H^k_{\mathrm{dR}}(M) \times H^{m-k}_{\mathrm{dR}}(M) :\to \mathbb{R}$,

$$([\omega], [\eta]) \mapsto \int_M \omega \wedge \eta$$

is well-defined and regular. The map $\Psi : H^k_{\mathrm{dR}}(M) \to (H^{m-k}_{\mathrm{dR}})^*$, $[\omega] \mapsto ([\eta] \mapsto \beta([\omega], [\eta])$, is an isomorphism.

4.6 Theorem (Poincaré duality, Pansu, 2008, [5, Lemma 13]). Let M be a complete oriented Riemannian manifold of dimension m. Let $p > 1$ and let q be Hölder conjugate to p. Let $\omega \in L^k_p(M)$. Then the following hold:

(i) $0 \neq [\omega] \in \bar{H}_p^k(M)$ if and only if there exists $\eta \in L_q^{m-k}(M)$ such that

$$\int_M \omega \wedge \eta \neq 0.$$

(ii) $0 \neq [\omega] \in H_p^k(M)$ if and only if there exists a sequence $\eta_j \in L_q^{m-k}(M)$ such that

$$\int_M \omega \wedge \eta_j \geq 1 \qquad \text{and} \qquad \|d\eta_j\|_{L_q(M)} \to 0.$$

(iii) As a consequence, we obtain

$$\bar{H}_p^k(M) = 0 \iff \bar{H}_q^{m-k}(M) = 0, \qquad T_p^k(M) = 0 \iff T_q^{m-k}(M) = 0.$$

4.7 Definition (Hölder pairing). Assume $1 \leq p, q \leq \infty$ are Hölder conjugate. The pairing $\beta : L_q^{m-k}(M) \times L_p^k(M) \to \mathbb{R}$,

$$(\omega, \eta) \mapsto \int_M \omega \wedge \eta$$

is called the *Hölder pairing of* M.

4.8 Theorem (Hölder duality for $L_p(M)$). For any Hölder conjugate $1 \leq p, q < \infty$ and $0 \leq k \leq m$, the Hölder pairing $\beta : L_q^{m-k}(M) \times L_p^k(M) \to \mathbb{R}$ is regular and the map $\Psi : L_q^{m-k}(M) \to L_p^k(M)^*$, $\omega \mapsto (\eta \mapsto \beta(\omega, \eta))$, is an isomorphism.

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