DIPLOMARBEIT

Der de Rham-Isomorphismus und die L_p-Kohomologie nicht-kompakter Riemannscher Mannigfaltigkeiten (The de Rham Isomorphism and the L_p-Cohomology of non-compact Riemannian Manifolds)

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Von

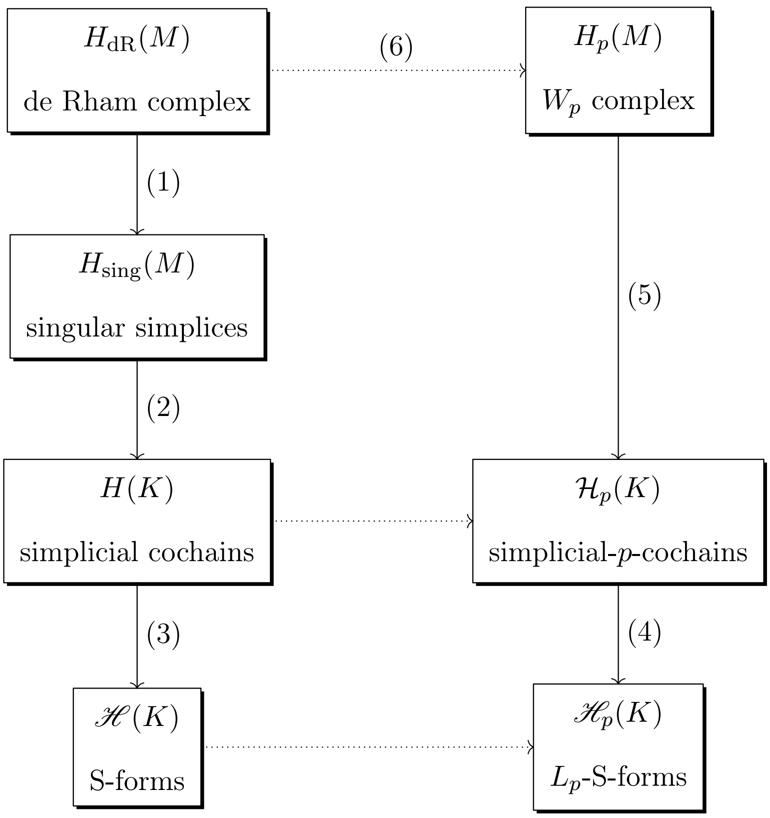
Nikolai Nowaczyk

Aus
Duisburg

Contents

Einleitung

Die vorliegende Arbeit behandelt die L_p-Kohomologie nicht-kompakter Riemannscher Mannigfaltigkeiten. Wir wollen die drei verschiedenen Ansätze L_p-Kohomologie zu definieren ausführlich behandeln und dann zeigen, dass alle drei isomorph sind. Bevor wir damit beginnen, ist es hilfreich, sich zunächst einen Überblick über alle relevanten Kohomologietheorien zu verschaffen. Es sei M eine glatte orientierte Riemannsche Mannigfaltigkeit ohne Rand der Dimension m, K ein Simplizialkomplex in einem $\mathbb{R}^n$ (siehe 2.2.6), $h : |K| \to M$ eine glatte Triangulierung und $1 \leq p \leq \infty$. Wir fügen in den folgenden Graphen einen Knoten für jede Kohomologietheorie ein und einen Pfeil zwischen zwei Knoten, falls es (unter gewissen sinnvollen technischen Voraussetzungen) einen Isomorphismus zwischen ihnen gibt.



(0.1)

Wir gehen davon aus, dass der Leser mit den klassischen Kohomologietheorien auf der linken Seite vertraut ist, insbesondere mit der de Rham Kohomologie glatter Mannigfaltigkeiten $H_{\mathrm{dR}}(M)$ (siehe [16, 15]), der singulären Kohomologie topologischer Räume $H_{\mathrm{sing}}(M)$ (siehe [25, 5.4]) und der simplizialen Homologie ([15, 13]) und Kohomologie von Simplizialkomplexen $H(K)$ (siehe Definition 2.2.22). Die Kohomologie von S-Formen und die L_p-Kohomologietheorien auf der rechten Seite werden in Kapitel 2 eingeführt.

Pfeil (1) ist der klassische *de Rham Isomorphismus*, welche von de Rham im Jahre 1931 konstruiert wurde. Er ist Gegenstand des *Satzes von de Rham*, der entweder mit Methoden der Garbentheorie (wie zum Beispiel in [29, 5]) oder mit etwas elementareren Methoden (siehe [16, 16]) bewiesen werden kann. Wir führen eine modifizierte Version des de Rham Isomorphismus in Kapitel 3.1 ein, welche für Anwendungen auf die L_p-Kohomologie beson-

ders geeignet ist.
Pfeil (2) ist ein klassischer Satz der algebraischen Topologie. Ein Beweis findet sich zum Beispiel in [15, 13].
Pfeil (3) ist Gegenstand eines sehr viel unbekannteren Satzes, welcher in [26] diskutiert wird. Wir notieren den Pfeil hier nur aus Gründen der Vollständigkeit.

Unser Hauptinteresse liegt auf Pfeilen (4) und (5). In Kapitel 2 werden wir alle nötigen Begriffe behandeln, die notwendig sind, um die L_p-Kohomologien $H_p(M)$, $\mathcal{H}_p(K)$, $\mathscr{H}_p(K)$ einzuführen und bereits einige ihrer grundlegenden Eigenschaften studieren. Hauptziel ist es zu zeigen, dass (unter einigen technischen Voraussetzungen) die Pfeile (4) und (5) existieren und Isomorphismen sind. Letztendlich erhalten wir das kommutative Diagramm

$$\begin{array}{ccc} & H_p(M) & \\ \nearrow & & \nwarrow^{(5)} \\ \mathcal{H}_p(K) & \xrightarrow{(4)} & \mathscr{H}_p(K). \end{array}$$

und der gestrichelte Pfeil liefert einen Isomorphismus, den man als Verallgemeinerung des klassischen de Rham Isomorphismus für die L_p-Kohomologie ansehen kann. Seine Existenz ist das Hauptresultat dieser Arbeit, dessen Beweis folglich in zwei Schritten besteht, der Existenz von Pfeil (4) und (5).
Um die Existenz des Isomorphismus (4) zu beweisen, werden wir eine Ko-Kettenabbildung w einführen, die *Whitney-Transformation.* Eine detaillierte Untersuchung dieser Abbildung in Abschnitt 3.2 wird ergeben, dass eine Modifikation des klassischen de Rham Isomorphismus ein Inverses zu (4) liefert. Die Details sind recht technisch, das Resultat ist Gegenstand von Haupsatz 3.2.8:

> *Sei K ein sternbeschränkter Simplizialkomplex und $L \subset K$ ein Unterkomplex. Für jedes $1 \leq p \leq \infty$ existieren wohldefinierte Ko-Kettenabbildungen*
>
> $$w : C_p^*(K, L) \rightleftarrows S_p^*(K, L) : I,$$
>
> *die zueinander inverse topologische Isomorphismen in der Kohomologie induzieren.*

Um den Isomorphismus (5) herleiten zu können, benötigen wir *Regularisierungsoperatoren* auf Mannigfaltigkeiten, die in Kapitel 5 detailliert behandelt werden. Einige Resultate lassen sich besonders klar mit Hilfe von *Strömen* formulieren. Daher werden wir in Kapitel 4 zunächst eine kurze Einführung in die Theorie der Ströme geben. Die beiden Kapitel 4 und 5 sind als Vorbereitung gedacht. Ihr Inhalt ist mehr oder weniger unabhängig vom Rest der Arbeit und die darin genannten Resultate sind auch für sich genommen nützlich. Wir werden die Regularisierungsoperatoren hauptsächlich anwenden, um Haupsatz 6.2.1 in Kapitel 6 beweisen zu können. Daraus folgt die Existenz von Pfeil (5) als Korollar 6.2.2:

> *Falls $h : |K| \to M$ die GKS-Bedingung (siehe Definition 6.1.2) erfüllt, dann induziert die Komposition*
>
> $$S_p(K) \longrightarrow S_p(M) \hookrightarrow W_p(M)$$
>
> *einen topologischen Isomorphismus $\mathscr{H}_p(K) \to H_p(M)$.*

Pfeil (6) existiert im Allgemeinen nicht, d.h. es gibt keinen Isomorphismus zwischen den Kohomologietheorien auf der linken und der rechten Seite des Diagramms (0.1). Falls wir uns allerdings auf kompakte Mannigfaltigkeiten beschränken, dann können die in Kapitel 5 entwickelten Regularisierungsoperatoren auch verwendet werden, um zu zeigen, dass in diesem Falle die L_p-Kohomologie mit der klassischen de Rham Kohomologie übereinstimmt (siehe 6.1.13).

Die gesamte Arbeit orientiert sich im Wesentlichen am Artikel "De Rham isomorphism of the L_p-cohomology of noncompact Riemannian manifolds" von by Gol'shtein, Kuz'minov und Shvedov, [5]. Obwohl dieser Artikel nur acht Seiten lang ist, skizziert er viele der grundlegenden Ideen. Leider fehlen zum einen viele der Voraussetzungen, die nötig sind, um die ganzen Begriffe und Sätze überhaupt formulieren zu können, und zum anderen werden viele technische Details ausgelassen. Es ist das Anliegen dieser Arbeit, eine möglichst detaillierte und in sich abgeschlossene Behandlung des Themas zu liefern.

Danksagungen. Ich möchte meinen zahlreichen Unterstützern danken, ohne die diese Arbeit nicht zustande gekommen wäre. Zunächst einmal Professor Dr. Werner Ballmann, meinem Diplomvater, der mich auf das Thema erst aufmerksam gemacht hat, sowie auch seiner gesamten Arbeitsgruppe, insbesondere Dr. Jan Swoboda. Er stand mir während der ganzen Zeit kompetent zur Seite, beantwortete geduldig meine zahllosen Fragen und inspirierte mich oft zu neuen Lösungswegen, wenn ich mal irgendwo stecken geblieben war. Danke auch an Professor Dr. Matthias Lesch für die Zweitkorrektur dieser Arbeit. Ich wurde während meines ganzen Studiums hier in Bonn finanziell und ideell gefördert von der Studienstiftung des deutschen Volkes. Sie hat mich insbesondere durch die Sommerakademien an entscheidender Stelle intellektuell und menschlich beflügelt. Schon mein ganzes Leben lang gefördert werde ich von meinen Eltern. Ohne die Gewissheit, dass sie hinter mir stehen, hätte ich das Mathematikstudium gar nicht erst beginnen können. Last but not least danke ich Jesko Hüttenhain für die tiefe Freundschaft, die sich zwischen uns während des gemeinsamen Studiums entwickelt hat.

Nikolai Nowaczyk,
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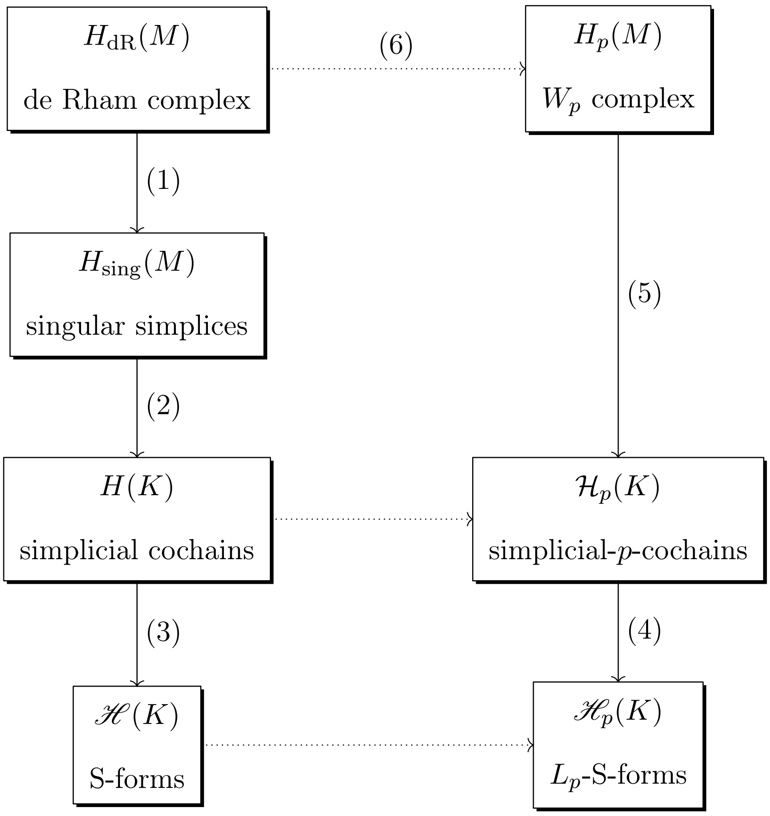
Preface

This thesis discusses the L_p-cohomology of noncompact Riemannian manifolds, in particular the three different approaches to define it and establishes isomorphisms between all of them. Before we start let us have an overview of all the various cohomology theories involved. Let M be a smooth oriented Riemannian manifold without boundary of dimension m, K be a simplicial complex in some $\mathbb{R}^n$ (see 2.2.6), $h : |K| \to M$ be a smooth triangulation and $1 \leq p \leq \infty$. In the following graph we will add a node for every cohomology theory of interest and an edge, if under certain reasonable assumptions there exists an isomorphism between them.



(0.2)

We assume the reader to be familiar with the classical cohomology theories on the left hand side, namely the de Rham cohomology of smooth manifolds $H_{\mathrm{dR}}(M)$ (c.f. [16, 15]), the singular cohomology of topological spaces $H_{\mathrm{sing}}(M)$ (c.f. [25, 5.4]), and the simplicial homolology ([15, 13]) and cohomology of simplicial complexes $H(K)$ (c.f. Definition 2.2.22). The cohomology of S-forms and the L_p-cohomology theories on the right hand side will be introduced in section 2.
Arrow (1) is the *de Rham isomorphism* and was established de Rham in 1931. It can be proven using sheaf theory (as done in [29, 5]) or by more elementary means (see for instance [16, 16]). We will give a self-contained introduction to a modified version the de Rham homomorphism in subsection 3.1 suited for L_p-cohomology theory.
Arrow (2) is a classical theorem from algebraic topology. A proof can be found in [15, 13].
Arrow (3) is a far less popular theorem discussed in [26]. We just enlist it here for reasons of completion.

Arrows (4) and (5) will be of our primary interest. Section 2 systematically introduces all the required definitions for the L_p-cohomology theories $H_p(M)$, $\mathcal{H}_p(K)$, $\mathscr{H}_p(K)$. Our ultimate goal is to prove that (under certain technical assumptions) the arrows (4) (see section 3) and (5) (see section 6) exist and are isomorphisms. In the end, the following diagram will commute

$$\begin{array}{ccccc} & & H_p(M) & & \\ & \nearrow & & \nwarrow^{(5)} & \\ \mathcal{H}_p(K) & & \xrightarrow{\quad (4) \quad} & & \mathscr{H}_p(K). \end{array}$$

and the dotted arrow represents an isomorphism one may consider a generalization of the classical de Rham isomorphism in the L_p-case. Its existence is the main result of this thesis.
The key idea to prove the existence of isomorphism (4) is to introduce a cochain map w, called *Whitney transformation*. A detailed study of this map in section 3.2 will reveal that the classical de Rham isomorphism may be modified slightly to a map $[I]$, which turns out to be the inverse of (4). The details are rather technical, the result the content of Main Theorem 3.2.8.
In order to be able to establish isomorphism (5) we require the notion of regularization operators on manifolds, which will be discussed in section 5 in detail. It will turn out to be nice to have the notion of currents available there. Therefore we will give a short introduction to currents in section 4 first. Both sections 4 and 5 are of a preparatory nature, their content is more or less independent of the rest of the thesis and may be useful for other purposes as well. We will utilize them to prove the existence of arrow (5) in section 6. The precise application of the regularization operators is the content of Main Theorem 6.2.1, which gives the desired result as an immediate Corollary 6.2.2.

In general arrow (6) does not exist, i.e. there is no isomorphism between the cohomology theories on the left hand side and on the right hand side of the diagram (0.2). However, if we restrict our attention to compact manifolds, the tools developed in section 5 may also be utilized to show that in this case L_p-cohomology and de Rham cohomology coincide (see 6.1.13).

The entire thesis is roughly based on the article "De Rham isomorphism of the L_p-cohomology of noncompact Riemannian manifolds" written by Gol'shtein, Kuz'minov and Shvedov, [5]. This article is only eight pages long, but sketches all the fundamental ideas we are going to present. Unfortunately many of the prerequisites, which are necessary to formulate all the notions and theorems are missing there as well as much of the technical details. It is the aim of this thesis to present a more detailled and self-contained discussion of the topic.

Acknowledgements. I would like to thank my many supporters whithout whom this thesis would not have been possible. First of all Professor Dr. Werner Ballman, my thesis adviser, who brought this topic to my attention, and his entire staff, in particular Dr. Jan Swoboda. He was at hand with competent help and advice, patiently answered my many questions and often inspired me to new ideas when I was stuck somewhere. Thanks to Professor Dr. Matthias Lesch too for co-correcting this thesis. During my entire studies here in Bonn I was financially and ideationally supported by the Studienstiftung des deutschen

Volkes. It inspired me in particular through its summer academies. During my entire life I was supported by my parents in countless ways. Without the certainty of their backup I could not have even stared my study of math. Last but least I would like to say thanks to Jesko Hüttenhain for the deep friendship that developed over our common studies.

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1 Tensor metrics

In this section we consider a Riemannian m-manifold (M,g) and investigate how to extend the Riemannian metric g to the various tensor bundles over M. For many theorems it will suffice to consider an m-dimensional inner product space (V,g) over $\mathbb{R}$ and then apply the result to all the (T_pM,g), $p \in M$. The theory for V is basically a nasty exercise in multi-linear algebra.
First, we fix some notation, since there are many slightly different conventions common in the literature. We will adopt most of these conventions from [17].

1.0.1 Definition. A (k,l)-*tensor* F on V is a $(k+l)$-fold multilinear map

$$(V^*)^l \times V^k \to \mathbb{R}.$$

We say F is k-fold *covariant* and l-fold *contravariant*. The space of all tensors of type (k,l) on V is denoted by

$$T^k_l V.$$

If M is a manifold, the set $T^k_l M$, defined by

$$T^k_l M := \coprod_{p \in M} T^k_l T_p M,$$

is the *tensor bundle over* M. It has a canonical topology and smooth structure such that $T^k_l M$ is a smooth vector bundle over M. Its smooth sections are denoted by

$$\mathcal{T}^k_l(M)$$

and are called *tensor fields* on M. In particular $\mathcal{T}(M) := \mathcal{T}^0_1(M)$ are the *vector fields* and $\mathcal{T}^*(M) := \mathcal{T}^1_0(M)$ are the *covector fields* on M.
We denote by $\Lambda^k(V)$ the set of all alternating tensors $F \in T^k_0(V)$. The corresponding bundle is denoted by

$$\Omega^k M := \coprod_{p \in M} \Lambda^k T_p M$$

and its smooth sections are called *differential forms of degree* k. The space of sections respectively compactly supported sections is denoted by

$$\Omega^k(M), \qquad \Omega^k_c(M).$$

The spaces

$$\Lambda(V) := \bigoplus_{k \in \mathbb{N}} \Lambda^k(V), \qquad \Omega(M) := \bigoplus_{k \in \mathbb{N}} \Omega^k(V)$$

are the *exterior algebra of* V *respectively* M.
Warning: Some authours consider our $\Lambda^k(V)$ as $\Lambda^k(V^*)$.

1.1 Reminder of musical operations

The metric can be extended very easily to the dual space V^* (sometimes we also denote the dual space by V') using musical operations (see also [17, 3]).

1.1.1 Definition (flat operator). The operator $\flat : TM \to T^*M$, given pointwise by $T_pM \to T_p^*M$, $X \mapsto X^\flat$, where $X^\flat : T_pM \to \mathbb{R}$, $Y \mapsto g(X,Y)$ is called *flat operator*.

1.1.2 Lemma (properties of $\flat$). Let φ be any chart for M. Denote by g_{ij} the coordinate matrix of g with respect to $\partial\varphi_i$ and by g^{ij} its inverse. The flat operator has the following properties:

(i) Locally, for any $X = X^i\partial\varphi_i \in \mathcal{T}(M)$, the flat operator may be calculated by

$$X^\flat = g_{ij}X^i d\varphi^j =: X_j d\varphi^j.$$

(ii) The coordinate matrix of $\flat$ with respect to $(\partial\varphi_1, \ldots, \partial\varphi_m)$ and $(d\varphi^1, \ldots d\varphi^n)$ is the matrix g_{ij} itself.

(iii) The flat operator is a diffeomorphism and its inverse $\sharp : T^*M \to TM$, the *sharp operator*, has local coordinate matrix g^{ij}.

(iv) Locally, for any $\omega = \omega_i d\varphi^i$

$$\omega^\sharp = g^{ij}\omega_j\partial\varphi_i.$$

Proof.

(i) Take any $Y = Y^j\partial\varphi_j$ and calculate

$$X^\flat(Y) = g(X,Y) = g_{ij}X^iY^j = g_{ij}X^i d\varphi^j(Y).$$

(ii) If $X = \partial\varphi_k$, then $X^i = \delta^{ki}$ and thus

$$(\partial\varphi_k)^\flat = g_{ij}\delta^{ki}d\varphi^j = g_{kj}d\varphi^j = g_{jk}d\varphi^j.$$

(iii) By hypothesis, the matrix g_{ij} is invertible.

(iv) We obtain

$$(g^{ik}\partial\varphi_k)^\flat = g^{ik}(\partial\varphi_k)^\flat = g^{ik}g_{kj}d\varphi^j = \delta_{ij}d\varphi^j = d\varphi^i$$

and thus

$$\omega^\sharp = \omega_i(d\varphi^i)^\sharp = \omega_i g^{ik}\partial\varphi_k = g^{ki}\omega_i\partial\varphi_k.$$ □

1.2 Main results

1.2.1 Theorem (tensor metric). Let (V, g) be an m-dimensional vector space over $\mathbb{R}$ and g be a inner product on V. There is a unique inner product $\langle\_,\_\rangle$ on each tensor bundle $T^k_l(V)$ with the property that whenever $E = (E_1, \ldots, E_m)$ is an orthonormal basis for V and $E^* = (E^1, \ldots, E^m)$ is the corresponding dual basis, then

$$T^k_l E := \{E_{j_1} \otimes \ldots \otimes E_{j_l} \otimes E^{i_1} \otimes \ldots \otimes E^{i_l} \mid \forall 1 \leq \nu \leq l : \forall 1 \leq \mu \leq k : j_\nu, i_\mu \in \{1, \ldots m\}\}$$

is an orthonormal basis for $T^k_l(V)$. Furthermore, this inner product satisfies the following properties:

(i) With respect to any other basis $(B_1, \dots B_m)$ this inner product is given by

$$\forall F, G \in T^k_l(V) : \langle F, G\rangle = g^{i_1 r_1} \dots g^{i_k r_k} g_{j_1 s_1} \dots g_{j_l s_l} F^{j_1 \dots j_l}_{i_1 \dots i_k} G^{s_1 \dots s_l}_{r_1 \dots r_k},$$

but is itself independent of the choice of basis.

(ii) If $F, G \in T^k_l(V)$, $F', G' \in T^{k'}_{l'}(V)$, there is a factorization

$$\langle F \otimes F', G \otimes G'\rangle = \langle F, G\rangle\langle F', G'\rangle.$$

(iii) The sharp operator $\sharp : T^*M \to TM$ is an isometry.

Furthermore this induces a inner product on $\bigoplus_{k,l\in\mathbb{N}} T^k_l(V)$ by declaring the summands to be mutually orthogonal.
Analogously, if (M, g) is a Riemannian manifold, this defines a fibre metric on all the $T^k_l M$ with analogous properties.

Proof. Let us call the defining property of $\langle \_, \_\rangle$ the (ONB)-property. Throughout the proof we will denote coordinates with respect to B without tildes and coordinates with respect to E with tildes.

Step 1 (uniqueness): First we will show that (ONB) implies all the other properties. In particular it will follow from (i) that such a inner product is unique. We will then use (i) to define it.

Step 1.1 ($\sharp$ is an isometry): Notice that by 1.1.2

$$(E^k)^\sharp = \tilde{g}^{ij}(E^k)_j E_i = \delta^{ij}\delta^k_j E_i = E_k.$$

This immediately implies for every covariant 1-tensors ω, η

$$\begin{aligned} \langle \omega^\sharp, \eta^\sharp\rangle &= \langle (\omega_i E^i)^\sharp, (\eta_j E^j)^\sharp\rangle = \omega_i\eta_j\langle (E^i)^\sharp, (E^j)^\sharp\rangle \\ &= \omega_i\eta_j\langle E_i, E_j\rangle = \omega_i\eta_j\delta^{ij} = \omega_i\eta_j\langle E^i, E^j\rangle = \langle \omega, \eta\rangle. \end{aligned} \tag{1.1}$$

Step 1.2 (factorization property and coordinate representation): First we calculate for any $1 \le \nu, \mu \le m$

$$\begin{aligned} &\langle B^\nu, B^\mu\rangle \overset{(1.1)}{=} \langle (B^\nu)^\sharp, (B^\mu)^\sharp\rangle \overset{1.1.2}{=} \langle g^{i_1 j_1}\delta_{\nu j_1} B_{i_1}, g^{i_2 j_2}\delta_{\mu j_2} B_{i_2}\rangle = g^{i_1 j_1}\delta_{\nu j_1} g^{i_2 j_2}\delta_{\mu j_2}\langle B_{i_1}, B_{i_2}\rangle \\ &= g^{i_1\nu} g^{i_2\mu} g_{i_1 i_2} = g^{i_2\mu}(g^{\nu i_1} g_{i_1 i_2}) = g^{i_2\mu}\delta_{\nu i_2} = g^{\nu\mu}. \end{aligned}$$

Thus $(g^{\nu\mu})$ is the coordinate matrix of $\langle \_, \_\rangle$ w.r.t. $(B^1, \dots, B^m)$ on V^*. Furthermore there are constants a^k_ν, b^ν_k such that

$$B^k = a^k_\nu E^\nu, \qquad B^k(E_l) = a^k_l, \qquad B_k = b^\nu_k E_\nu, \qquad B_k(E^l) = b^l_k.$$

Having this in mind, we calculate on the one hand

$$\begin{aligned} &\langle B_{j_1} \otimes \dots \otimes B_{j_l} \otimes B^{i_1} \otimes \dots B^{i_k}, B_{s_1} \otimes \dots \otimes B_{s_l} \otimes B^{r_1} \otimes \dots B^{r_k}\rangle \\ &= \langle b^{\nu_1}_{j_1} E_{\nu_1} \otimes \dots \otimes b^{\nu_l}_{j_l} E_{\nu_l} \otimes a^{i_1}_{\lambda_1} E^{\lambda_1} \otimes \dots \otimes a^{i_k}_{\lambda_k} E^{\lambda_k}, b^{\mu_1}_{s_1} E_{\mu_1} \otimes \dots \otimes b^{\mu_l}_{s_l} E_{\mu_l} \otimes a^{r_1}_{\eta_1} E^{\eta_1} \otimes \dots a^{r_k}_{\eta_k} E^{\eta_k}\rangle \\ &\overset{(ONB)}{=} b^{\nu_1}_{j_1} \dots b^{\nu_l}_{j_l} a^{i_1}_{\lambda_1} \dots a^{i_k}_{\lambda_k} b^{\mu_1}_{s_1} \dots b^{\mu_l}_{s_l} a^{r_1}_{\eta_1} \dots a^{r_k}_{\eta_k} \delta_{\nu_1\mu_1} \dots \delta_{\nu_l\mu_l} \delta^{\lambda_1\eta_1} \dots \delta^{\lambda_k\eta_k} \\ &= \sum_{\nu_1,\dots,\nu_l,\lambda_1,\dots\lambda_k} b^{\nu_1}_{j_1} \dots b^{\nu_l}_{j_l} a^{i_1}_{\lambda_1} \dots a^{i_k}_{\lambda_k} b^{\nu_1}_{s_1} \dots b^{\nu_l}_{s_l} a^{r_1}_{\lambda_1} \dots a^{r_k}_{\lambda_k} \\ &= \sum_{\nu_1,\dots,\nu_l,\lambda_1,\dots\lambda_k} b^{\nu_1}_{j_1} b^{\nu_1}_{s_1} \dots b^{\nu_l}_{j_l} b^{\nu_l}_{s_l} a^{i_1}_{\lambda_1} a^{r_1}_{\lambda_1} \dots a^{i_k}_{\lambda_k} a^{r_k}_{\lambda_k} \end{aligned}$$

and on the other hand

$$\begin{aligned}
&\langle B_{j_1}, B_{s_1}\rangle \dots \langle B_{j_l}, B_{s_l}\rangle \langle B^{i_1}, B^{r_1}\rangle \dots \langle B^{i_k}, B^{r_k}\rangle \\
=&\langle b^{\nu_1}_{j_1} E_{\nu_1}, b^{\mu_1}_{s_1} E_{\mu_1}\rangle \dots \langle b^{\nu_l}_{j_l} E_{\nu_l}, b^{\mu_l}_{s_l} E_{\mu_l}\rangle \langle a^{i_1}_{\lambda_1} E^{\lambda_1}, a^{r_1}_{\eta_1} E^{\eta_1}\rangle \dots \langle a^{i_k}_{\lambda_k} E^{\lambda_k}, a^{r_k}_{\eta_k} E^{\eta_k}\rangle \\
=&b^{\nu_1}_{j_1} b^{\mu_1}_{s_1} \dots b^{\nu_l}_{j_l} b^{\mu_l}_{s_l} a^{i_1}_{\lambda_1} a^{r_1}_{\eta_1} \dots a^{i_k}_{\lambda_k} a^{r_k}_{\eta_k} \langle E_{\nu_1}, E_{\mu_1}\rangle \dots \langle E_{\nu_l}, E_{\mu_l}\rangle \langle E^{\lambda_1}, E^{\eta_1}\rangle \dots \langle E^{\lambda_k}, E^{\eta_k}\rangle \\
\overset{\text{(ONB)}}{=}& b^{\nu_1}_{j_1} b^{\mu_1}_{s_1} \dots b^{\nu_l}_{j_l} b^{\mu_l}_{s_l} a^{i_1}_{\lambda_1} a^{r_1}_{\eta_1} \dots a^{i_k}_{\lambda_k} a^{r_k}_{\eta_k} \delta_{\nu_1\mu_1} \dots \delta_{\nu_l\mu_l} \delta^{\lambda_1\eta_1} \dots \delta^{\lambda_k\eta_k} \\
=& \sum_{\nu_1 \dots \nu_l, \lambda_1, \dots \lambda_k} b^{\nu_1}_{j_1} b^{\nu_1}_{s_1} \dots b^{\nu_l}_{j_l} b^{\nu_l}_{s_l} a^{i_1}_{\lambda_1} a^{r_1}_{\lambda_1} \dots a^{i_k}_{\lambda_k} a^{r_k}_{\lambda_k}.
\end{aligned}$$

These expressions agree and therefore we have proven

$$\begin{aligned}
&\langle B_{j_1} \otimes \dots \otimes B_{j_l} \otimes B^{i_1} \otimes \dots B^{i_k}, B_{s_1} \otimes \dots \otimes B_{s_l} \otimes B^{r_1} \otimes \dots B^{r_k}\rangle \\
&= \langle B_{j_1}, B_{s_1}\rangle \dots \langle B_{j_l}, B_{s_l}\rangle \langle B^{i_1}, B^{r_1}\rangle \dots \langle B^{i_k}, B^{r_k}\rangle \\
&= g_{j_1 s_1} \dots g_{j_l s_l} g^{i_1 r_1} \dots g^{i_k r_k}.
\end{aligned}$$

This finally implies that the metric has the desired form and satisfies the factorization property. Notice that the expression we have derived so far, proves uniqueness, since it no longer depends on the fibre metric in $T^k_l M$, but only on the initial inner product g and the basis B.

Step 2 (existence): To show existence, we would like to define the inner product by (i). The expression is obviously bilinear and symmetric. To see that it is positive definit assume $F = G \in T^k_l B$, which implies $F^{j_1 \dots j_l}_{i_1 \dots i_k} = \delta^{j_1\nu_1} \dots \delta^{j_l\nu_l} \delta_{i_1\mu_1} \dots \delta_{i_k\mu_k}$ for some $1 \leq \nu_1, \dots, \nu_l, \mu_1, \dots, \mu_k \leq m$. This implies

$$\begin{aligned}
\langle F, F\rangle &= g^{i_1 r_1} \dots g^{i_k r_k} g_{j_1 s_1} \dots g_{j_l s_l} F^{j_1 \dots j_l}_{i_1 \dots i_k} F^{s_1 \dots s_l}_{r_1, \dots r_k} \\
&= g^{i_1 r_1} \dots g^{i_k r_k} g_{j_1 s_1} \dots g_{j_l s_l} \delta^{j_1\nu_1} \dots \delta^{j_l\nu_l} \delta_{i_1\mu_1} \dots \delta_{i_k\mu_k} \delta^{s_1\nu_1} \dots \delta^{s_l\nu_l} \delta_{r_1\mu_1} \dots \delta_{r_k\mu_k} \\
&= g^{\mu_1\mu_1} \dots g^{\mu_k\mu_k} g_{\nu_1\nu_1} \dots g_{\nu_l\nu_l} > 0,
\end{aligned}$$

since the diagonal entries of a positive definit matrix are always positive. Therefore $\langle \_, \_ \rangle$ is positive definit on a basis, thus on the entire space. In addition if $B = E$ is an ONB and $F, G \in T^k_l E$ such that analogously

$$\tilde{F}^{j_1 \dots j_l}_{i_1 \dots i_k} = \delta^{j_1\nu_1} \dots \delta^{j_l\nu_l} \delta_{i_1\mu_1} \dots \delta_{i_k\mu_k}, \qquad \tilde{G}^{s_1 \dots s_l}_{r_1 \dots r_k} = \delta^{s_1\alpha_1} \dots \delta^{s_l\alpha_l} \delta_{r_1\beta_1} \dots \delta_{r_k\beta_k},$$

we obtain

$$\begin{aligned}
\langle F, G\rangle &= \tilde{g}^{i_1 r_1} \dots \tilde{g}^{i_k r_k} \tilde{g}_{j_1 s_1} \dots \tilde{g}_{j_l s_l} \tilde{F}^{j_1 \dots j_l}_{i_1 \dots i_k} \tilde{G}^{s_1 \dots s_l}_{r_1, \dots r_k} \\
&= \delta^{i_1 r_1} \dots \delta^{i_k r_k} \delta_{j_1 s_1} \dots \delta_{j_l s_l} \delta^{j_1\nu_1} \dots \delta^{j_l\nu_l} \delta_{i_1\mu_1} \dots \delta_{i_k\mu_k} \delta^{s_1\alpha_1} \dots \delta^{s_l\alpha_l} \delta_{r_1\beta_1} \dots \delta_{r_k\beta_k} \\
&= \delta^{\mu_1\beta_1} \dots \delta^{\mu_k\beta_k} \delta_{\nu_1\alpha_1} \dots \delta_{\nu_l\alpha_l},
\end{aligned}$$

which is precisely the (ONB)-property in coordinates.
In case of a vector space V we are done. A manifold may be covered by coordinate domains, on which we may use (i) to define the metric. By uniqueness they must agree wherever the domains overlap and the entire construction does not depend on the choice of charts. □

This rather complicated product has a simpler form when applied to alternating tensors and wedge products. Since there are different conventions regarding the wedge product in the literature, we introduce it here for later reference. Our conventions agree with [16, 12].

1.2.2 Definition (alternator, wedge product). The map $\mathrm{Alt} : T^k(V) \to T^k(V)$ defined by

$$\mathrm{Alt}(T) := \frac{1}{k!} \sum_{\pi \in \mathfrak{S}_k} \mathrm{sgn}(\pi)^{\pi} T$$

is the *alternator*. Here $\mathfrak{S}_k$ denotes the symmetric group of k-permutations and the tensor ${}^{\pi}T$ is given by its action on vectors by ${}^{\pi}T(X_1, \ldots, X_k) := T(X_{\pi(1)}, \ldots X_{\pi(k)})$. The image $\mathrm{im}\,\mathrm{Alt}\left(T^k(V)\right) = \Lambda^k(V)$ is precisely the set of alternating tensors on V.
The map $\wedge : T^k(V) \times T^l(V) \to \Lambda^{k+l}(V)$

$$(\omega, \eta) \mapsto \frac{(k+l)!}{k!l!} \mathrm{Alt}(\omega \otimes \eta)$$

is the *wedge product*.

1.2.3 Corollary (metric of alternating tensors). The metric $\langle \_, \_ \rangle$ from Theorem 1.2.1 satisfies

$$\langle v^1 \wedge \ldots \wedge v^k, w^1 \wedge \ldots \wedge w^k \rangle = k! \det(\langle v_i, w_j \rangle)$$

on arbitrary decomposable alternating tensors $v^1 \wedge \ldots \wedge v^k, w_1 \wedge \ldots \wedge w_k \in \Lambda^k(V)$.

Proof. Let $b_1 \ldots, b_n \in V$ be an ONB. Then by 1.2.1 $b^1, \ldots, b^n \in V^*$ is an ONB for V^*. Consider two tensors of the form $b^{i_1} \wedge \ldots \wedge b^{i_k}, b^{j_1} \wedge \ldots \wedge b^{j_k} \in \Lambda^k(V)$, where the $i_1, \ldots, i_k$ are all mutally distinct as well as the $j_1, \ldots, j_k$. Define $I := \{i_1, \ldots, i_k\}$, $J := \{j_1, \ldots, j_k\}$ and distinguish two cases.
Case 1 ($I \neq J$): This means that there exists at least one index $i_\nu \notin J$ and at least one $j_\mu \notin I$. Using the determinant property of the wedge product (c.f. [17, 12.8e)]), multlinearity and the factorization property from 1.2.1, we obtain

$$\begin{aligned}
\langle b^{i_1} \wedge \ldots \wedge b^{i_k}, & b^{j_1} \wedge \ldots \wedge b^{j_k} \rangle \\
&= \left\langle \sum_{\sigma \in \mathfrak{S}_k} \mathrm{sgn}(\sigma) b^{\sigma(i_1)} \otimes \ldots \otimes b^{\sigma(i_k)}, \sum_{\tau \in \mathfrak{S}_k} \mathrm{sgn}(\tau) b^{\tau(j_1)} \otimes \ldots \otimes b^{\tau(j_k)} \right\rangle \\
&= \sum_{\sigma \in \mathfrak{S}_k} \sum_{\tau \in \mathfrak{S}_k} \mathrm{sgn}(\tau) \mathrm{sgn}(\sigma) \langle b^{\sigma(i_1)}, b^{\tau(j_1)} \rangle \ldots \langle b^{\sigma(i_k)}, b^{\tau(j_k)} \rangle = 0.
\end{aligned}$$

The last equality holds since there is at least the factor $\langle b^{\sigma(i_\nu)}, b^{\tau(j_\nu)} \rangle = 0$. On the other hand $\det(\langle b^{i_r}, b^{j_s} \rangle) = 0$ as well, since row ν is identically zero. Thus in this case, the statement is true.
Case 2 ($I = J$): In that case there exists a permutation $\pi \in \mathfrak{S}_k$ such that $(\pi(i_1), \ldots, \pi(i_k)) = (j_1, \ldots, j_k)$. This implies

$$\begin{aligned}
\langle b^{i_1} \wedge \ldots \wedge b^{i_k}, b^{j_1} \wedge \ldots \wedge b^{j_k} \rangle &= \langle b^{i_1} \wedge \ldots \wedge b^{i_k}, b^{\pi(i_1)} \wedge \ldots \wedge b^{\pi(i_k)} \rangle \\
&= \mathrm{sgn}(\pi) \langle b^{i_1} \wedge \ldots \wedge b^{i_k}, b^{i_1} \wedge \ldots \wedge b^{i_k} \rangle \\
&= \mathrm{sgn}(\pi) \sum_{\sigma \in \mathfrak{S}_k} \sum_{\tau \in \mathfrak{S}_k} \mathrm{sgn}(\tau) \mathrm{sgn}(\sigma) \underbrace{\langle b^{\sigma(i_1)}, b^{\tau(i_1)} \rangle \ldots \langle b^{\sigma(i_k)}, b^{\tau(i_k)} \rangle}_{=0 \text{ ,unless } \sigma = \tau} \\
&= \mathrm{sgn}(\pi) \sum_{\sigma \in \mathfrak{S}_k} \langle b^{\sigma(i_1)}, b^{\sigma(i_1)} \rangle \ldots \langle b^{\sigma(i_k)}, b^{\sigma(i_k)} \rangle \\
&= \mathrm{sgn}(\pi) k!.
\end{aligned}$$

On the other hand, the determinant on the right hand side also equals $\mathrm{sgn}(\pi)$.

Thus in both cases the statement is valid for a basis of $\Lambda^k(V)$ and thus on all of $\Lambda^k(V)$ by multilinearity. □

There is an alternative way of extending the inner product directly to the space of alternating tensors.

1.2.4 Theorem (extension to exterior algebra). Let $(V, \langle \_, \_ \rangle)$ be a real inner product space of dimension m. For any $0 \leq k \leq m$ there exists exactly one inner product $\langle \_, \_ \rangle_{\Lambda^k} : \Lambda^k(V) \times \Lambda^k(V) \to \mathbb{R}$ such that for any ONB $C = (c_1, \ldots, c_m)$ with respect to $\langle \_, \_ \rangle$, the basis

$$\Lambda^k C := \{c^{i_1} \wedge \ldots \wedge c^{i_k} \mid 1 \leq i_1 < \ldots < i_k \leq m\}$$

is an ONB with respect to $\langle \_, \_ \rangle_{\Lambda^k}$. This scalar prouduct is given as the unique bi-additive extension of

$$\langle v^1 \wedge \ldots \wedge v^k, w^1 \wedge \ldots \wedge w^k \rangle_{\Lambda^k} = \det\left(\langle v^i, w^j \rangle\right)$$

where $v^1, \ldots, v^k, w^1, \ldots, w^k \in V^*$.

Proof.

Step 1 (existence): First of all, we discuss the case $k = 1$: Define $\langle \_, \_ \rangle$ on V^* by declaring the sharp operator $\sharp : V^* \to V$ to be an isometry, i.e.

$$\forall \omega, \eta \in V^* : \langle \omega, \eta \rangle := \langle \omega^\sharp, \eta^\sharp \rangle.$$

For any ONB $c_1, \ldots, c_m$ of V, the corresponding dual basis $c^1, \ldots, c^m$ of V^* satisfies

$$c^j = \langle \_, c_j \rangle = (c_j)^\flat.$$

Therefore $c^1, \ldots, c^m$ is an ONB as well.
Now we discuss the general case: Certainly there exists an ONB B of V. Since $\Lambda^k B$ is a basis of $\Lambda^k(V)$, it suffices to define

$$\langle b^{i_1} \wedge \ldots \wedge b^{i_k}, b^{j_1} \wedge \ldots \wedge b^{j_k} \rangle_{\Lambda^k} := \det\left(\langle b^{i_r}, b^{j_s} \rangle\right)$$

and extend this bilinearly onto $\Lambda^k(V)$. Since

$$\langle b^{i_1} \wedge \ldots \wedge b^{i_k}, b^{i_1} \wedge \ldots \wedge b^{i_k} \rangle_{\Lambda^k} = \det\left(\langle b^{i_r}, b^{i_s} \rangle\right) = \det\left(\delta_{i_r, i_s}\right) = 1 > 0,$$

$\langle \_, \_ \rangle_{\Lambda^k}$ is positive definit, thus a inner product for which $\Lambda^k B$ is an ONB. If $C = (c_1, \ldots, c_n)$ is any other ONB in of V, then

$$\det\left(\langle c^{i_r}, c^{j_s} \rangle\right) = \det\left(\langle b^{i_r}, b^{j_s} \rangle\right),$$

by the discussion of the case $k = 1$. Thus $\Lambda^k C$ is an ONB w.r.t. $\langle \_, \_ \rangle_{\Lambda^k}$ as well.

Step 2 (uniqueness): Let g be any other inner product satisfying the required properties. If $(b_1, \ldots, b_n)$ be an ONB of V, then $\Lambda^k B$ is a $\langle \_, \_ \rangle$-ONB and a g-ONB. Thus g and $\langle \_, \_ \rangle_{\Lambda^k}$ are equal on an ONB of $\Lambda^k(V)$ and thus equal on all of $\Lambda^k(V)$. □

1.2.5 Convention. It is customary to denote both the metrics given in 1.2.1 and 1.2.4 by the same symbol $\langle \_, \_\rangle$ as the original metric. This usually should not cause any confusion, since we have already shown in 1.2.3 that they only differ by a constant $k!$. But in case we would like to stress, which metric is meant, we will denote the *tensor metric* of a space V obtained from 1.2.1 and the *exterior metric* obtain from 1.2.4 above by

$$\langle \_, \_\rangle_{T^k(V)}, \qquad \text{respectively}, \qquad \langle \_, \_\rangle_{\Lambda^k(V)}.$$

1.2.6 Corollary. Let $(V, \langle \_, \_\rangle_V)$ be an inner product space and denote by $|\_|_V$ the induced norm. Then the dual space V' satisfies $T^1(V) = V' = \Lambda^1(V)$ and the norms satisfy

$$\|\_\|_{T^1(V)} = \|\_\|_{V'} = \|\_\|_{\Lambda^1(V)},$$

where $\|\_\|_{V'}$ is the usual operator norm on on V'.

Proof. The equality $\|\_\|_{T^1(V)} = \|\_\|_{\Lambda^1(V)}$ follows directly from 1.2.3. So let $v' \in V'$ be arbitrary and choose $v \in V$ such that $v^\sharp = v'$. Then

$$\|v'\|_{V'} = \max_{\|w\|_V=1} |v'(w)| = \max_{\|w\|_V=1} |v^\sharp(w)| = \max_{\|w\|_V=1} |\langle v, w\rangle| \overset{(*)}{=} \|v\|_V \overset{1.2.1(iii)}{=} \|v^\sharp\|_{T^1(V)}.$$

The equality (*) follows from the Cauchy-Schwarz inequality: One hand

$$|\langle v, w\rangle| \leq \|v\|_V \|w\|_V = \|v\|_V$$

and on the other hand, by considering $w := v/\|v\|_V$, we obtain $|\langle v, w\rangle| = \|v\|_V$. □

1.2.7 Definition (pullback). Let $f : X \to Y$ be a linear map. We call the induced map $f^* : T^k(Y) \to T^k(X)$, where

$$\forall \omega \in T^k(Y) : \forall x_1, \ldots, x_k \in X : f^*(\omega)(x_1, \ldots, x_k) := \omega(f(x_1), \ldots, f(x_k)),$$

the *pullback of* f. In particular, the induced map on the dual spaces $f' : Y' \to X'$ is the *dual map.*

1.2.8 Theorem (norms of pullbacks). Let $f : (X, \langle \_, \_\rangle_X) \to (Y, \langle \_, \_\rangle_Y)$ be a linear map between inner product spaces. Let $m := \dim X$, $n := \dim Y$ and assume

$$\exists C > 0 : \forall x \in X : \|f(x)\|_Y \leq C\|x\|_X. \tag{1.2}$$

Then for any $1 \leq k \leq n$

$$\forall \omega \in T^k(Y) : \|f^*(\omega)\|_{T^k(X)} \leq \binom{n}{k} C^k \|\omega\|_{T^k(Y)}.$$

Proof.

Step 1 ($k = 1$): We calculate

$$\forall y' \in Y' : \forall x \in X : |f'(y')(x)| = |y'(f(x))| \leq \|y'\|_{Y'} \|f(x)\|_Y \overset{(1.2)}{\leq} C\|y'\|_{Y'} \|x\|_X$$

and therefore by

$$\forall y' \in Y' : \|f'(y')\|_{T^1(X)} \leq C\|y'\|_{T^1(Y)}. \tag{1.3}$$

Now 1.2.6 yields the result.

Step 2 (for a basis): Notice that the factorization property 1.2.1(ii) implies that

$$\forall F \otimes G \in T^k(X) \otimes T^l(X) : \|F \otimes G\|_{T^{k+l}(X)} = \|F\|_{T^k(X)}\|G\|_{T^l(X)}.$$

Therefore, we obtain for any $v^1 \otimes \ldots \otimes v^k \in T^k(Y)$

$$\begin{aligned}\|f^*(v^1 \otimes \ldots \otimes v^k)\|_{T^k(X)} = \|f'(v^1) \otimes \ldots \otimes f'(v^k)\|_{T^k(X)} &\overset{1.2.6}{=} \|f'(v^1)\|_{X'} \ldots \|f'(v^k)\|_{X'} \\ &\overset{(1.3)}{\leq} C^k\|v^1\|_{Y'} \ldots \|v^k\|_{Y'} = C^k\|v^1 \otimes \ldots \otimes v^k\|_{T^k(Y)}.\end{aligned}$$

Step 3 (general case): In particular, we may choose an ONB B of Y. Then T^kB is an ONB of $T^k(Y)$. By the previous step, the estimate holds on this basis with constant C^k. Therefore the statement follows from Lemma 1.2.9 below. □

1.2.9 Lemma. Let $f : (X, \langle \_, \_\rangle_X) \to (Y, \langle \_, \_\rangle_Y)$ be a linear map between finite dimensional inner product spaces. Assume that there exists an ONB $B = (b_1, \ldots, b_m)$ of X such that

$$\exists C > 0 : \forall 1 \leq i \leq m : \|f(b_i)\|_Y \leq C\|b_i\|_X = C. \tag{1.4}$$

Then

$$\forall x \in X : \|f(x)\|_Y \leq mC\|x\|_X.$$

Proof. Let $x \in X$ be arbitrary. If we expand x w.r.t. B, we obtain (using the Cauchy-Schwarz inequality) that

$$x = \sum_{i=1}^m x^i b_i, \qquad |x^i| = |\langle x, b_i\rangle| \leq \|x\|_X\|b_i\|_X = \|x\|_X \tag{1.5}$$

and therefore

$$\begin{aligned}\|f(x)\|_X = \Big\|\sum_{i=1}^m x^i f(b_i)\Big\|_X &\leq \sum_{i=1}^m |x^i|\|f(b_i)\|_X \\ &\overset{(1.4)}{\leq} \sum_{i=1}^m |x^i|C \overset{(1.5)}{\leq} \sum_{i=1}^m C\|x\|_X = mC\|x\|_X.\end{aligned}$$ □

1.2.10 Corollary. Let f be as in Theorem 1.2.8 above. Then the induced map $f^* : \Lambda^k(Y) \to \Lambda^k(X)$ satisfies

$$\forall \omega \in Y : \|f^*(\omega)\|_{\Lambda^k(X)} \leq \binom{n}{k} C^k\|\omega\|_{\Lambda^k(Y)}.$$

Proof. This follows directly by combining 1.2.1 with 1.2.8. □

1.2.11 Corollary. Let f be as in Theorem 1.2.8 above. For any linearly independent system $b_1, \ldots, b_k$ define

$$\operatorname{vol}(b_1, \ldots, b_k) := \sqrt{\det(\langle b_i, b_j\rangle_X)}$$

to be the k-volume of the parallelepiped spanned by $b_1, \ldots, b_k$ (for a linearly dependent system this is zero anyway). We obtain

$$\operatorname{vol}(f(b_1), \ldots, f(b_k)) \leq \binom{m}{k} C^k \operatorname{vol}(b_1, \ldots, b_k).$$

Proof. We would like to apply Corollary 1.2.10. The problem is that f^* maps into the wrong direction and that we want to work on the vector space itself rather than on the dual space. But this problem can be easily solved by passing to the bi-dual space and by applying 1.2.10 to $f' : Y' \to X'$ instead of f: First of all consider the canonical isomorphism $i_X : X \to X''$, $x \mapsto (x' \mapsto x'(x))$. Define $\langle \_, \_ \rangle : X'' \times X'' \to \mathbb{R}$, $(x_1'', x_2'') \mapsto \langle i_X^{-1}(x_1''), i_X^{-1}(x_2'')\rangle_X$. Then i_X is an isometry and we obtain the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_Y} \\ X'' & \xrightarrow{f''} & Y'' \end{array} \tag{1.6}$$

The simple calculation

$$\forall x'' \in X'' : \forall y' \in Y' : (f')^*(x'')(y') = x''(f'(y')) = f''(x'')(y') \tag{1.7}$$

shows that $(f')^* = f'' : X'' \to Y''$. Unwinding the definitions and applying 1.2.10 to f', we obtain

$$\begin{aligned} \operatorname{vol}(f(b_1), \dots, f(b_k))) &= \sqrt{\det \langle f(b_i), f(b_j)\rangle_Y} = \sqrt{\det \langle i_Y(f(b_i)), i_Y(f(b_j))\rangle_{Y''}} \\ &\overset{(1.6)}{=} \sqrt{\det \langle f''(i_X(b_i)), f''((i_X(b_j))\rangle_{Y''}} \\ &\overset{(1.7)}{=} \sqrt{\det \langle (f')^*(i_X(b_i)), (f')^*((i_X(b_j))\rangle_{Y''}} \\ &\overset{1.2.10}{\leq} \binom{m}{k} C^k \sqrt{\det \langle i_X(b_i), i_X(b_j)\rangle_{X''}} \\ &= \binom{m}{k} C^k \sqrt{\det \langle b_i, b_j\rangle_X} = \binom{m}{k} C^k \operatorname{vol}(b_1, \dots, b_k). \end{aligned}$$

□

1.2.12 Theorem. Let $f : (X, \langle \_, \_ \rangle_X) \to (Y, \langle \_, \_ \rangle_Y)$ be an isometry between inner product spaces. Then $f^* : (\Lambda^k(Y), \langle \_, \_ \rangle_{\Lambda^k Y}) \to (\Lambda^k(X), \langle \_, \_ \rangle_{\Lambda^k X})$ is an isometry as well.

Proof. By basic linear algebra it suffices to check that f^* maps an ONB to an ONB. Therefore let $C = (c_1, \dots, c_m)$ be an ONB of Y. Since $f : X \to Y$ is an isometry, $f' : Y' \to X'$ is an isometry as well. By construction the $c^I = c^{i_1} \wedge \dots \wedge c^{i_k}$, where $1 \leq i_1 < \dots i_k \leq m$, are an ONB of $\Lambda^k(Y)$. Consequently

$$\begin{aligned} \langle f^*(c^{i_1} \wedge \dots \wedge c^{i_k}), f^*(c^{j_1} \wedge \dots \wedge c^{j_k})\rangle &= \langle f'(c^{i_1}) \wedge \dots \wedge f'(c^{i_k}), f'(c^{j_1}) \wedge \dots \wedge f'(c^{j_k})\rangle \\ &= \det(\langle f'(c^{i_\nu}), f'(c^{i_\mu})\rangle) = \det(\langle f(c_{i_\nu}), f(c_{i_\mu})\rangle) \\ &= \det(\langle c_{i_\nu}, c_{i_\mu}\rangle) = \delta_{IJ}, \end{aligned}$$

since if $I = J$, clearly this expression equals 1. If $I \neq J$ the matrix $(\langle c_{i_\nu}, c_{i_\mu}\rangle)$ has a zero column and therefore its determinant equals zero. □

1.3 Applications to manifolds

The theorems of the preceding section allow us to control the distortion of volumes under a diffeomorphism, if we are able to control the operator norm of its push-forward. Before we elaborate on this, here is a short reminder on the change of variables formula on Riemannian manifolds (following notes taken from a lecture by Professor Ballmann). In this subsection we will assume all manifolds to be oriented and Riemannian.

1.3.1 Definition. Let $X_1, \dots, X_m \in T_pM$ be a positive basis. We denote by

$$X_1 \wedge \ldots \wedge X_m := \Big\{ \sum_{i=1}^m t_i X_i \mid t_1, \dots, t_m \in [0,1] \Big\}$$

the *parallelepiped spanned by* $X_1, \dots, X_m$.
Let M and N be Riemannian manifolds, $F : M \to N$ be a diffeomorphism and $p \in M$. Then $\operatorname{Jac} F : M \to \mathbb{R}$,

$$p \mapsto \frac{\operatorname{vol}(F_*|_p(X_1) \wedge \ldots \wedge F_*|_p(X_m))}{\operatorname{vol}(X_1 \wedge \ldots \wedge X_m)},$$

is the *Jacobian of* F.

1.3.2 Lemma. Under this hypothesis,

$$\operatorname{vol}(X_1 \wedge \ldots \wedge X_m) = \sqrt{\det(\langle X_i, X_j \rangle)}$$

and $\operatorname{Jac}(p)$ does not depend on the choice of basis $X = (X_1, \dots, X_m)$. Furthermore

$$\forall p \in M : \operatorname{Jac}(F^{-1})(F(p)) = (\operatorname{Jac}(F)(p))^{-1}.$$

1.3.3 Theorem (transformation theorem for Riemannian manifolds). Let M, N be Riemannian manifolds, $F : M \to N$ be a diffeomorphism and let $f : N \to \mathbb{R}$ be integrable. Then $(f \circ F) \cdot \operatorname{Jac} F : M \to \mathbb{R}$ is integrable and

$$\int_M (f \circ F) \operatorname{Jac} F = \int_N f.$$

1.3.4 Corollary (volume distortion). Let $F : (M, g) \to (N, h)$ be a diffeomorphism between Riemannian m-manifolds and let $f : N \to \mathbb{R}$.

(i) If in addition

$$\exists C > 0 : \forall p \in M : \forall v \in T_pM : \|F_*|_p(v)\|_h \le C_1 \|v\|_g,$$

then

$$\int_N f d_h V \le C_1^m \int_M f \circ F d_g V.$$

In particular

$$\operatorname{vol}_h(N) \le C_1^m \operatorname{vol}_g(M).$$

(ii) In case

$$\exists C > 0 : \forall q \in N : \forall w \in T_qN : \|F_*^{-1}|_q(w)\|_g \le C_2 \|w\|_h,$$

then

$$\int_M f \circ F \le C_2^m \int_N f.$$

In particular

$$\operatorname{vol}_g(M) \le C_2^m \operatorname{vol}_h(N).$$

Proof.

(i) First assume that there exists a chart $\varphi : M \to V \subset \mathbb{R}^m$ for M. We calculate

$$\begin{aligned}\operatorname{Jac} F(p) &= \frac{\operatorname{vol}(F_*|_p(\partial\varphi_1) \wedge \ldots \wedge F_*|_p(\partial\varphi_m))}{\operatorname{vol}(\partial\varphi_1 \wedge \ldots \wedge \partial\varphi_m)} \\ &\overset{1.3.2}{=} \frac{\sqrt{\det(h(F_*|_p(\partial\varphi_i), F_*|_p(\partial\varphi_j)))}}{\sqrt{\det(g(\partial\varphi_i, \partial\varphi_j))}} \overset{1.2.11}{\leq} C_1^m.\end{aligned}$$

This implies

$$\int_N f d_h V \overset{1.3.3}{=} \int_M f \circ F \operatorname{Jac} F d_g V \leq C_1^m \int_M f \circ F d_g V.$$

The general case follows from the definition of the integral. Choosing $f \equiv 1$, we obtain the statement for the volume.

(ii) Analogously, for any $F(p) = q \in N$

$$(\operatorname{Jac} F(p))^{-1} = \operatorname{Jac}(F^{-1})(q) \leq C_2^m,$$

thus

$$\int_M f \circ F = \int_M f \circ F \operatorname{Jac} F \circ \operatorname{Jac} F^{-1} \leq C_2^m \int_M f \circ F \operatorname{Jac} F = C_2^m \int_N f. \qquad \square$$

1.3.1 Bounded diffeomorphisms and equivalence of Finsler metrics

As another application we discuss the norms induced by Riemannian metrics and their extensions to the tensor bundles. We will also study diffeomorphisms with bounded derivatives and their interplay with these norms. This will become important in 6.1.2.

1.3.5 Definition (Finsler metric). A continuous map $|\_| : TM \to \mathbb{R}$ is a *Finsler metric*, if

(i) $|\_|$ is smooth (away from the zero section),

(ii) for any $p \in M$, $|\_| : T_pM \to \mathbb{R}$ is a norm.

If g is a Riemannian metric on M, the *induced norm*

$$\forall p \in M : \forall X \in T_pM : |X|_g := \sqrt{g_p(X, X)}$$

clearly is a Finsler metric.
Let $C_1, C_2 > 0$. Two finsler metrics $(|\_|_1, |\_|_2)$ are (C_1, C_2)-*equivalent*, if

$$\forall X \in TM : C_1|X|_1 \leq |X|_2 \leq C_2|X|_1.$$

Two Riemannian metrics (g, h) are (C_1, C_2)-*equivalent*, if $(|\_|_g, |\_|_h)$ are (C_1, C_2)-equivalent.

Notice that the constants uniformly control the two different norms in all tangent spaces. Such an equivalence is a substantial restriction only if the manifold is noncompact.

1.3.6 Definition (bounded diffeomorphism). Let $F : (M, g) \to (N, h)$ be a smooth map and let $C > 0$. Then $F_* : TM \to TN$ is C-*bounded*, if $\|F_*\| \leq C$, where $\|\_\|$ denotes the operator norm induced by $|\_|_g$ and $|\_|_h$. Somewhat more explicitely this means

$$\forall p \in M : \forall X \in T_pM : h_{F(p)}(F_*|_pX, F_*|_pX) \leq C^2 g_p(X, X).$$

We also say that F is C-bounded, if F_* is C-bounded.
Let $C_1, C_2 > 0$. A diffeomorphism $F : (M, g) \to (N, h)$ is (C_1, C_2)-*bounded*, if F is C_1-bounded and F^{-1} is C_2-bounded.

1.3.7 Theorem (properties of bounded maps). Let $F : (M, g) \to (N, h)$ be a diffeomorphism.

(i) If F is an isometry, then F is $(1, 1)$-bounded.
(ii) If F is (C_1, C_2)-bounded, then (F^*h, g) are (C_1^{-1}, C_2)-equivalent.
(iii) If F is (C_1, C_2)-bounded and in addition $\tilde{h}$ is a metric on N such that $(h, \tilde{h})$ are (C_3, C_4)-equivalent, then $(g, F^*\tilde{h})$ are $(C_2^{-1}C_3, C_1C_4)$-equivalent.

Proof.

(i) This is clear.
(ii) By hypothesis, for any $p \in M$ and any $X \in T_pM$ we calculate on the one hand

$$(F^*h)|_p(X, X) = h_{F(p)}(F_*|_pX, F_*|_pX) \leq C_1^2 g_p(X, X),$$

and on the other hand

$$\begin{aligned} g_p(X, X) &= g_{F^{-1}(F(p))}(F_*^{-1}|_{F(p)}(F_*|_p(X)), F_*^{-1}|_{F(p)}(F_*|_p(X))) \\ &\leq C_2^2 h_{F(p)}(F_*|_p(X)), (F_*|_p(X)) = C_2^2 (F^*h)|_p(X, X). \end{aligned}$$

Consequently

$$C_1^{-1}|X|_{F^*h} \leq |X|_g \leq C_2|X|_{F^*h}.$$

(iii) Using the second part, we calculate

$$\begin{aligned} (F^*\tilde{h})_p(X, X) &= \tilde{h}_{F(p)}(F_*|_pX, F_*|_pX) \leq C_4^2 h_{F(p)}(F_*|_pX, F_*|_pX) \\ &= C_4^2 (F^*h)_p(X, X) \leq C_4^2 C_1^2 g_p(X, X), \end{aligned}$$

and

$$\begin{aligned} g_p(X, X) &\leq C_2^2 (F^*h)|_p(X, X) = C_2^2 h_{F(p)}(F_*|_pX, F_*|_pX) \\ &\leq C_2^2 C_3^{-2} \tilde{h}_{F(p)}(F_*|_pX, F_*|_pX) = C_2^2 C_3^{-2} (F^*\tilde{h})_p(X, X). \end{aligned}$$ □

1.3.8 Theorem (Bounded maps and equivalent norms).

(i) Two Riemannian metrics (h, g) on M are (C_1^{-1}, C_2)-equivalent, if and only if $\mathrm{id} : (M, g) \to (M, h)$ is (C_1, C_2)-bounded.
(ii) If $F : (M, g) \to (N, h)$ is (C_1, C_2)-bounded, then $F^* : T^*M \to T^*N$ is (C_1, C_2)-bounded as well, where the cotangent spaces are endowed with the operator norms induced by g and h.

(iii) If $F_* : TM \to TN$ is (C_1, C_2)-bounded, then $F_* : T^k_l M \to T^k_l N$,

$$\forall T \in T^k_l(M) : \forall \eta^1, \dots, \eta^l \in T^*N : \forall Y_1, \dots, Y_k \in TN :$$
$$F_*(T)(\eta^1, \dots, \eta^l, X_1, \dots, X_k) := T(F^*\eta^1, \dots, F^*\eta^l, F_*^{-1}Y_1, \dots, F_*^{-1}Y_k),$$

is $(2^{k+l}C_1^l C_2^k, 2^{k+l}C_1^k C_2^l)$-bounded, where $T^k_l M$, $T^k_l N$ are endowed with the metrics induced by $(T^k_l h, T^k_l g)$ via 1.2.1.

(iv) Let (h, g) be two Riemannian metrics on M. Again endow $T^k_l M$ with the induced metrics via 1.2.1. If (h, g) are (C_1, C_2)-equivalent on M (i.e. on TM), then the induced metrics $(T^k_l g, T^k_l h)$ are $(\tilde{C}_1, \tilde{C}_2) := (2^{-k-l}C_1^l C_2^{-k}, 2^{k+l}C_1^{-k}C_2^l)$ - equivalent on $T^k_l M$.

Proof.

(i) By definition the conditions

$$|\operatorname{id}_*(X)|_h \le C_1|X|_g, \qquad |\operatorname{id}_*^{-1}(X)|_g \le C_2|X|_h$$

are equivalent to

$$C_1^{-1}|X|_h \le |X|_g \le C_2|X|_h.$$

(ii) By definition $F_* : (T_pM, g) \to (T_{F(p)}N, h)$ and $F^* : T_{F(p)}N \to T_pM$ is the operator dual to F_*. By a standard theorem $\|F^*\| = \|F_*\|$.

(iii) It suffices to consider the situation on an arbitrary tangent space $V := T_pM$. By 1.2.6, the norms on V^* induced by 1.2.1 are identical to the operator norms. Therefore by combining (i) and (ii), we obtain the conclusion for $T^1_0 M$.
Now we prove the statement for decomposable tensors: Assume

$$T = T_J \otimes T^I = T_{j_1} \otimes \dots \otimes T_{j_l} \otimes T^{i_1} \otimes \dots \otimes T^{i_k} \in T^k_l(V)$$

and calculate (using the rules from 1.2.1)

$$\begin{aligned}
|F_*(T)|_{T^k_l h} &= |F_*(T_{j_1} \otimes \dots \otimes T_{j_l} \otimes T^{i_1} \otimes \dots \otimes T^{i_k})|_{T^k_l h} \\
&= |F_*T_{j_1} \otimes \dots \otimes F_*T_{j_l} \otimes (F^*)^{-1}T^{i_1} \otimes \dots \otimes (F^*)^{-1}T^{i_k}|_{T^k_l h} \\
&= |F_*T_{j_1}|_{T^k_l h} \dots |F_*T_{j_l}|_{T^k_l h} |(F^*)^{-1}T^{i_1}|_{T^k_l h} \dots |(F^*)^{-1}T^{i_k}|_{T^k_l h} \\
&\le C_1^l C_2^k |T_{j_1}|_{T^k_l g} \dots |T_{j_l}|_{T^k_l g} |T^{i_1}|_{T^k_l g} \dots |T^{i_k}|_{T^k_l g} = C_1^l C_2^k |T|_{T^k_l g}.
\end{aligned}$$

In the same fashion, we obtain $|F_*^{-1}(T)|_{T^k_l g} \le C_1^k C_2^l |T|_{T^k_l h}$. Now the conclusion follows from 1.2.9.

(iv) Let (g, h) be (C_1, C_2)-equivalent. By (i) the map $\operatorname{id}_* : (TM, g) \to (TN, h)$ is (C_1^{-1}, C_2)-bounded. By (iii) the map $\operatorname{id}_* : (T^k_l M, T^k_l g) \to (T^k_l N, T^k_l h)$ is $(2^{k+l}C_1^{-l}C_2^k, 2^{k+l}C_1^{-k}C_2^l)$-bounded. By (i) $(T^k_l h, T^k_l g)$ are $(2^{-k-l}C_1^l C_2^{-k}, 2^{k+l}C_1^{-k}C_2^l)$ - equivalent. □

2 L_p-cohomology theories

In this section we will systematically built up three different approaches to L_p-cohomology, namely the L_p-cohomology of differential forms on manifolds, the L_p-cohomology of simplicial complexes in $\mathbb{R}^n$ and the L_p-cohomology of so called S-forms - these are beeings in between. We are not aiming at a full treatment of all these theories, we merely want to introduce notation, definitions and prove some theorems, which ensure that this all makes sense.

2.1 L_p-cohomology of differential forms

The first approach is to establish an L_p-theory for differential forms on manifolds analogous to L_p-functions on $\mathbb{R}^m$. We will assume that (M, g) is a possibly non-compact oriented Riemannian manifold without boundary and employ the following definitions and conventions.

2.1.1 Convention (exterior direct sums). We will frequently define $\mathbb{Z}$-indexed systems of vector spaces $(V^k)_{k\in\mathbb{Z}}$. In that case, the space V is understood to be

$$V := \bigoplus_{k\in\mathbb{Z}} V^k.$$

2.1.2 Definition (set of measure zero). A subset $A \subset M$ has *measure zero*, if for any chart (U, φ) of M, the set $\varphi(A\cap U)$ has Lebesgue measure zero in $\mathbb{R}^m$. (Since this property is invariant under diffeomorphisms, this notion is well-defined.)

2.1.3 Definition (locally p-integrable). Let $\omega : M \to T^kM$ be any section. Then ω is *measurable*, if in any chart all the component functions of ω are Lebesgue-measurable. We denote by $L^k(M)$ the space of all measurable k-forms on M. For any $\omega \in L^k(M)$, $1 \leq p < \infty$ and any domain of integration $N \subset M$, we denote

$$\|\omega\|_{L_p^k(N)} := \left(\int_N |\omega|^p dV\right)^{\frac{1}{p}}, \qquad \|\omega\|_{L_\infty^k}(N) := \operatorname{ess\,sup}_{x\in N} |\omega|(x).$$

Here, $|\omega|$ is defined pointwise by

$$\forall x \in M : |\omega|(x) = \|\omega(x)\|_{\Lambda^k(T_xM)}$$

using 1.2.4. We say ω is *locally p-integrable*, $1 \leq p \leq \infty$, if for any compact subset $K \subset M$ $\|\omega\|_{L_p^k(K)} < \infty$. The space of all measurable locally p-integrable differential forms of degree k on M is denoted by $L_{p,\mathrm{loc}}^k(M)$.

2.1.4 Definition (p-integrable). Let $1 \leq p \leq \infty$. Then

$$L_p^k(M) := \{\omega \in L_{p,\mathrm{loc}}^k(M) \mid \|\omega\|_{L_p^k(M)} < \infty\}$$

are the *p-integrable forms.*

2.1.5 Convention. The space $L_p^0(\mathbb{R}^n)$ is the usual L^p-space of functions. This space is usually identified with the space of L^p-classes, i.e. equivalence classes of functions modulo equality up to sets of measure zero. We will employ this convention on $L(M)$ and all its subspaces as well. Having this in mind, we may generalize a classical theorem.

2.1.6 Theorem (completeness of L_p-spaces). For every $1 \leq p \leq \infty$ and every $0 \leq k \leq m$ the space $L_p^k(M)$ is a Banach space.

In many ways the calculus for measurable differential forms is analogous to the calculus of smooth forms. We will discuss some aspects in the following.

2.1.7 Lemma (wedge). Let $\omega \in L^k(M)$, $\eta \in L^l(M)$. Then the *wedge product* $\omega \wedge \eta \in L^{k+l}(M)$ defined by

$$\forall p \in M : (\omega \wedge \eta)_p := \omega_p \wedge \eta_p,$$

is well-defined.

Proof. By definition

$$\omega|_p \wedge \eta_p = \frac{1}{k!l!} \sum_{\sigma \in \mathfrak{S}_{k+l}} \operatorname{sgn}(\sigma)(\omega_p \otimes \eta_p)^\sigma,$$

thus it suffices to check that the tensor product of two measurable forms is well-defined. By choosing local coordinates φ, we see that

$$\omega \otimes \eta = (\omega_i d\varphi^i) \otimes (\eta_j d\varphi^j) = \omega_i \eta_j \, d\varphi^i \otimes d\varphi^j. \tag{2.1}$$

Now let $\omega' \sim \omega$, i.e. there exists a set $E_\omega \subset M$ of measure zero such that

$$\forall p \in M \setminus E_\omega : \omega_p = \omega'_p$$

and analogously for η. Then the local representation (2.1) implies

$$\forall p \in M \setminus (E_\omega \cup E_\eta) : (\omega' \otimes \eta')_p = (\omega \otimes \eta)_p.$$

Since $E_\omega \cup E_\eta \subset M$ has measure zero as well, this implies that $\omega \wedge \eta$ is well-defined. It is measurable since the product of measurable functions is measurable. □

Because this pointwise defined operation is well-defined, all the standard theorems concerning the wedge product carry over to the measurable case as well.

2.1.8 Theorem. The wedge product $\wedge : L(M) \times L(M) \to L(M)$ satisfies the following properties:

(i) $\wedge$ is bilinear,

(ii) associative,

(iii) graded anti-commutative.

These generalized wedge products can be utilized to generalize yet another classical construction from the calculus of smooth forms on manifolds.

2.1.9 Definition (weak differential). Let $\omega \in L_{p,\mathrm{loc}}^k(M)$ and $\omega' \in L_{p,\mathrm{loc}}^{k+1}(M)$. Then ω' is a *weak differential* of ω if

$$\forall \eta \in \Omega_c^{m-k-1}(M) : \int_M \omega' \wedge \eta = (-1)^{k+1} \int_M \omega \wedge d\eta.$$

In that case we denote $d\omega := \omega'$, c.f. 2.1.14. The space of all those forms is denoted by $W_{p,\mathrm{loc}}^k(M)$.

2.1.10 Definition (exterior Sobolev spaces). Employing the notation

$$\|\omega\|^p_{W^k_p(M)} := \|\omega\|^p_{L^k_p(M)} + \|d\omega\|^p_{L^{k+1}_p(M)}, \quad \|\omega\|_{W^k_\infty(M)} := \max\{\|\omega\|_{L^k_\infty(M)}, \|\omega\|_{L^{k+1}_\infty(M)}\},$$

we define the *(exterior) Sobolev spaces*

$$W^k_p(M) := \{\omega \in W^k_{p,\mathrm{loc}}(M) \mid \|\omega\|_{W^k_p(M)} < \infty\}.$$

2.1.11 Remark. Sometimes a more asymetric generalization of these spaces is used: If $1 \leq p, q \leq \infty$, define

$$W^k_{p,q}(M) := \{\omega \in W^k_{p,\mathrm{loc}}(M) \mid \omega \in L^k_p(M), d\omega \in L^{k+1}_q(M)\}.$$

This space is then endowed with the norm

$$\|\omega\|^2_{W_{p,q}(M)} := \|\omega\|^2_{L_p(M)} + \|d\omega\|^2_{L_q(M)}.$$

Thus $W_{p,p}(M) = W_p(M)$ as vector spaces and the norms are equivalent. We will restrict our study to W_p.

There is a classical result from functional analysis (see [9, Theorem 1.2.5] for instance), which is very important for the uniqueness of weak differentials.

2.1.12 Lemma (fundamental lemma of the calculus of variaions for functions). Let $U \subset \mathbb{R}^n$ be open. An L_p-class $f \in L_p(U)$ is zero if and only if

$$\forall \varphi \in \mathcal{C}^\infty_c(U) : \int_U f(x)\varphi(x)dx = 0.$$

This lemma generalizes to forms.

2.1.13 Lemma (fundamental lemma of the calculus of variaions for forms). Let $\omega \in L^k_{p,\mathrm{loc}}(M)$ be arbitrary. Then

$$\omega = 0 \ a.e. \iff \forall \eta \in \Omega^{m-k}_c(M) : \int_M \omega \wedge \eta = 0.$$

Proof. Only the direction "$\Leftarrow$" requires proof. Let $\varphi : U \to U'$ be any positive chart. Then ω can be expressed locally by $\omega = \sum_I \omega_I d\varphi^I$, where the sum is taken over multi indices $I = (0 \leq i_1 < \ldots < i_k \leq m)$ of length k. Let J be any such index. Then there exists a complementary index $J' = (j'_1, \ldots, j'_{n-k})$ such that

$$d\varphi^J \wedge d\varphi^{J'} = d\varphi^{JJ'} = \pm d\varphi^1 \wedge \ldots d\varphi^m = \pm \det(g_{ij})^{-\frac{1}{2}} d_g V. \tag{2.2}$$

Let $\eta \in \mathcal{C}^\infty_c(U')$ be arbitrary. Since $\varphi^*(\eta)d\varphi^{J'} \in \Omega^{m-k}_c(M)$, we obtain by hypothesis

$$\begin{aligned} 0 &= \int_U \omega \wedge (\varphi^*(\eta)d\varphi^{J'}) = \int_U \Big(\sum_I \omega_I d\varphi^I\Big) \wedge (\varphi^*(\eta)d\varphi^{J'}) = \sum_I \int_U \omega_I \varphi^*(\eta) d\varphi^I \wedge d\varphi^{J'} \\ &= \int_U \omega_J \varphi^*(\eta) d\varphi^J \wedge d\varphi^{J'} \overset{(2.2)}{=} \pm \int_U \omega_J \varphi^*(\eta) \det(g_{ij})^{-\frac{1}{2}} d_g V = \pm \int_{U'} \varphi_*(\omega_J)\eta. \end{aligned}$$

Since this holds for any η, Lemma 2.1.12 above implies $\varphi_*(\omega_J) = 0$ a.e. in U'. By definition $\omega_J = 0$ a.e. in U. This procedure can be executed on all the component functions ω_J and we obtain $\omega = 0$ a.e. in U. Since M can be covered by countably many of those charts and the countable union of sets of measure zero is again a set of measure zero, we obtain that ω is zero a.e. □

2.1.14 Corollary. The weak differential of a form $\omega \in L^k_{p,\mathrm{loc}}(M)$ is uniquely determined (if it exists). If ω is smooth, the weak differential equals the exterior differential.

Proof. Uniqueness follows from Lemma 2.1.13 above. Any $\omega \in \Omega^k(M)$ automaticaly satisfies $\omega \in L^k_{p,\mathrm{loc}}(M)$, $d\omega \in L^{k+1}_{p,\mathrm{loc}}(M)$. Using Stokes' theorem and the Leibniz rule we obtain

$$\forall \eta \in \Omega^{m-k-1}_c(M) : 0 = \int_{\partial M} \omega \wedge \eta = \int_M d(\omega \wedge \eta) = \int_M d\omega \wedge \eta + (-1)^k \int_M \omega \wedge d\eta.$$

Thus the statement follows from 2.1.13. □

2.1.15 Convention. Due to this corollary we no longer distinguish between weak and exterior differential. For any form $\omega \in L^k_{p,\mathrm{loc}}(M)$ we denote by $d\omega$ the *differential* (provided it exists).

2.1.16 Warning. From the choice of terminology one might believe that a form ω on M is in $W^k_p(M)$ if and only if all all its coefficient functions locally belong to the classical Sobolev space.[1] This is wrong! For example take $M := \mathbb{R}^2$ and any two functions $f, g \in L_p(\mathbb{R}^2)$, which are both weakly differentiable and whose derivatives satisfy $\partial_y f, \partial_x g \notin L_p(\mathbb{R}^2)$, but instead $\partial_y f = \partial_x g$. Then f, g are not in the classical Sobolev space over $\mathbb{R}^2$, but the form $\omega := f dx + g dy$ satisfies

$$\begin{aligned}
&\|\omega\|_{L^1_p(\mathbb{R}^2)} \leq \|f dx\|_{L^1_p(\mathbb{R}^2)} + \|g dy\|_{L^1_p(\mathbb{R}^2)} = \|f\|_{L_p(\mathbb{R}^2)} + \|g\|_{L_p(\mathbb{R}^2)} < \infty, \\
&dw = (\partial_y f - \partial_x g) dx \wedge dy = 0.
\end{aligned}$$

Thus $\omega \in W^1_p(M)$.
Somewhat more generally, we can say that every closed weakly differentiable form $\omega \in L^k_p(M)$ is automatically in $W^k_p(M)$ and $\|\omega\|_{W^k_p(M)} = \|\omega\|_{L^k_p(M)}$.

Nevertheless the following property still holds in this setup.

2.1.17 Lemma (completeness of exterior Sobolev spaces). For every $1 \leq p \leq \infty$ and every $0 \leq k \leq m$ the map $d : W^k_p(M) \to W^{k+1}_p(M)$ is a bounded linear operator between Banach spaces.

Proof. See [4, 1.3]. □

2.1.18 Theorem (Hölder Inequality). Let $1 \leq p, q \leq \infty$, $\omega \in L^k_p(M)$, $\eta \in L^l_q(M)$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

(i) $\omega \wedge \eta \in L^{k+l}_r(M)$,

(ii) $\|\omega \wedge \eta\|_{L_r(M)} \leq \|\omega\|_{L_r(M)} \|\eta\|_{L_r(M)}$,

(iii) if $\omega \in W^k_p(M)$, $\eta \in W^l_q(M)$, then $\omega \wedge \eta \in W^{k+l}_r(M)$ and

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

Proof. See [4, 1.4]. □

[1] These spaces play a central role in the study of pseudodifferential operators on vector bundles and are discussed in [13, III] for example.

We will require the last statement only in the following weaker version, whose proof is straightforward.

2.1.19 Theorem (weak Leibniz rule). Let $\omega \in W^k_{p,\mathrm{loc}}(M)$, $\eta \in \Omega^l(M)$. Then $\omega \wedge \eta \in W^{k+l}_{p,\mathrm{loc}}(M)$ and

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

in the weak sense.

Proof. Choose any $u \in \Omega_c^{m-k-l-1}(M)$ and notice that the Leibniz rule for smooth forms implies

$$\int_M \omega \wedge d(\eta \wedge u) = \int_M \omega \wedge d\eta \wedge u + (-1)^l \int_M \omega \wedge \eta \wedge du. \tag{2.3}$$

Notice further that $\eta \wedge u \in \Omega_c^{m-k-1}(M)$. Since ω is weakly differentiable, this implies

$$\int_M d\omega \wedge \eta \wedge u = (-1)^{k+1} \int_M \omega \wedge d(\eta \wedge u). \tag{2.4}$$

Therefore by the definition of the weak differential

$$\begin{aligned}(-1)^{k+l+1} \int_M \omega \wedge \eta \wedge du &\overset{(2.3)}{=} (-1)^{k+1} \int_M \omega \wedge d(\eta \wedge u) + (-1)^k \int_M \omega \wedge d\eta \wedge u \\ &\overset{(2.4)}{=} \int_M d\omega \wedge \eta \wedge u + \int_M ((-1)^k \omega \wedge d\eta) \wedge u = \int_M (d\omega \wedge \eta + (-1)^k \omega \wedge d\eta) \wedge u.\end{aligned}$$

This proves the statement. □

2.1.20 Definition (L_p-cohomology). The exterior Sobolev spaces $W^k_p(M)$ together with the weak differential $d^k : W^k_p(M) \to W^{k+1}_p(M)$ assemble to a cochain complex

$$(W_p(M), d)$$

called the *L_p-complex of M*. Its cohomology groups

$$H^k_p(M) := \frac{\ker d^k}{\operatorname{im} d^{k-1}}$$

are called *L_p-cohomology of M*. This is a $\mathbb{Z}$-indexed system of vector spaces endowed with the ordinary quotient semi-norm $\|\_\|_{H^k_p}$ induced by $\|\_\|_{W^k_p}$. The spaces

$$\bar{H}^k_p(M) := \frac{\ker d^k}{\overline{\operatorname{im} d^{k-1}}} \cong \frac{H^k_p(M)}{\{x \in H^k_p(M) \mid \|x\|_{H^k_p} = 0\}} \tag{2.5}$$

are called *reduced L_p-cohomology of M*. For any $1 \leq p \leq \infty$, we also define

$$Z^k_p(M) := \ker\left(d^k : W^k_p(M) \to W^{k+1}_p(M)\right), \qquad \text{the } \textit{closed } k\textit{-forms,}$$
$$B^k_p(M) := \operatorname{im}\left(d^k : W^{k-1}_p(M) \to W^k_p(M)\right), \qquad \text{the } \textit{exact } k\textit{-forms.}$$

2.1.21 Remark (cochain complex properties). If $\mathrm{Ban}_{\mathbb{R}}$ is the category of real Banach spaces and continuous linear operators, we may regard (W_p, d) as an element of $\mathrm{Ch}(\mathrm{Ban}_{\mathbb{R}})$, the category of cochain complexes over $\mathrm{Ban}_{\mathbb{R}}$: By Lemma 2.1.17, $W_p^k(M) \in \mathrm{Ban}_{\mathbb{R}}$ and it follows from

$$\begin{aligned}\forall \omega \in W_p^k(M) : \|d\omega\|^p_{W_p^{k+1}(M)} &= \max\{\|d\omega\|^p_{W_p^{k+1}(M)}, \|dd\omega\|^p_{W_p^{k+2}(M)}\} = \|d\omega\|^p_{W_p^{k+1}(M)} \\ &\leq \max\{\|\omega\|^p_{W_p^k(M)}, \|d\omega\|^p_{W_p^{k+1}(M)}\} = \|\omega\|^p_{W_p^k(M)}\end{aligned}$$

that $\|d\| \leq 1$. However $d(W_p^k(M)) \subset W_p^{k+1}(M)$ is not necessarily closed as we will see in the example below. Thus although $W_p^{k+1}(M)$ is a Banach space, $H_p^{k+1}(M)$ is in general not even a normed space anymore, but only a semi-normed vector space.

2.1.22 Remark (alternative description of reduced L_p-cohomology). The problem described above is a general functional analytic phenomenon: Let $(X, \|\_\|_X)$ be a Banach space and $U \subset X$ be a linear subspace. Then we may consider the algebraic quotient space X/U. Let $\pi_U : X \to X/U$ be the canonical projection and define $\|\_\|_{X/U} : X/U \to \mathbb{R}$ by

$$\|\pi_U(x)\|_{X/U} := \inf_{u \in U} \|x - u\|_X.$$

Then $\|\_\|_{X/U}$ is always a semi-norm (even if X is incomplete), but if U is not closed, the space

$$N := \{\pi_U(x) \in X/U \mid \|\pi_U(x)\|_{X/U} = 0\}$$

might not be trivial. There are two possible ways to fix that problem: The first one is to factor out N again and obtain the space $(X/U)/N$, which is a Banach space again by construction. The other one is to take the closure of U and consider $X/\bar{U}$. These are more or less the same spaces, see Lemma 2.1.23 below for the details. So we may either use the completion $(X/U)/N$ or the somewhat simpler space $X/\bar{U}$. The isomorphism in (2.5) is meant in this way.

Factoring out a bit more in order to obtain a Banach space is a two-edged sword: On the one hand a Banach space is always nice in order to do calculus. On the other hand, this considerably changes the notion of an exact form: A form is reduced exact, if and only if it can be written as a limit of exact forms. But the form itself might not be exact.

2.1.23 Lemma (double quotients). Let $(X, \|\_\|_X)$ be a Banach space and $U \subset X$ be a linear subspace. Let $\pi_U : X \to X/U$ be the canonical projection onto the algebraic quotient and define the quotient semi-norm $\|\_\|_{X/U} : X/U \to \mathbb{R}$ by

$$\|\pi_U(x)\|_{X/U} := \inf_{u \in U} \|x - u\|_X.$$

In general, the space

$$N := \{\pi_U(x) \in X/U \mid \|\pi_U(x)\|_{X/U} = 0\}$$

is not trivial. Denote by $\pi_N : X/U \to (X/U)/N$ the canonical projection and endow $(X/U)/N$ with the quotient semi-norm $\|\_\|_{(X/U)/N}$ as well. Then $\|\_\|_{(X/U)/N}$ is a norm and $(X/U)/N$ is a Banach space as well.

Define $\pi := \pi_N \circ \pi_U : X \to (X/U)/N$. Then $\ker \pi = \bar{U}$ and we obtain a Banach space isomorphism $X/\bar{U} \to (X/U)/N$, which makes the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\pi_U} & X/U & \xrightarrow{\pi_N} & (X/U)/N \\ {\scriptstyle \pi_{\bar{U}}}\downarrow & & & \nearrow_{\sim} & \\ X/\bar{U} & & & & \end{array} \tag{2.6}$$

commute.

Proof. The first statements follow from general functional analysis.

Step 1 ($\bar{U} \subset \ker \pi$): Let $x \in \bar{U}$. Then there exists a sequence $u_i \in U$ such that $u_i \to x$ in X. By continuity $\pi_U(u_i) \to \pi_U(x)$ in X/U. But $\pi_U(u_i) = 0$ for every $i \in \mathbb{N}$, thus $\pi_U(x) = 0$ as well. This implies $0 = \pi_N(\pi_U(x)) = \pi(x)$, which implies $x \in \ker \pi$, thus $\bar{U} \subset \ker \pi$.

Step 2 ($\ker \pi \subset \bar{U}$): Conversely, if $x \in \ker \pi$, we obtain by definition

$$0 = \pi(x) = \pi_N(\pi_U(x)) \Rightarrow 0 = \|\pi_N(\pi_U(x))\|_{(X/U)/N} = \inf_{\pi_U(v) \in N} \|\pi_U(x) - \pi_U(v)\|_{X/U}.$$

Consequently, there exists a sequence $\pi_U(v_j) \in N$ such that

$$\pi_U(v_j) \xrightarrow[X/U]{j\to\infty} \pi_U(x) \ . \tag{2.7}$$

By definition of N

$$\forall j \in \mathbb{N} : 0 = \|\pi_U(v_j)\|_{X/U} = \inf_{u \in U} \|v_j - u\|_X.$$

Consequently, for any $j \in \mathbb{N}$, there exists a sequence $(u_{ij})_{i \in \mathbb{N}} \in U$ such that

$$u_{ij} \xrightarrow[X]{i\to\infty} v_j \ ,$$

which implies $v_j \in \bar{U}$. Now let $k \in \mathbb{N}$ be arbitrary. By (2.7) there exists a $j_k \in \mathbb{N}$ such that

$$\inf_{u \in U} \|v_{j_k} - x - u\|_X = \|\pi_U(v_{j_k} - x)\|_{X/U} < \frac{1}{2k}.$$

By definition of the infimum, there exists $u_k \in U$ such that

$$\|v_{j_k} - x - u_k\|_X < \frac{1}{k}.$$

Consequently $\bar{u}_k := v_{j_k} - u_k$ is a sequence in $\bar{U}$ such that

$$\bar{u}_k \xrightarrow[X]{k\to\infty} x \ .$$

This implies $x \in \bar{U}$.

Step 3: We have constructed a surjective continuous map $\pi = \pi_N \circ \pi_U : X \to (X/U)/N$, which induces an isomorphism $X/\ker \pi \to (X/U)/N$ as in (2.6) by the fundamental theorem on homomorphisms. By the open map theorem it is a Banach space isomorphism. $\square$

2.1.24 Example (L_1-cohomology of the half-line). The following example illustrates that the phenomenon described in the Remark 2.1.22 above really occurs and that L_p-cohomology can be very different from the classical de Rham cohomology. For simplicity let $p = 1$, define $M :=]1, \infty[\subset \mathbb{R}$ and remember that

$$t \mapsto t^{-s} \in L_1^0(M) \Leftrightarrow s > 1. \tag{2.8}$$

Similiarly, for an antiderivative of this function, we obtain

$$t \mapsto \frac{1}{-s+1} t^{-(s-1)} \in L_1^0(M) \Leftrightarrow s > 2. \tag{2.9}$$

(i) Define $f : M \to \mathbb{R}$ by $f(t) := t^{-2}$ and $\omega := f\,dt \in L_1^1(M)$. Since $\dim M = 1$, $d\omega = 0$, thus ω is closed and $\omega \in W_1^1(M)$. An antiderivative is easily seen to be $F : M \to \mathbb{R}$, $t \mapsto -t^{-1}$. So in the classical de Rham cohomology we would conclude that $\omega = dF$ is exact. Since every smooth function on M has an antiderivative, we obtain $H^1_{\mathrm{dR}}(M) = 0$. The crucial observation here is that

$$\forall c \in \mathbb{R} : F + c \notin L_1^0(M),$$

i.e. no antiderivative of ω is integrable. Thus $0 \neq [\omega] \in H_1^1(M)$.

(ii) Now consider the sequence $g_n := t^{-(2+\frac{1}{n})}$ and $G_n := \frac{-1}{1+\frac{1}{n}} t^{-(1+\frac{1}{n})}$. We notice that for all $n \in \mathbb{N}$

$$2 + \frac{1}{n} > 1 \Longrightarrow g_n \in L_1^0(M), \qquad 1 + \frac{1}{n} > 1 \Longrightarrow G_n \in L_1^0(M).$$

Thus for all $n \in \mathbb{N}$, $\omega_n := g_n dt = dG_n \in W_1^1(M)$ and $G_n \in W_1^0(M)$. Consequently $0 = [\omega_n] \in H_1^1(M)$. We calculate

$$\|\omega - \omega_n\|_{W_1^1(M)} = \|f - g_n\|_{L_1(M)} = \int_1^\infty |f(t) - g_n(t)| dt = \int_1^\infty t^{-2} - t^{-(2+\frac{1}{n})} dt$$
$$= -t^{-1} + \frac{n}{n+1} t^{-(1+\frac{1}{n})} \Big|_1^\infty = \lim_{R\to\infty} -\frac{1}{R} + \frac{n}{n+1} \frac{1}{R^{1+\frac{1}{n}}} + 1 - \frac{n}{n+1} = 1 - \frac{n}{n+1} \longrightarrow 0.$$

This shows $\omega_n \to \omega$ in $W_1^1(M)$. So altogether, we have found a non-exact W_1-form ω, which is a W_1^1-limit of exact forms ω_n. In particular $d : W_1^0(M) \to W_1^1(M)$ is not a closed operator.

(iii) It is also remarkable that $M =]1, \infty[$ is homotopy equivalent to the one point space $\{*\}$. Clearly $H_1^1(\{*\}) = 0$, so L_p-cohomology is not homotopy invariant (and therefore does not satisfy the Eilenberg-Steenrod axioms of a (co-)homology theory, c.f. [25, 4.8]). But de Rham cohomology is homotopy invariant, thus $H^1_{\mathrm{dR}}(M) = 0$, but $H_1^1(M) \neq 0$. To make things worse, we will illustrate how to employ equations (2.8) and (2.9) to show that $H_1^1(M)$ is not even finitely generated. These equations directly show that for each $0 < \varepsilon < 1$ the form $\omega_\varepsilon := t^{-(1+\varepsilon)} dt \in W_1^1(M)$ is L_1-closed, but not exact, since its antiderivative $F_\varepsilon := -\frac{1}{\varepsilon} t^{-\varepsilon} \notin L_1^0(M)$. Clearly $\{\omega_\varepsilon \mid 0 < \varepsilon < 1\}$ is linearly independent. We claim that they all represent different L_1-cohomology classes. Therefore consider $0 < \delta < \varepsilon < 1$. Clearly

$$\lim_{t\to\infty} \frac{\varepsilon}{\delta} t^{\varepsilon - \delta} = \infty,$$

thus there exists $t_0 > 1$ such that

$$\forall t \geq t_0 : \frac{\varepsilon}{\delta} t^{\varepsilon-\delta} \geq 2 \Leftrightarrow \varepsilon t^{\varepsilon} \geq 2\delta t^{\delta} \Leftrightarrow \frac{1}{\varepsilon} t^{-\varepsilon} \leq \frac{1}{2\delta} t^{-\delta} \Leftrightarrow -\frac{1}{\varepsilon} t^{-\varepsilon} \geq -\frac{1}{2\delta} t^{-\delta}.$$

This implies

$$\forall t \geq t_0 : |F_\varepsilon(t) - F_\delta(t)| = F_\varepsilon(t) - F_\delta(t) = -\frac{1}{\varepsilon} t^{-\varepsilon} + \frac{1}{\delta} t^{-\delta} \geq -\frac{1}{2\delta} t^{-\delta} + \frac{1}{\delta} t^{-\delta} = \frac{1}{2\delta} t^{-\delta},$$

which is clearly not in $L_1^0([t_0, \infty[)$ by (2.8). Since $F_\varepsilon - F_\delta$ is an anti-derivative of $\omega_\varepsilon - \omega_\delta$ and any other anti-derivative differs from $F_\varepsilon - F_\delta$ only by a constant, no anti-derivative of $\omega_\varepsilon - \omega_\delta$ is in $L_1^0(M)$. Consequently $[\omega_\varepsilon] \neq [\omega_\delta] \in H_1^1(M)$.

2.1.1 Morphisms

2.1.25 Remark. In category theory, a functor is defined on objects and morphisms of a certain category and one usually wants that a cohomology theory is functorial. In our present case, this is not so easy, since we have not yet defined a suitable class of morphisms. If $F : M \to N$ is a smooth map, then it functorially induces a pull-back $F^* : \Omega(N) \to \Omega(M)$. In general $F^* : L(N) \to L(M)$ is not even well-defined: Assume that $F = \iota : M \hookrightarrow N$ is an inclusion of a submanifold with M with dimension $m < n$. Then M is a set of measure zero in N. But forms $\omega \in L(N)$ may be changed arbitrarily on a set of measure zero. So there is no chance of defining $\iota^*\omega = \omega|_M$ directly.

2.1.26 Definition (zero-preserving). A map $F : M \to N$ is *zero-preserving* if for any set of measure zero $E \subset N$, the set $F^{-1}(E) \subset M$ is a set of measure zero as well.

2.1.27 Definition. Let $F : (M, h) \to (N, h)$ be zero-preserving and smooth. Let $\omega \in L^k(N)$ be any differential form. Then $F^*\omega \in L^k(M)$ defined by

$$\forall p \in M : \forall X_1, \ldots, X_k \in \mathcal{T}(M) : (F^*\omega)_p(X_1, \ldots, X_k) := \omega_{F(p)}(F_*X_1, \ldots, F_*X_k)$$

is the *pull-back of* ω *along* F. Notice that this gives a well-defined map $F^* : L(N) \to L(M)$.

2.1.28 Theorem. Let $F : (M, g) \to (N, h)$.

(i) If F is C_1-bounded (c.f. 1.3.6) and zero-preserving, then $F^* : L_\infty(N) \to L_\infty(M)$ is bounded.

(ii) If F is a (C_1, C_2)-bounded diffeomorphism, then $F^* : L_p(N) \to L_p(M)$, is a Banach space isomorphism for all $1 \leq p < \infty$.

(iii) If F is an isometry, then $F^* : L_p(N) \to L_p(M)$ is an isometry for all $1 \leq p \leq \infty$.

Proof.

(i) We calculate

$$\begin{aligned}\|F^*(\omega)\|_{L_\infty(M)} &= \operatorname*{ess\,sup}_{x \in M} |F^*(\omega)|(x) \overset{1.2.10}{\leq} \binom{n}{k} C_1^k \operatorname*{ess\,sup}_{x \in M} (|\omega| \circ F)(x) \\ &= \binom{n}{k} C_1^k \operatorname*{ess\,sup}_{y \in F(M)} |\omega|(y) \leq \binom{n}{k} C_1^k \, \|\omega\|_{L_\infty(N)}.\end{aligned}$$

(ii) In this case F is zero-preserving by 1.3.4. For any $\omega \in L^k_p(N)$ we calculate

$$\int_M |F^*(\omega)|^p d_gV \overset{1.2.10}{\leq} \binom{n}{k}^p C_1^{kp} \int_M |\omega|^p \circ F d_gV \overset{1.3.4}{\leq} \binom{n}{k}^p C_1^{kp} C_2^m \int_N |\omega|^p dV_h.$$

Since F^{-1} is a (C_2, C_1)-bounded diffeomorphism, the result follows.

(iii) In this case, we have to sharpen the inequalities derived so far: For any $p \in M$

$$\begin{aligned}\operatorname{Jac} F(p) &\overset{1.3.1}{=} \frac{\operatorname{vol}(F_*|_p(X_1) \wedge \ldots \wedge F_*|_p(X_m))}{\operatorname{vol}(X_1 \wedge \ldots X_m)} \\ &\overset{1.3.2}{=} \frac{\sqrt{\det(\langle F_*|_p(X_i), F_*|_p(X_j)\rangle}}{\sqrt{\det(\langle X_i, X_j\rangle)}} \overset{1.2.12}{=} 1,\end{aligned} \tag{2.10}$$

since F is an isometry. Therefore in case $1 \leq p < \infty$

$$\begin{aligned}\|F^*(\omega)\|^p_{L_p(M)} &= \int_M |F^*(\omega)|^p d_gV \overset{1.2.12}{=} \int_M |\omega|^p \circ F d_gV \\ &\overset{(2.10)}{=} \int_M |\omega|^p \circ F \operatorname{Jac} F d_gV \overset{1.3.3}{=} \int_N |\omega|^p dV_h = \|\omega\|^p_{L_p(N)}\end{aligned}$$

and in case $p = \infty$

$$\begin{aligned}\|F^*(\omega)\|_{L_\infty(M)} &= \operatorname*{ess\,sup}_{x \in M} |F^*(\omega)|(x) \overset{1.2.12}{=} \operatorname*{ess\,sup}_{x \in M} (|\omega| \circ F)(x) \\ &= \operatorname*{ess\,sup}_{y \in N} |\omega|(y) = \|\omega\|_{L_\infty(N)}.\end{aligned}$$

□

2.1.29 Theorem. Let $F : (M, g) \to (N, h)$ be a diffeomorphism.

(i) For any $\omega \in W^k_{p,\text{loc}}(N)$, the form $F^*\omega$ is weakly differentiable and satisfies

$$(d \circ F^*)(\omega) = (F^* \circ d)(\omega).$$

In particular $F^* : W_{p,\text{loc}}(N) \to W_{p,\text{loc}}(M)$.

(ii) If in addition F is (C_1, C_2)-bounded, then $F^* : W_p(N) \to W_p(M)$ is a Banach space isomorphism.

(iii) If F is an isometry, then $F^* : W_p(N) \to W_p(M)$ is an isometry.

Proof.

(i) By definition of the weak differential and the diffeomorphism invariance of the integral, any $\eta \in \Omega^{m-k-1}_c(M)$ satisfies

$$\begin{aligned}(-1)^{k+1} \int_M F^*(d\omega) \wedge \eta &= (-1)^{k+1} \int_M F^*(d\omega \wedge F^{-1*}(\eta)) \\ &= (-1)^{k+1} \int_N \pm d\omega \wedge F^{-1*}(\eta) = \pm \int_N \omega \wedge dF^{-1*}(\eta) \\ &= \pm \int_N \omega \wedge F^{-1*}(d\eta) = \int_M F^*(\omega \wedge F^{-1*}(d\eta)) \\ &= \int_M F^*(\omega) \wedge d\eta,\end{aligned}$$

thus $dF^*(\omega) = F^*(d\omega)$ in the weak sense. In particular $F^*(\omega) \in W^k_{p,\text{loc}}(M)$.

(ii) This follows by combining (i) with 2.1.28.

(iii) By Theorem 2.1.28 $F^* : L_p(N) \to L_p(M)$ is an isometry and by (ii) it restricts to a map $F^* : W_p(N) \to W_p(M)$, which commutes with d. Consequently for any $1 \leq p < \infty$

$$\begin{aligned}\|F^*(\omega)\|^p_{W^k_p(M)} &= \|F^*(\omega)\|^p_{L_p(M)} + \|dF^*(\omega)\|^p_{L_p(M)} \\ &= \|\omega\|^p_{L_p(N)} + \|F^*(d\omega)\|^p_{L_p(M)} = \|\omega\|^p_{W_p(N)}\end{aligned}$$

and for $p = \infty$

$$\begin{aligned}\|F^*(\omega)\|_{W^k_\infty(M)} &= \max\{\|F^*(\omega)\|_{L_\infty(M)}, \|dF^*(\omega)\|_{L_\infty(M)}\} \\ &= \max\{\|\omega\|_{L_\infty(M)}, \|F^*(d\omega)\|_{L_\infty(M)}\} = \|\omega\|_{W_\infty(N)}.\end{aligned}$$ □

2.1.30 Theorem (isometry Invariance). Assume $F : (M, g) \to (N, h)$ is an isometry between Riemannian manifolds. Then F induces an isometry $[F] : H_p(N) \to H_p(M)$ of semi-normed spaces.

Proof. By Theorem 2.1.29, F induces isometries on the cochain level $W_p(N) \to W_p(M)$. Now consider any cohomology class $[\omega] \in H^k_p(N)$ and calculate

$$\begin{aligned}\|[F]([\omega])\|_{H^k_p(M)} &= \|[F^*(\omega)]\|_{H^k_p(M)} = \inf_{\eta \in W^{k-1}_p(M)} \|F^*(\omega) - d\eta\|_{W^k_p(M)} \\ &\overset{2.1.29}{=} \inf_{\eta \in W^{k-1}_p(M)} \|F^*(\omega) - F^*(dF^{-1^*}(\eta))\|_{W^k_p(M)} \\ &= \inf_{\eta \in W^{k-1}_p(M)} \|\omega - dF^{-1^*}(\eta)\|_{W^k_p(N)} \\ &= \inf_{\phi \in W^{k-1}_p(N)} \|\omega - d\phi\|_{W^k_p(N)} = \|[\omega]\|_{H^k_p(N)}.\end{aligned}$$ □

2.2 L_p-cohomology of simplicial complexes

In this subsection we will introduce the L_p-cohomology of simplicial complexes. First, we unfortunately will have to establish lots of notation concerning simplices and simplicial complexes in $\mathbb{R}^n$. We will assume the reader to be familiar with these constructions and try to keep this as brief as possible. Second, we have to pass from the classical simplicial homology to cohomology. This is done by dualization and has conceptual reasons: It would be strange to compare a homology theory with cohomology theories. Third, we will introduce L_p-norms on cochains and obtain our desired L_p-cohomology theory.

2.2.1 Simplicial Complexes

2.2.1 Definition (simplex). The $k+1$ points $x_0, \ldots, x_k \in \mathbb{R}^n$ are in *general position*, if the set $\{x_1 - x_0, \ldots, x_k - x_0\} \subset \mathbb{R}^n$ is linearly independent. In that case their convex hull

$$\sigma := \langle x_0, \ldots, x_k \rangle := \left\{ \sum_{i=0}^k \beta_i x_i \;\middle|\; \forall 0 \leq i \leq k : \beta_i \in [0,1] \text{ and } \sum_{i=0}^k \beta_i = 1 \right\} \subset \mathbb{R}^n$$

is a *simplex* in $\mathbb{R}^n$ *spanned* by the *vertices* $x_0, \dots, x_k$. The integer k is the *dimension* of σ and we also say that σ is a k-simplex. The tupel $(\beta_0, \dots, \beta_k)$ associated to a point $x = \sum_{i=0}^k \beta_k x_k \in \sigma$ are the *barycentric coordinates of* x. Since $x_0, \dots, x_k$ are in general position, the barycentric coordinates of x are well-defined.

Any simplex σ can be seen as a topological space $|\sigma|$ by endowing it with the subspace topology inherited from $\mathbb{R}^n$. The vertices of σ are characterized as all the points of $|\sigma|$, which are not the midpoint of a line in $|\sigma|$ that is not itself a point. Therefore the tuple of vertices of σ may be recovered from $|\sigma|$ up to its order. Notice that for any permutation $\pi \in \mathfrak{S}_{k+1}$ and any k-simplex $\sigma = \langle x_0, \dots, x_k \rangle$, we always have

$$\pi\sigma := \langle x_{\pi(0)}, \dots, x_{\pi(k)} \rangle = \langle x_0, \dots, x_k \rangle$$

as an equality of sets. But the barycentric coordinate functions $\pi(\beta)_0, \dots, \pi(\beta)_k$ of $\pi\sigma$ satisfy

$$(\beta_{\pi(0)}, \dots, \beta_{\pi(k)}) = (\pi(\beta)_0, \dots, \pi(\beta)_k)$$

as an equality of tuples. We say σ is a *topological simplex*, if we want to stress the fact that we regard it only as a topological space with no canonical order of the vertices.

A simplex τ spanned by any subset of $\{x_0, \dots, x_k\}$ is a *face* of σ, which we denote by $\tau \leq \sigma$. If $\tau \neq \sigma$, we call τ a *proper face* of σ and denote $\tau < \sigma$. The $(k-1)$-dimensional faces of σ are the *boundary faces*. In particular $\partial^i \sigma := \langle x_0, \dots, \hat{x}_i, \dots x_k \rangle := \langle x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_k \rangle$ is the *i-th boundary face* of σ .

2.2.2 Definition (oriented simplex). Denote by $\mathfrak{S}_{k+1}$ the *group of permutations on* $\{0, \dots, k\}$. Then $\operatorname{sgn} : \mathfrak{S}_{k+1} \to \{+1, -1\}$ is a group homomorphism and its kernel $\mathfrak{A}_{k+1}$, the *alternating group*, is a normal subgroup of index two. Consequently, the quotient $\mathfrak{O}_{k+1} := \mathfrak{S}_{k+1}/\mathfrak{A}_{k+1}$ is a group consisting of the two equivalance classes $[+1_{k+1}], [-1_{k+1}]$. These are called *orientations*. Denote by $\delta_{k+1} : \mathfrak{O}_{k+1} \to \mathfrak{O}_{k+1}$ the uniquely determined nontrivial group homomorphism, by δ_{k+1}^i its i-th power, and by $\partial_{k+1} : \mathfrak{O}_{k+1} \to \mathfrak{O}_k$, $[\pm 1_{k+1}] \mapsto [\pm 1_k]$. For any $0 \leq i \leq k+1$, the map $\partial^i := \partial_{k+1}^i := \delta_k^i \circ \partial_{k+1} : \mathfrak{O}_{k+1} \to \mathfrak{O}_k$ is the *i-th boundary map*.

Let $x_0, \dots, x_k \in \mathbb{R}^n$ be in general position. An *oriented simplex* is a tuple $(\sigma, [\pi])$ consisting of a simplex $\sigma = \langle x_0, \dots, x_k \rangle$ and an orientation $[\pi] \in \mathfrak{O}_{k+1}$. For any $0 \leq i \leq k$, we call

$$\partial^i(\sigma, [\pi]) := (\partial^i \sigma, \partial^i [\pi])$$

the *i-th boundary of* $(\sigma, [\pi])$ *with induced orientation.* This inductively induces an orientation on all the faces of the simplex.

For any $\pi \in \mathfrak{S}_{k+1}$, we call $(x_{\pi(0)}, \dots, x_{\pi(k)})$ an *ordering of the vertices.* Two orderings $(x_{\pi(0)}, \dots, x_{\pi(k)})$, $(x_{\pi'(0)}, \dots, x_{\pi'(k)})$ are equivalent, if $[\pi] = [\pi'] \in \mathfrak{O}_{k+1}$. Define $[x_{\pi(0)}, \dots, x_{\pi(k)}] := (\langle x_0, \dots, x_k \rangle, [\pi])$ and $[x_{\pi(0)}, \dots, x_{\pi(k)}]^{-1} := (\langle x_0, \dots, x_k \rangle, \delta_{k+1}([\pi]))$.
If $\tau_i := \langle y_0, \dots, y_{k-1} \rangle := \langle x_0, \dots, \hat{x}_i, \dots, x_k \rangle$ is the i-th boundary face, the induced orientation may be written with this notation by

$$\partial^i [x_0, \dots, x_k] = [x_0, \dots, \hat{x}_i, \dots, x_k]^{(-1)^i} = (\tau_i, \delta_k^i([+1_k])).$$

2.2.3 Definition (incidence coefficient). For any oriented $(k+1)$-simplex $[\sigma]$ and any oriented k-simplex $[\tau]$ the integer

$$[\sigma : \tau] := \begin{cases} +1, & \tau \leq \sigma \textit{ and } [\tau] \textit{ has the orientation induced by } [\sigma], \\ -1, & \tau \leq \sigma \textit{ and } [\tau] \textit{ has the orientation opposite to the one induced by } [\sigma] \ , \\ 0, & \text{otherwise,} \end{cases}$$

is the *incidence coefficient of* $[\tau]$ *and* $[\sigma]$. In case τ and σ are not oriented, this number is just supposed to be $+1$ if $\tau \leq \sigma$ and 0 otherwise.

2.2.4 Definition (standard simplex). Let $k \in \mathbb{N}$, define $e_0 := 0 \in \mathbb{R}^k$ and let $e_i := (0, \dots, 1, \dots, 0) \in \mathbb{R}^k$, $1 \leq i \leq k$, be the canonical basis of $\mathbb{R}^k$. The set

$$\Delta_k := \langle e_0, \dots, e_k \rangle \subset \mathbb{R}^k$$

is the *k-dimensional* standard simplex. If we require it to be oriented, we assume that $(\Delta_k, [\mathrm{id}_{k+1}])$ is the chosen orientation. We also consider it as a subset of all $\mathbb{R}^n$ with $n \geq k$. Besides beeing a subset of $\mathbb{R}^k$, we consider Δ_k as a smooth manifold with corners.

2.2.5 Definition (standard atlas). Let $\sigma := \langle x_0, \dots, x_k \rangle$ be a k-simplex. For any fixed $0 \leq i \leq k$ the map $B^i : |\sigma| \to \Delta_k$,

$$B^i(x) := B^i\Big(\sum_{j=0}^{k} \beta_j x_j\Big) = (\beta_0, \dots, \widehat{\beta_i}, \dots \beta_k) \in \mathbb{R}^k$$

is a *standard chart on* σ. These assemble to a smooth atlas.

2.2.6 Definition (simplicial complex). A countable set K of topological simplices in some $\mathbb{R}^n$ is a *simplicial complex*, if the following conditions are satisfied:

(i) For any $\sigma \in K$: $\tau \leq \sigma \Longrightarrow \tau \in K$.

(ii) For any $\sigma, \tau \in K$: $\sigma \cap \tau \neq \emptyset \Longrightarrow \sigma \cap \tau \leq \sigma$.

For any integer k we define the *k-skeleton* K^k to be the set of all simplices in K with dimension less or equal to k, and $K^{(k)}$ to be the set of all simplices in K with dimension precisely k. Since $K^k = 0$, if $k > n$, there is a well-defined number

$$\dim K := \max\{k \in \mathbb{N} \mid K^{k+1} = K^k\},$$

the *dimension* of K.
A subset $L \subset K$ is a *subcomplex* if L is itself a simplicial complex. In that case, we call (K, L) a *pair of complexes.*
Notice that the 0-simplices of a complex K are precisely the vertices of all the simplices in K. We will assume that the vertices of K are labeled $\{x_i\}_{i \in \mathbb{N}}$.

2.2.7 Definition (closure / star / link). Let K be a simplicial complex and $S \subset K$ be an arbitrary subset.

(i) The *closure* of S is the smallest simplicial complex $\mathrm{cl}(S)$ such that $S \subset \mathrm{cl}(S)$.

(ii) The *star* of S in K is the set

$$\mathrm{st}(S) := \mathrm{st}_K(S) := \{\sigma \in K \mid \exists \tau \in S : \tau \leq \sigma\}.$$

See figure 2.2.1 for a visualization.

(iii) The *link* of S in K is

$$\mathrm{lk}(S) := \mathrm{lk}_K(S) := \mathrm{cl}(\mathrm{st}_K(S)) \setminus \mathrm{st}_K(S).$$

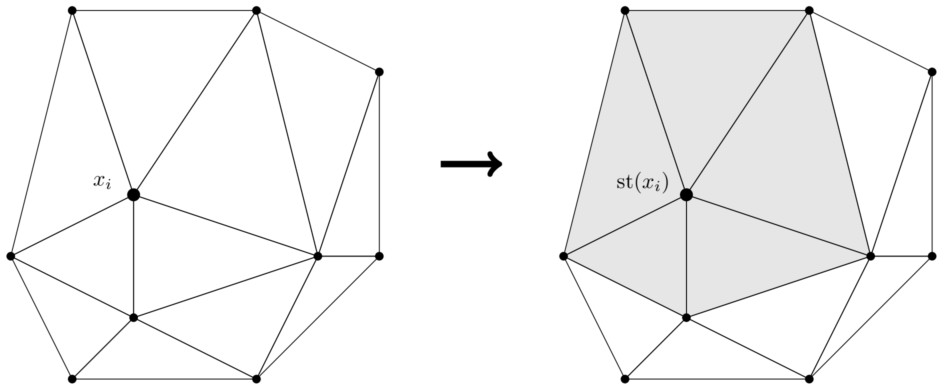


Figure 2.1: Forming stars

2.2.8 Definition (star-bounded). A simplicial complex K is *star-bounded* with *star bound* N, if the stars of all the simplices in K contain no more than N simplices, i.e.

$$\exists N \in \mathbb{N} : \forall \sigma \in K : \sharp \operatorname{st}_K(\sigma) \leq N.$$

2.2.9 Definition (geometric realization). Let K be a simplicial complex in $\mathbb{R}^n$. The set

$$|K| := \bigcup_{\sigma \in K} |\sigma| \subset \mathbb{R}^n$$

is the *geometric realization* of K.
A subset $P \subset \mathbb{R}^n$ is a *polyhedron*, if there exists a simplicial complex K such that $P = |K|$.

2.2.10 Definition (triangulation). A topological space X is *triangulable*, if there exists a simplicial complex K and a homeomorphism $h : |K| \to X$. The homeomorphism h is a *triangulation.*
If M is a smooth manifold, we say M is *smoothly triangulable*, if there exists a triangulation $h : |K| \to M$ such that for any $\sigma \in K$, h restricts to a smooth map $|\sigma| \to M$ between manifolds with corners. We call h a *smooth triangulation.*

2.2.11 Definition (barycentric coordinate functions). Let K be a simplicial complex and $\{(x_i)_{i\in\mathbb{N}}\}$ be its vertices. On every single simplex $\sigma = \langle x_{i_0}, \ldots, x_{i_k}\rangle \in K$ the baricentric coordinates define functions $\beta_{i_0}, \ldots, \beta_{i_k} : |\sigma| \to [0,1]$. The collection of all these functions $\beta_i : |K| \to [0,1]$ are the *barycentric coordinate functions of* K, where we set $\beta_i(x) := 0$, if $x \in |K| \setminus \operatorname{st}(x_i)$.

2.2.12 Lemma (properties of barycentric coordinate functions). The barycentric coordinate functions defined above satisfy

$$\operatorname{supp} \beta_i \subset \operatorname{st}(x_i), \qquad \sum_{i\in\mathbb{N}} \beta_i = 1, \qquad \sum_{i\in\mathbb{N}} d\beta_i = 0,$$

where all the sums are locally finite.

2.2.13 Definition (barycentric subdivision). Let $\sigma = \langle x_0, \ldots, x_k \rangle \subset \mathbb{R}^n$ be a k-simplex. Then

$$b_\sigma := \frac{1}{k+1} \sum_{i=0}^{k} x_i \in \operatorname{Int} \sigma$$

is the *barycenter of* σ.
For any other point $x \in \mathbb{R}^n$ we define

$$x * \sigma := \langle x, x_0, \ldots, x_k \rangle$$

to be the *cone* on σ from x (provided $\{x, x_0, \ldots, x_k\}$ are in general position as well).
For any simplicial complex K we the define the complex $B(K)$, called the *barycentric subdivision of* K, inductively as follows: If $\dim K = 0$, then $B(K) := K$. Now assume $B(K)$ has been defined for all simplicial complexes of dimension $\leq k$. Define

$$B(K) := B(K^k) \cup \bigcup_{\sigma \in K^{(k+1)}, \tau < \sigma} b_\sigma * \tau,$$

where K is a simplicial complex of dimension $k+1$. We sometimes say $B(K)$ is the *first barycentric subdivision of* K and $B^l(K) := B(B(\ldots B(K)))$ is the *l-th barycentric subdivision of* K.

2.2.2 Simplicial maps

We would like to construct a category of simplicial complexes. Therefore we have to define an appropriate class of morphisms. In this we will roughly follow some exercises in [15, 5.4,5.2].

2.2.14 Definition (affine). A map $F : X \to Y$ between vector spaces is *affine*, if there exist a linear map $A : X \to Y$ and $b \in Y$ such that $F = A + b$. We say F is an *affine isomorphism* if F is affine and A is an isomorphism (we will see below that the decomposition $F = A + b$ is unique and therefore this is well-defined.)

2.2.15 Lemma. Let $F : X \to Y$ be affine.

(i) The representation $F = A + b$ is unique and may be recovered from F by

$$b = F(0) \qquad\qquad A = F - b = F - F(0).$$

(ii) Any linear map is affine.

(iii) A composition of affine maps is affine. If $F = A + b : X \to Y$, $G = B + c : Y \to Z$ are affine, the composition $G \circ F : X \to Y$ is given by

$$G \circ F = B \circ A + B(b) + c.$$

(iv) If F is an affine isomorphism, it is bijective and the inverse $G : Y \to X$ is an affine isomorphism as well, which has the representation

$$G = A^{-1} - A^{-1}(b).$$

Proof.

(i) Since A is linear, $b = A(0) + b = F(0)$, thus by definition $A = F - b = F - F(0)$.

(ii) Just choose $b = 0$.

(iii) We calculate for any $x \in X$

$$(G \circ F)(x) = G(A(x) + b) = B(A(x)) + B(b) + c.$$

(iv) Let $x \in X$ and $y := F(x)$. This implies

$$y = F(x) = A(x) + b \Leftrightarrow y - b = A(x) \Leftrightarrow x = A^{-1}(y) - A^{-1}(b). \qquad \square$$

2.2.16 Definition (simplicial map). A map $f : \sigma \to \tau$, where $\sigma, \tau \subset \mathbb{R}^n$ are a k- and an l-simplex, is *simplicial*, if there exists an affine map $F : \mathbb{R}^n \to \mathbb{R}^n$ such that $F|_\sigma = f$ and $F(\mathrm{cl}(\sigma)^{(0)}) \subset \mathrm{cl}(\tau)^{(0)}$ (i.e. F maps the vertices of σ to vertices of τ).

2.2.17 Lemma (properties of simplicial maps). Let $\sigma, \tau \subset \mathbb{R}^n$ be a k- and an l-simplex.

(i) For any map $f^{(0)} : \mathrm{cl}(\sigma)^{(0)} \to \mathrm{cl}(\tau)^{(0)}$ there exists a unique simplicial map $f : \sigma \to \tau$ such that $f|_{\mathrm{cl}(\sigma)^{(0)}} = f^{(0)}$.

(ii) In case $k = l$ there exists a simplicial homeomorphism $f : \sigma \to \tau$.

Proof. Let $\sigma = \langle \sigma_0, \dots, \sigma_k \rangle$, $\tau = \langle \tau_0, \dots, \tau_l \rangle$.

(i) We may identify the map $f^{(0)} : \{\sigma_0, \dots, \sigma_k\} \to \{\tau_0, \dots, \tau_l\}$ with the corresponding map $f_0 : \{0, \dots, k\} \to \{0, \dots, l\}$ defined by the relation $f^{(0)}(\sigma_i) = \tau_{f_0(i)}$, $1 \le i \le k$.

STEP 1 (uniqueness): Assume $f = F|_\sigma$ is simplicial and $F = A + b : \mathbb{R}^n \to \mathbb{R}^n$ is affine such that $F|_{\mathrm{cl}(\sigma)^{(0)}} = f^{(0)}$. Let $x = \sum_{i=0}^k \beta_i \sigma_i \in \sigma$ be arbitrary. We calculate

$$\begin{aligned} f(x) &= F\Big(\sum_{i=0}^k \beta_i \sigma_i\Big) = A\Big(\sum_{i=0}^k \beta_i \sigma_i\Big) + b = \sum_{i=0}^k \beta_i A(\sigma_i) + b \\ &= \sum_{i=0}^k \beta_i (A(\sigma_i) + b) - \sum_{i=0}^k \beta_i b + b = \sum_{i=0}^k \beta_i F(\sigma_i) = \sum_{i=0}^k \beta_i f^{(0)}(\sigma_i) \in \tau. \end{aligned}$$

STEP 2 (existence): We will construct the desired map F step by step.

$$\begin{array}{ccc} \sigma & \xrightarrow{F} & \tau \\ {\scriptstyle G_\sigma}\downarrow & & \uparrow{\scriptstyle F_\tau} \\ \Delta_k & \xrightarrow{F_\Delta} & \Delta_l \end{array}$$

Let $A_\sigma : \mathbb{R}^k \to \mathbb{R}^n$ be the map $e_i \mapsto \sigma_i - \sigma_0$. This map is an isomorphism onto its image V. Let $\pi : \mathbb{R}^n \twoheadrightarrow V$ be the canonical projection and define $G_\sigma : \mathbb{R}^n \to \mathbb{R}^k$ by

$$G_\sigma(x) := A_\sigma^{-1}(\pi(x - \sigma_0)) = A_\sigma^{-1}(\pi(x)) - A_\sigma^{-1}(\pi(\sigma_0))$$

Then G_σ is an affine map, satisfying $G_\sigma(\sigma_i) = e_i$. By uniqueness, this implies that it sends σ to $\Delta_k \subset \mathbb{R}^k$. Define $F_\tau = A_\tau + b_\tau : \mathbb{R}^l \to \mathbb{R}^n$ by setting $A_\tau(e_j) := \tau_j - \tau_0$, $1 \le j \le l$, $b_\tau := \tau_0$. Then $F_\tau|_{\Delta_l} : \Delta_l \to \tau$ is a simplicial homeomorphism. Define the linear map $F_\Delta : \mathbb{R}^k \to \mathbb{R}^l$ by setting $F_\Delta(e_i) := e_{f_0(i)}$, $1 \le i \le k$. The composition $F := F_\tau \circ F_\Delta \circ G_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is affine and satisfies

$$F(\sigma_i) = F_\tau(F_\Delta(G_\sigma(\sigma_i))) = F_\tau(F_\Delta(e_i)) = F_\tau(e_{f_0(i)}) = \tau_{f_0(i)} - \tau_0 + \tau_0 = f^{(0)}(\sigma_i).$$

By uniqueness $f := F|_\sigma : \sigma \to \tau$ is the simplicial map we are looking for.

(ii) We just define $f^{()} : \mathrm{cl}(\sigma)^{(0)} \to \mathrm{cl}(\tau)^{(0)}$, $\sigma_i \mapsto \tau_i$, $1 \le i \le k = l$. This map is obviously bijective. Denote its inverse by $g^{()}$. Both maps induce a unique simplicial map $f : \sigma \to \tau$, $g : \tau \to \sigma$. Clearly $g \circ f : \sigma \to \sigma$ is a simplicial map such that $(g \circ f)^{(0)} := \mathrm{id} : \mathrm{cl}(\sigma)^{(0)} \to \mathrm{cl}(\sigma)^{(0)}$. Now $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ is clearly affine and its restriction to $\mathrm{cl}(\sigma)^{(0)}$ is $(g \circ f)^{(0)}$. By the uniqueness, this implies $g \circ f = \mathrm{id} : |\sigma| \to |\sigma|$. The same holds for $f \circ g$ and therefore the statement is proven. □

2.2.18 Definition (simplicial map). Let K, L be simplicial complexes. A continous map $f : |K| \to |L|$ whith the property that for every $\sigma \in K$ there exists $\tau \in L$ such that $f|_\sigma : \sigma \to \tau$ is a simplicial map in the sense of 2.2.16, is a *simplicial map (between complexes)*. In that case we say the induced map $f^{(0)} : K^{(0)} \to L^{(0)}$ is the *vertex map of* f. We say K and L are (simplicially) isomorphic, if there exists a simplicial homeomorphism $f : |K| \to |L|$.

2.2.19 Lemma (properties of simplicial maps). Let K, L be simplicial complexes.

(i) Let $f_0 : K^{(0)} \to L^{(0)}$ be any map with the property that whenever $\{x_{i_0}, \dots, x_{i_k}\}$ are the vertices of a simplex in K, $\{f_0(x_{i_0}), \dots, f_0(x_{i_k})\}$ are the vertices of a simplex in L (possibly with repetitions). Then there exists a unique simplicial map $f : K \to L$ such that $f^{(0)} = f_0$.

(ii) Let f_0 be as above with the additional property that f_0 is bijective and $\{x_{i_0}, \dots, x_{i_k}\}$ are vertices of a simplex in K if and only if $\{f_0(x_{i_0}), \dots, f_0(x_{i_k})\}$ are vertices of a simplex in L. Then K and L are isomorphic.

Proof.

(i) The hypothesis ensures that for any $\sigma \in K$, there exists $\tau \in L$ such that $f_0(\mathrm{cl}(\sigma^{(0)})) \subset \mathrm{cl}(\tau)^{(0)}$. Thus by Lemma 2.2.17 there exists a unique simplicial map $f_\sigma : \sigma \to \tau$ such that $f_\sigma^{(0)} = f_0|_{\mathrm{cl}(\sigma^{(0)})}$. Define $f : |K| \to |L|$ by setting $f(x) := f_\sigma(x)$ if $x \in \mathrm{Int}\,\sigma$. Since $|K|$ is the disjoint union of the interior of its simplices (if $\dim \sigma = 0$, then $\mathrm{Int}\,\sigma = \sigma$), this is well-defined. By construction this map is simplicial. It is continuous, since whenever we consider interior points of a simplex, f is the restriction of an affine, hence continuous, map. If two simplices meet at a common face $\rho \in \sigma \cap \tau$, then $f_\sigma|_\rho = f_\rho = f_\tau|_\rho$. Thus f is globally continuous.

(ii) This follows directly from the hypothesis, part (i) and Lemma 2.2.17(ii). □

2.2.20 Corollary. On a star bounded simplicial complex there exists only a finite number of isomorphism classes of stars of simplices.

Proof. Let K be a star-bounded simplicial complex of dimension n with star bound N and consider a simplex $\sigma \in K$. The complex $\mathrm{cl}(\mathrm{st}_K(\sigma))$ is finite as well and contains by definition less than N simplices of dimension less or equal to n. Consider all simplicial complexes built out of less than N standard simplices of dimension less or equal n. These are only finitely many and any $\mathrm{cl}(\mathrm{st}_K(\sigma))$ has to be isomorphic to at least one of them by successive application of Lemma 2.2.17,(ii). □

2.2.21 Definition (galactic cover). Let $\{x_i\}_{i \in \mathbb{N}}$ be a fixed counting of the vertices of the star-bounded simplicial complex K. A simplicial isomorphism class $[\mathrm{st}_K(x_i)]$ of a star

of a vertex $x_i \in K$ is a *galaxy.* If $\mathrm{st}_K(x_j) \in [\mathrm{st}_K(x_i)]$, we say that x_i and x_j (or even that i and j) *belong to the same galaxy.*
Let K be a star-bounded simplicial complex with star bound N. As we just pointed out in 2.2.20 the set $\{[\mathrm{st}_K(x)]\}_{x\in K^{(0)}}$ of galaxies is finite. Therefore there exists a finite representation system, i.e. some number $G = G(N)$ and vertices $x_1, \dots, x_G \in K^{(0)}$, such that all the galaxies of K are given by $[\mathrm{st}_K(x_1)], \dots, [\mathrm{st}_K(x_G)]$. Since

$$K = \bigcup_{\nu=1}^{G} [\mathrm{st}_K(x_\nu)],$$

we call the later one a *galactic cover* of K.

2.2.3 Simplicial cohomology

2.2.22 Definition (simplicial homology). Let K be a simplicial complex and R be a commutative ring with unit. For any set S let $R\langle S\rangle$ be the free module generated by S over R. Define

$$C_k(K, R) := R\langle\{(\sigma, [\pm 1_k]) \mid \sigma \in K^{(k)}\}\rangle/\!\sim,$$

where $(\sigma, [+1_k]) \sim -(\sigma, [-1_k])$. In other words: We take all the topological simplices in K, choose both possible orientations, take all these oriented simplices, form the free module and then identify. The module $C_k(K, R)$ is the *k-th simplicial chain group of K with coefficients in R.*
For any $\sigma = \langle x_{i_0}, \dots, x_{i_k}\rangle \in K$, we denote by $[\sigma]$ the oriented simplex obtained by defining $[\sigma] := (\sigma, [+1])$, if $i_0 < \dots < i_k$. We will make no notational distinction between an oriented simplex $[\sigma]$ and the equivalence class $[\sigma] \in C_k(K, R)$, i.e. we will write $[\sigma^{-1}] = -[\sigma]$.
The map $d_k : C_k(K, R) \to C_{k-1}(K, R)$ is defined as the linear extension of

$$[\sigma] = [x_{i_0}, \dots, x_{i_k}] \mapsto \sum_{\nu=0}^{k} (-1)^\nu [x_{i_0}, \dots, \hat{x}_{i_\nu}, \dots, x_{i_k}].$$

These groups and maps assemble to a chain complex of R-modules (C_*, d_*). The homology groups

$$H_*(K, R) := H_*(C_*(K, R))$$

are the *simplicial homology groups with coefficients in R.* We will write $C_*(K) := C_*(K, \mathbb{R})$.

2.2.23 Definition (simplicial cohomology). Let K be a simplicial complex and R be a commutative ring with unit. We call

$$C^k(K, R) := \mathrm{Hom}_R(C_k(K, R), R)$$

the *k-th simplicial cochain group of K with coefficients in R* . The map $d^k : C^k(K, R) \to C^{k+1}(K, R)$ is defined by

$$d^k(c^k)([\sigma_{k+1}]) := c_k(d_k([\sigma_{k+1}])).$$

These groups and maps assemble to a cochain complex of R-modules (C^*, d^*). The cohomology groups

$$H^*(K, R) := H^*(C^*(K, R))$$

are the *simplicial cohomology groups with coefficients in R*. We will always write $C^*(K) := C^*(K, \mathbb{R})$. For any subcomplex $L \subset K$ we call

$$C^k(K, L) := \{c^k \in C^k(K) \mid \forall \sigma \in L : c^k([\sigma]) = 0\}$$

the *simplicial cochains relative L*. These assemble to a cochain complex $C^*(K, L)$ as well and its cohomology

$$H^*(K, L) := H^*(C^*(K, L))$$

is the *simplicial cohomology relative L*.

2.2.24 Convention (dualized bases). We would like to identify elements in the chain groups with elements in the cochain groups. In our standard situation K will represent a triangulation of a noncompact manifold, which implies that it is not finite. So caution should be exercised.

Let V be a possibly infinite dimensional $\mathbb{R}$-vector space. Zorn's Lemma still provides us with a basis $B = (b_i)_{i \in I}$ of V. We define the *dualized basis* $B^* := (b^i)_{i \in I}$, where $b^i \in V^*$ is the unique linear extension, defined by $b^i(b_j) := \delta^i_j$. It is a well know fact from linear algebra that if I is finite, B^* is in fact a basis for V^*, usually called the *dual basis*. In that case, any element $\varphi \in V^*$ has a unique representation

$$\varphi = \sum_{i \in I} \varphi(b_i) b^i.$$

It is also a well-known fact that this is wrong if I is not finite. Consider for example the element $\varphi \in V^*$, defined by the linear extension of $\varphi(b_i) := 1$, $i \in I$. In that case, the sum above is not finite and the equation makes no sense in V^*. However the dualized basis B^* still exists even if it is not a basis for V^* anymore. Since B is a basis of V, any $v \in V$ has a unique representation

$$v = \sum_{j \in I} v^j b_j = \sum_{j \in I} b^j(v) b_j,$$

where all, but finitely many v^j are zero. Denote by $I(v) \subset I$ the finite subset of all indices, where $v^j \neq 0$. This implies that for any $i \in I$

$$b^i(v) = b^i\Big(\sum_{j \in I} v^j b_j\Big) = \sum_{j \in I} v^j b^i(b_j) = \sum_{j \in I(v)} v^j \delta^i_j.$$

Therefore $i \notin I(v) \Rightarrow b^i(v) = 0$. Thus $b^i(v) = 0$ for all but finitely many $i \in I$. This gives rise to the following construction: For any system $(\lambda_i \in \mathbb{R})_{i \in I}$, we define $\sum_{i \in I} \lambda_i b^i$ to be the map $V \to \mathbb{R}$, given by

$$v \mapsto \Big(\sum_{i \in I} \lambda_i b^i\Big)(v) := \sum_{i \in I} \lambda_i b^i(v) = \sum_{i \in I(v)} \lambda_i b^i(v).$$

We have just shown that the sum on the right-hand side is always finite and thus well-defined. It is clear from this definition that $\sum_{i \in I} \lambda_i b^i \in V^*$. For any $\varphi \in V^*$, the equation

$$\varphi = \sum_{i \in I} \varphi(b_i) b^i : V \to \mathbb{R} \tag{2.11}$$

is valid in the sense that

$$\forall v \in V : \Big(\sum_{i \in I} \varphi(b_i) b^i\Big)(v) = \sum_{i \in I(v)} \varphi(b_i) b^i(v) = \sum_{i \in I(v)} \varphi(b_i) v^i = \varphi\Big(\sum_{i \in I(v)} v^i b_i\Big) = \varphi(v).$$

If $[\sigma] \in C_k(K)$ is a generator, we denote by $[\sigma]^* \in C^k(K)$ its corresponding dual.

2.2.25 Lemma (coboundary formula). Let K be a simplicial complex in $\mathbb{R}^n$ with vertices $(x_i)_{i\in\mathbb{N}}$. Let $[\sigma] = [x_{i_0}, \dots, x_{i_k}] \in C_k(K)$, $[\tau] = [x_{j_0}, \dots, x_{j_{k+1}}] \in C_{k+1}(K)$ and $I' := I \setminus \{i_0, \dots, i_k\}$. Then

$$d([\sigma]^*)([\tau]) = \begin{cases} (-1)^r & , \text{ if } \{i_0, \dots, i_k\} \subset \{j_0, \dots, j_{k+1}\} \\ 0 & , \text{ otherwise} \end{cases}, \tag{2.12}$$

where $0 \leq r \leq k+1$ such that $(i_0, \dots, i_k) = (j_0, \dots, \widehat{j_r}, \dots, j_{k+1})$. With Convention 2.2.24 in power, we may expand

$$d[\sigma]^* = \sum_{i \in I'} [x_i, x_{i_0}, \dots, x_{i_k}]^*.$$

Proof. Consider

$$d([\sigma]^*)([\tau]) = [\sigma]^*(d([\tau])) = \sum_{\nu=0}^{k+1} (-1)^\nu [x_{i_0}, \dots, x_{i_k}]^* \left([x_{j_0}, \dots, \widehat{x_{j_\nu}}, \dots, x_{j_{k+1}}]\right).$$

This expression is zero by definition, unless $\{i_0, \dots, i_k\} \subset \{j_0, \dots, j_{k+1}\}$. Otherwise there exists $0 \leq r \leq k+1$ such that

$$[x_{i_0}, \dots, x_{i_k}] = [x_{j_0}, \dots, \widehat{x_{j_r}}, \dots, x_{j_{k+1}}],$$

in which case

$$d([\sigma]^*)([\tau]) = (-1)^r.$$

For such an index $J = (j_0, \dots, j_{k+1})$ we write this r as $r = r(J)$. Having this in mind and using (2.11) we calculate

$$\begin{aligned} d[\sigma]^* = \sum_{\tau \in K^{(k+1)}} d([\sigma]^*)([\tau])[\tau]^* &= \sum_{J=(j_0<\dots<j_{k+1})} d([\sigma]^*)([x_{j_0}, \dots, x_{j_k}])([x_{j_0}, \dots, x_{j_{k+1}}]^*) \\ \overset{(2.12)}{=} \sum_{\substack{J=\{j_0<\dots<j_{k+1}\} \\ \supset\{i_0,\dots,i_k\}}} &(-1)^{r(J)} [x_{j_0}, \dots, x_{j_{r(J)}}, \dots x_{j_{k+1}}]^* = \sum_{i \in I'} [x_i, x_{i_0}, \dots, x_{i_k}]^*. \end{aligned}$$ □

2.2.26 Definition (simplicial L_p-cochains). Let K be a simplicial complex. We call

$$\forall 1 \leq p < \infty : C_p^k(K) := \left\{ c \in C^k(K) \;\middle|\; \|c\|_{C_p^k(K)} := \Big(\sum_{\sigma \in K^{(k)}} |c([\sigma])|^p \Big)^{\frac{1}{p}} < \infty \right\},$$
$$C_\infty^k(K) := \Big(c \in C^k(K) \mid \|c\|_{C_\infty^k(K)} := \sup_{\sigma \in K^{(k)}} |c([\sigma])| < \infty \Big),$$

the *k-th simplicial L_p-cochain group of K*.
If $L \subset K$ is a subcomplex, we call

$$C_p^k(K, L) := \{c \in C_p^k(K) \mid \forall \sigma \in L^{(k)} : c([\sigma]) = 0\}$$

the *k-th simplicial L_p-cochain group of K relative L.*

2.2.27 Lemma. Let K be a star-bounded simplicial complex with star bound N and $k \in \mathbb{N}$ be arbitrary. Then for any $1 \leq p < \infty$ and any $c \in C_p^k(K)$

$$\|dc\|_{C_p^{k+1}(K)} \leq (k+2)\sqrt[p]{N}\|c\|_{C_p^k(K)}.$$

For any $c \in C_\infty^k(K)$

$$\|dc\|_{C_\infty^{k+1}}(K) \leq (k+2)\|c\|_{C_\infty^k(K)}.$$

In particular, for any $1 \leq p \leq \infty$, the coboundary d^k is a bounded linear operator $C_p^k(K) \to C_p^{k+1}(K)$ and $(C_p^*(K), d^*)$ is a well-defined cochain complex.

Proof. In case $p < \infty$, we calculate:

$$\begin{aligned}
\|dc\|^p_{C_p^{k+1}(K)} &= \sum_{\sigma \in K^{(k+1)}} |(dc)([\sigma])|^p = \sum_{\sigma \in K^{(k+1)}} |c(d[\sigma])|^p = \sum_{\sigma \in K^{(k+1)}} \Big| \sum_{\tau \in K^{(k)}} [\sigma : \tau] c([\tau]) \Big|^p \\
&\leq \sum_{\sigma \in K^{(k+1)}} (k+2)^p \sum_{\tau \in K^{(k)}} |[\sigma : \tau] c(\tau)|^p \\
&= (k+2)^p \sum_{\tau \in K^{(k)}} |c(\tau)|^p \sum_{\sigma \in K^{(k+1)}} |[\sigma : \tau]| \\
&\leq (k+2)^p \sum_{\tau \in K^{(k)}} |c([\tau])|^p (\sharp \operatorname{st}_K(\tau)) \leq (k+2)^p N \|c\|^p_{C_p^k(K)}.
\end{aligned}$$

In case $p = \infty$, we calculate similarly:

$$\begin{aligned}
\|dc\|_{C_\infty^{k+1}(K)} &= \sup_{\sigma \in K^{(k+1)}} |(dc)([\sigma])| = \sup_{\sigma \in K^{(k+1)}} |c(d[\sigma])| = \sup_{\sigma \in K^{(k+1)}} \Big| \sum_{\tau \in K^{(k)}} [\sigma : \tau] c([\tau]) \Big| \\
&\leq \sup_{\sigma \in K^{(k+1)}} \sum_{\tau \in K^{(k)}} |[\sigma : \tau]||c([\tau])| \leq (k+2) \sup_{\sigma \in K^{(k+1)}} \sup_{\tau \in K^{(k)}} |[\sigma : \tau]||c([\tau])| \\
&= (k+2) \sup_{\tau \in K^{(k)}} |c([\tau])| \leq (k+2)\|c\|_{C_\infty^k(K)}.
\end{aligned}$$

□

2.2.28 Definition (simplicial L_p-cohomology). Let K be a star-bounded simplicial complex. Then we call

$$\mathcal{H}_p^*(K) := H^*(C_p^*(K), d^*)$$

the *simplicial L_p-cohomology of K*. We denote its closed and exact forms by $\mathcal{Z}_p(K)$ respectively $\mathcal{B}_p(K)$. The norm on $C_p^k(K)$ induces a semi-norm on $\mathcal{H}_p^k(K)$. We call

$$\bar{\mathcal{H}}_p^*(K) := \frac{\mathcal{H}_p^*(K)}{\{x \in H_p^*(K) \mid \|x\|_{H_p^*(K)} = 0\}}$$

the *reduced simplicial L_p-cohomology.*
For any subcomplex $L \subset K$, we call

$$\mathcal{H}_p^*(K, L) := H^*(C_p^*(K, L))$$

the *simplicial L_p-cohomology relative L.*

2.2.29 Lemma (long exact sequence). The short sequence

$$0 \longrightarrow C_p(K, L) \longrightarrow C_p(K) \longrightarrow C_p(L) \longrightarrow 0$$

is exact and induces a long exaxt sequence in cohomology.

Proof. This is clear from the definition of $C_p(K, L)$. □

2.3 L_p-cohomology of S-forms

Finally we introduce the cohomology of S-forms. This cohomology theory is developed for technical reasons and should be thought of as a theory between the other two.

2.3.1 Simplicial metrics

Before we can define S-forms, there is one technical obstacle to overcome that will turn out to be of vital importance later. At the moment K is a simplicial complex in some $\mathbb{R}^n$, i.e. $|K| \subset \mathbb{R}^n$ as a set. We would like to define a "Riemannian metric" on $|K|$, which is different from the Euclidean one.

2.3.1 Definition (simplicial Riemannian Metric). Let K be a simplicial complex. For any $\sigma \in K$ we think of σ as a smooth manifold with corners. A system of Riemannian metrics

$$g = \{g(\sigma) \in \mathcal{T}^2(\sigma) \mid \sigma \in K\}$$

is a *simplicial Riemannian metric* or just an "*S-metric*", if whenever $\tau \leq \sigma$ and $j_{\tau,\sigma} : \tau \hookrightarrow \sigma$ is the inclusion, then $j^*_{\tau,\sigma} g(\sigma) = g(\tau)$.

2.3.2 Remark. So, an S-metric attatches Riemannian metrics to every simplex in the complex in a compatible way. We would like to have an S-metric g_S on K such that for every s-simplex $\sigma \in K$ any simplicial isomorphism $\sigma \to \Delta_s$ onto the standard simplex is an isometry. This will have the convenient effect that the g_S-volume of σ equals the Euclidean volume of the standard simplex Δ_s. Notice that it is very easy to construct such a metric for the complex $\mathrm{cl}(\sigma)$: Just choose any standard chart $B : |\sigma| \to \Delta_s \subset \mathbb{R}^n$ as in 2.2.5, denote by g the Euclidean metric in $\mathbb{R}^n$ and define $g_S := B^* g$. Now we could try to inductively construct an S-metric on an arbitrary complex, but this would be rather nasty to carry out in detail.

2.3.3 Remark (modified standard simplex). In the following it will be convenient to slightly change the definition of the standard simplex: Renumber the coordinates of a point in $\mathbb{R}^{n+1}$ to $x = (x_0, \ldots, x_n)$, and denote by $\{\tilde{e}_i \mid 0 \leq i \leq n\}$ the canonical basis of $\mathbb{R}^{n+1}$. Define

$$\tilde{\Delta}_k := \Big\{ \sum_{i=0}^{k} \beta_i e_i \mid \forall 0 \leq i \leq k : \beta_i \in [0,1] \text{ and } \sum_{i=0}^{k} \beta_i = 1 \Big\} \subset \mathbb{R}^{k+1}.$$

This definition has the advantage that all boundary faces of $\tilde{\Delta}_k$ have the same volume and are isometric to $\tilde{\Delta}_{k-1}$. Notice that this is not true for Δ_k: For example Δ_2 has two faces of length 1, namely $\langle 0, e_1 \rangle$ and $\langle 0, e_2 \rangle$. But $\langle e_1, e_2 \rangle$ has length $\sqrt{2}$ and $\Delta_1 = \langle 0, e_1 \rangle$ has length 1. For matters of geometry, this is very inconvenient. Of course this definition has the disadvantage that $\tilde{\Delta}_k \subset \mathbb{R}^{k+1}$, i.e. we are troubled with a seemingly superflous additional dimension.

2.3.4 Definition (Hilbert space l^2). Let l^2 be the space of all real square summable sequences, i.e.

$$\mathbb{R}^\infty := l^2 := \{y \in \mathbb{R}^{\mathbb{N}} \mid \|y\|^2_{l^2} := \sum_{i=0}^{\infty} y_i^2 < \infty\}.$$

For any $n \in \mathbb{N}$, we may identify $\mathbb{R}^n$ as a subspace of l^2, by identifying the vector $(x_0, \ldots, x_{n-1})$ with the sequence $(x_0, \ldots, x_{n-1}, 0, 0, \ldots)$. For any $i \in \mathbb{N}$ define $\tilde{e}_i \in l^2$ to be the sequence $(\tilde{e}_i)_j = \delta_{ij}$. Notice that l^2 is a Hilbert space with the inner product

$$\forall y, y' \in l^2 : g(y, y') := \sum_{i=0}^{\infty} y_i y'_i.$$

For technical reasons it might be useful to temporarily think of l^2 as a smooth Riemannian Hilbert manifold[2].

We think of

$$\tilde{\Delta}_\infty := \Big\{ \sum_{i=0}^{n} \beta_i \tilde{e}_i \mid n \in \mathbb{N}, \beta_i \geq 0, \sum_{i=0}^{n} \beta_i = 1 \Big\} \subset \mathbb{R}^\infty$$

as the infinite dimensional standard simplex. For any $N \in \mathbb{N}$

$$\tilde{\Delta}_\infty^N := \Big\{ \sum_{i=0}^{n} \beta_i \tilde{e}_i \in \Delta_\infty \mid n \leq N \Big\}$$

may be thought of as $\tilde{\Delta}_N \subset \mathbb{R}^{N+1}$.

2.3.5 Theorem (S-metric). Let K be a simplicial complex and let $\{x_i\}_{i \in \mathbb{N}}$ be an arbitrary but fixed counting of its vertices. Define

$$K_N := \{\langle x_{i_0}, \ldots, x_{i_k} \rangle \in K \mid i_0, \ldots, i_k \leq N, k \leq \dim K\}$$

and let $f_N : K_N \to \tilde{\Delta}_\infty^N$ be the unique simplicial map determined by $x_i \mapsto \tilde{e}_i$, $0 \leq i \leq N$. (c.f. 2.2.19(i)). This defines a map $f : |K| \to \tilde{\Delta}_\infty$ by $x \mapsto f_N(x)$, if $x \in K_N$. This is a well-defined embedding and $g_S := f^* g$ is an S-metric on K. This metric satisfies the following properties.

(i) For any k-simplex $\sigma \in K$, $\mathrm{vol}_S(\sigma) := \mathrm{vol}_{g_S}(\sigma) = \mathrm{vol}(\tilde{\Delta}_k) = 1/k! =: v_k$.

(ii) Let $i, j \in \mathbb{N}$ and assume that $\mathrm{st}_K(x_i)$ and $\mathrm{st}_K(x_j)$ are simplicially isomorphic. Then any simplicial isomorphism $\psi_{ij} : \mathrm{st}_K(x_i) \to \mathrm{st}_K(x_j)$ is a Riemannian isometry with respect to g_S.

Proof.

(i) Let $\sigma = \langle x_{i_0}, \ldots, x_{i_k} \rangle$ and assume $i_0, \ldots, i_k \leq N$. By construction f is a Riemannian isometry onto its image. Therefore $\mathrm{vol}(\sigma) = \mathrm{vol}(f(\sigma))$. Now

$$f(\sigma) = \langle e_{i_0}, \ldots, e_{i_k} \rangle \subset \mathbb{R}^{N+1},$$

which is the standard simplex up to a permutation of the vertices. But such a permutation is clearly realized by an isometry. Thus $\mathrm{vol}(f(\sigma)) = \mathrm{vol}(\tilde{\Delta}_k)$.

(ii) Certainly there exists $N \in \mathbb{N}$ such that $i, j \leq N$. Therefore we consider $f_N : K_N \to \tilde{\Delta}_\infty^N \subset \mathbb{R}^{N+1}$. Since $\mathrm{st}_K(x_i)$ and $\mathrm{st}_K(x_j)$ are simplicially isomorphic, there exists some number r, and indices such that $\mathrm{cl}(\mathrm{st}_K(x_i))^{(0)} = \{x_{i_0}, \ldots, x_{i_r}\}$, $\mathrm{cl}(\mathrm{st}_K(x_j))^{(0)} =$

[2] A seperable Hausdorff space M in which every point has an open neighborhood that is homeomorphic to an open set of some Hilbert space H is a *Hilbert manifold*. Analogously to the finite dimensional case, we say M is a *smooth* Hilbert manifold, if it is endowed with a maximal atlas such that all transition functions are smooth.

$\{x_{j_0}, \ldots, x_{j_r}\}$. The simplicial isomorphism ψ_{ij} is uniquely determined by some permutation $\pi \in \mathfrak{S}_{r+1}$, $\pi(i_\nu) = j_\nu$, $0 \leq \nu \leq r$, (c.f. 2.2.19(i)). This corresponds to an isometry $\Pi_{ij} : \tilde{\Delta}^N_\infty \to \tilde{\Delta}^N_\infty$, which maps $e_{i_\nu} \mapsto e_{j_\nu}$, $0 \leq \nu \leq r$, and is the identity everywhere else. Define $f_i := f_N|\operatorname{st}_K(x_i)$ and $f_j := f_N|\operatorname{st}_K(x_j)$. We obtain the commutative diagram

$$\begin{array}{ccc} \operatorname{st}_K(x_i) & \xrightarrow{f_i} & \tilde{\Delta}^N_\infty \\ \downarrow{\scriptstyle \psi_{ij}} & & \downarrow{\scriptstyle \Pi_{ij}} \\ \operatorname{st}_K(x_j) & \xrightarrow{f_j} & \tilde{\Delta}^N_\infty \end{array}$$

Since f_i and f_j are Riemannian isometries onto their images, ψ_{ij} is an isometry as well. □

2.3.6 Convention. If not explicitely stated otherwise, we will always assume that K is endowed with the standard S-metric described above.

2.3.2 S-Forms

2.3.7 Remark (restriction Operators). The aim of this section is to introduce the notion of S-Forms, so we would like to start with their definition in 2.3.9 right away. The problem is that we cannot write down condition (2.14) at this point.
In subsection 2.1.1, we already noticed that a map $F : M \to N$ does not necessarily induce a well-defined map $F^* : W_p(N) \to W_p(M)$ and therefore restricted our attention to zero-preserving maps. These maps turned out to form an appropriate class of morphisms. Let $\sigma \subset \mathbb{R}^n$ be a k-simplex and $\tau < \sigma$ be one of its boundary faces. Consider the inclusion $j = j_{\tau,\sigma} : \tau \hookrightarrow \sigma$. Then $\tau \subset \sigma$ is a set of measure zero in σ, but $j^{-1}(\tau) = \tau$ is certainly not a set of measure zero in τ. Therefore we cannot apply the theory developed in 2.1.1 directly to j.
Discussing the entire theory necessary to define j^* would take us too far afield. Nevertheless we will elaborate the very short sketch given in [5, p. 191] at least a bit further: Our ultimate goal is to define a bounded linear operator $j^*_{\tau,\sigma} : W_\infty(\sigma) \to W_\infty(\tau)$. The idea is to factor this operator into

$$\begin{array}{ccc} W_\infty(\sigma) & \xrightarrow{j^*_{\tau,\sigma}} & W_\infty(\tau) \\ \downarrow{\scriptstyle \gamma_{\sigma,U}} & \nearrow{\scriptstyle j^*_{\tau,U}} & \\ W_\infty(U) & & \end{array} \tag{2.13}$$

Here U is an open set in the affine hull of σ. For the definition of $\gamma_{\sigma,U}$ and $j^*_{\tau,U}$ article [5] refers to [6]. This article is in Russian[3], but the important part roughly translates to:

> *Lemma 4: Let E be an n-dimensional Riemannian manifold without boundary, $D \subset E$ be a smooth submanifold of dimension n. Then there exists an open set $U \subset E$, $D \subset U$ and bounded linear operators $A : L^k_p(D) \to L^k_p(U)$, $A(W^k_{p,q}(D)) \subset W^k_{p,q}(U)$, $1 \leq p, q \leq \infty$ such that $A(\omega)|_D = \omega$.*

[3] Thanks to Wassilij Gnedin and Valentin Krasontovich for helping with the translation.

Thus one may define γ by a formal application of this Lemma, i.e. define $\gamma_{\sigma,U} := A$. In the situation we consider here, E is the affine hull of σ, U corresponds to U, D corresponds to $\operatorname{Int}\sigma$ and $p = q = \infty$ (remember 2.1.11 for the definition of $W_{p,q}$).
Alternatively it is not hard to obtain this result directly: By definition of the S-metric σ is isometric to a standard simplex, i.e. a very simple submanifold. It is certainly possible to extend a form $\omega \in W_\infty(\tilde{\Delta}_k)$ to a form $\tilde{\omega}$ on some small open neighbourhood $U \supset \tilde{\Delta}_k$ in the affine hull of $\tilde{\Delta}_k$ such that even

$$\operatorname*{ess\,sup}_{x\in U} |\tilde{\omega}(x)| = \operatorname*{ess\,sup}_{x\in\tilde{\Delta}_k} |\omega(x)|,$$

i.e. $\tilde{\omega} \in W_\infty(U)$.
The much more critical part is the construction of $j^*_{\tau,U}$: Notice that although j is not zero-preserving, j is certainly Lipschitz continuous. For a Lipschitz continuous map $f : X \to Y$ and in case $p = \infty$, it is possible to define an operator $f^* : W^k_\infty(Y) \to W^k_\infty(X)$ by setting $f^*\omega := \eta$, where η is the unique form satisfying

$$\int_{f|_\rho} \omega = \int_\rho \eta,$$

for any smooth simplex $\rho \subset X$. This equation from [6, p. 56] is an integral representation. Integral representations are studied extensively in [6]. The existence and uniqueness of this form is based on Whitney's work, in particular [31, X.8.A]. Combining these results one may obtain the following theorem from [6]:

> *Let E be an open Riemannian manifold and let $D \subset \operatorname{Int} E$ be a compact submanifold. Let $j : D \to E$ be the identity embedding. Then there exists a bounded linear operator $j^* : W_\infty(E) \to W_\infty(D)$.*

In general this is not possible for any $1 \leq p < \infty$.
Using these theorems, we define $j^*_{\tau,\sigma}$ by (2.13) and remark that it does not depend on the chosen extension operator.

Summing up, we obtain the following lemma.

2.3.8 Lemma. For any simplex σ and any face $\tau \leq \sigma$ there exists a bounded linear restriction operator $j^*_{\tau,\sigma} : W_\infty(\sigma) \to W_\infty(\tau)$.

2.3.9 Definition (S-form). Let K be a simplicial complex. For any two simplices $\tau, \sigma \in K$, $\tau \leq \sigma$ consider the inclusion map $j_{\tau,\sigma} : \tau \hookrightarrow \sigma$. A collection of forms

$$\theta := \{\theta(\sigma) \in W^k_\infty(\sigma) \mid \sigma \in K\}$$

such that

$$\forall \tau \leq \sigma \in K : j^*_{\tau,\sigma}(\theta(\sigma)) = \theta(\tau), \tag{2.14}$$

is a *simplicial differential form of degree k* or just an "*S-form*". The space of all these S-forms of degree k on K is denoted by $S^k(K)$. For any S-form $\theta := \{\theta(\sigma)\}_{\sigma\in K}$ of degree k, the collection $d\theta := \{d\theta(\sigma)\}_{\sigma\in K}$ is an S-form of degree $k+1$. Thus the S-forms assemble to a cochain complex $(S^*(K), d^*)$, the *cochain complex of S-forms on K*.

2.3.10 Definition (p-summable S-forms). Let $\theta \in S^k(K)$ and $1 \le p < \infty$. We say θ is *p-summable*, if

$$\|\theta\|^p_{S^k_p(K)} := \sum_{\sigma \in K} \|\theta(\sigma)\|^p_{W^k_\infty(\sigma)} < \infty, \quad \text{respectively} \quad \|\theta\|_{S^k_\infty(K)} := \sup_{\sigma \in K} \|\theta(\sigma)\|_{W^k_\infty(\sigma)} < \infty.$$

The set of all p-summable S-forms is denoted by $S^*_p(K)$. (Notice that K has only countably many simplices and since $\|\theta(\sigma)\|_{W^k_\infty(\sigma)} \ge 0$, the norm does not depend on the order of summation.)

2.3.11 Lemma. For every $1 \le p \le \infty$ the $S^*_p(K)$ assemble to a cochain complex of Banach spaces.

Proof. The fact that $S^k_p(K)$ is a vector space follows from the fact that l^p, the space of p-summable sequences is a vector space. The following proof that $S^k_p(K)$ is complete is also very similiar to the one that l^p is complete.

Step 1 (completeness): Take an enumeration $\{\sigma_i\}_{i \in \mathbb{N}}$ of all the simplices in K. Let $(\theta_n) \in S^k_p(K)$ be a Cauchy-Sequence and let $\varepsilon > 0$. Then there exists $N_0 \in \mathbb{N}$ such that

$$\forall n, m \ge N_0 : \|\theta_n - \theta_m\|_{S^k_p(K)} < \varepsilon. \tag{2.15}$$

This implies in particular that

$$\forall i \in \mathbb{N} : \|\theta_n(\sigma_i) - \theta_m(\sigma_i)\|_{W^k_\infty(\sigma_i)} \le \|\theta_n - \theta_m\|_{S^k_p(K)} < \varepsilon.$$

Thus, for any $i \in \mathbb{N}$, $(\theta_n(\sigma_i))_{n \in \mathbb{N}}$ is a Cauchy sequence in $W^k_\infty(\sigma_i)$. Since $W^k_\infty(\sigma_i)$ is a Banach space (c.f. 2.1.17),

$$\exists \theta(\sigma_i) \in W^k_\infty(\sigma_i) : \ \theta_n(\sigma_i) \xrightarrow[W^k_\infty(\sigma_i)]{n \to \infty} \theta(\sigma_i) \ . \tag{2.16}$$

Since for any $\tau \le \sigma$ the restriction $j^*_{\tau,\sigma} : W_\infty(\sigma) \to W_\infty(\tau)$ is continuous, these forms assemble to an S-form $\{\theta(\sigma)\}_{\sigma \in K}$ of degree k.
For any fixed $l \in \mathbb{N}$, we obtain

$$\forall n, m \ge N_0 : \Big(\sum_{i=1}^{l} \|\theta_n(\sigma_i) - \theta_m(\sigma_i)\|^p_{W^k_\infty(\sigma_i)} \Big)^{1/p} \le \|\theta_n - \theta_m\|_{S^k_p(K)} \overset{(2.15)}{\le} \varepsilon.$$

By (2.16) and the fact that the sum is finite, we may take the limit $m \to \infty$ here in order to obtain

$$\forall n \ge N_0 : \Big(\sum_{i=1}^{l} \|\theta_n(\sigma_i) - \theta(\sigma_i)\|^p_{W^k_\infty(\sigma_i)} \Big)^{1/p} \le \varepsilon.$$

Since l was arbitrary, this implies

$$\forall n \ge N_0 : \|\theta_n - \theta\|_{S^k_p(K)} \le \varepsilon.$$

From this we obtain on the one hand that $\theta = (\theta - \theta_{N_0}) + \theta_{N_0} \in S^k_p(K)$ and on the other hand

$$\theta_n \xrightarrow[S^k_p(K)]{n \to \infty} \theta \ .$$

Step 2 (continuity of d): First of all notice that for any simplex $\sigma \in K^{(k)}$ the k-form $\omega := \theta(\sigma)$ satisfies

$$\begin{aligned}\|d\omega\|_{W_\infty^{k+1}(\sigma)} &= \max\{\|d\omega\|_{L_\infty^{k+1}(\sigma)}, \|dd\omega\|_{L_\infty^{k+2}(\sigma)}\} = \|d\omega\|_{L_\infty^{k+1}(\sigma)} \\ &\leq \max\{\|\omega\|_{L_\infty^{k}(\sigma)}, \|d\omega\|_{L_\infty^{k+1}(\sigma)}\} = \|\omega\|_{W_\infty^{k}(\sigma)}.\end{aligned}$$

Thus the operator norm of $d : W_\infty^k(\sigma) \to W_\infty^{k+1}(\sigma)$ is less or equal to one.
Now if $p < \infty$, we calculate

$$\|d\theta\|_{S_p^{k+1}}^p = \sum_{\sigma\in K} \|d\theta(\sigma)\|_{W_\infty^{k+1}}^p \leq \sum_{\sigma\in K} \|\theta(\sigma)\|_{W_\infty^k}^p = \|\theta\|_{S_p^k}^p.$$

If $p = \infty$, we have

$$\|d\theta\|_{S_\infty^{k+1}}^p = \sup_{\sigma\in K} \|d\theta(\sigma)\|_{W_\infty^{k+1}(\sigma)} \leq \sup_{\sigma\in K} \|\theta(\sigma)\|_{W_\infty^{k}(\sigma)} = \|\theta\|_{S_\infty^k}^p.$$

□

2.3.12 Definition (S-form cohomology). We denote by

$$\mathscr{H}_p^k(K) := H^k(S_p(K), d)$$

the L_p*-cohomology of S-forms on* K. The corresponding closed and exact forms are denoted by $\mathscr{Z}_p(K)$ and $\mathscr{B}_p(K)$.

2.3.13 Lemma. Let $h : |K| \to M$ be a smooth triangulation and $\omega \in W_{\infty,\mathrm{loc}}^k(M)$. For any simplex $\sigma \in K$, define $\theta(\sigma) := (h|_\sigma)^*(\omega)$. Then $\varphi_h(\omega) := \{\theta(\sigma)|\sigma \in K\}$ is an S-form on K and the map $\varphi := \varphi_h : W_{\infty,\mathrm{loc}}^k(M) \to S^k(K)$ is an isomorphism of vector spaces and $\varphi_h : W_{\infty,\mathrm{loc}}(M) \to S(K)$ is an isomorphism of cochain complexes.

Proof. We assume K is a simplicial complex in $\mathbb{R}^n$. Since $\dim M = m$ and h is a diffeomorphism, this implies $\dim K = m$.

Step 1 ($\varphi(\omega)$ is an S-form): Let $\tau < \sigma \in K$. Since the triangulation h is smooth, there exists a smooth continuation $h_U : U \to M$ of $h|_\sigma$ to some open neighbourhood U of the simplex σ in its affine hull. Consequently

$$\theta(\tau) = (h|_\tau)^*\omega = j_{\tau,U}^*((h_U)^*(\omega)) = (h_U \circ j_{\tau,U})^*\omega = (h|_\sigma \circ j_{\tau,\sigma})^*\omega = j_{\tau,\sigma}^*(h|_\sigma^*(\omega)) = j_{\tau,\sigma}^*(\theta(\sigma)).$$

Step 2 (injectivity): Let $\theta := \varphi(\omega) = 0$. Then for any open subset $U = \operatorname{Int}\sigma$, where σ is an m-dimensional simplex, $0 = h|_U^*(\omega)$. Since h is a diffeomorphism, this implies $\omega|_{h(U)} = 0$, which implies altogether $\omega = 0$ a.e.

Step 3 (surjectivity / cochain map): Let $\theta \in S^k(K)$ be an arbitrary S-form. For any m-simplex $\sigma \in K$, we define $\omega|_{h(\operatorname{Int}\sigma)} := (h^{-1}|_{h(\operatorname{Int}\sigma)})^*(\theta(\sigma))$. Now since σ is compact, h^{-1} restricts to a diffeomorphism $h^{-1}|_{h(\operatorname{Int}\sigma)} : W_\infty^k(h(\operatorname{Int}\sigma)) \to W_\infty^k(\operatorname{Int}\sigma)$ with bounded operator norms. Therefore Theorem 2.1.29 implies $\omega|_{h(\operatorname{Int}\sigma)} \in W_\infty(h(\operatorname{Int}\sigma))$ and $d\omega|_{h(\operatorname{Int}\sigma)} = (h^{-1}|_{h(\operatorname{Int}\sigma)})^*(d\theta(\sigma))$. By patching together these forms, we obtain a globally defined form $\omega \in W_{\infty,\mathrm{loc}}^k(M)$, since $M \setminus \bigcup_{\sigma\in K} h(\operatorname{Int}\sigma)$ is a set of measure zero. This also shows that d commutes with φ_h. □

2.3.14 Lemma. Let K be a star-bounded simplicial complex and $L \subset K$ be a subcomplex. The map $j^* := j_{L,K}^* : S_p^k(K) \to S_p^k(L)$, $\{\theta(\sigma)\}_{\sigma\in K} \mapsto \{\theta(\tau)\}_{\tau\in L}$, is an epimorphism.

Proof.

Step 1: We will show that there exists a bounded linear operator $\gamma : S_p(\partial\sigma) \to W_\infty(\sigma)$ such that for every $\tau \leq \sigma$: $j^*_{\tau,\sigma}\gamma(\theta) = \theta(\tau)$. To see this notice that $S_p(\partial\sigma) \cong W_{\infty,\mathrm{loc}}(\partial\sigma) = W_\infty(\partial\sigma)$ by Lemma 2.3.13. Furthermore $(\sigma, \partial\sigma) \cong (B^n, S^{n-1})$, where $B^n \subset \mathbb{R}^n$ is the closed unit ball and S^{n-1} is the unit sphere. The commutative diagram

$$\begin{array}{ccccc} S_p(\partial\sigma) & \longrightarrow & W_\infty(\sigma) & \overset{\sim}{\longleftrightarrow} & W_\infty(B^n) \\ & \overset{\sim}{\nwarrow\!\!\searrow} & \uparrow & & \uparrow \gamma \\ & & W_\infty(\partial\sigma) & \overset{\sim}{\longleftrightarrow} & W_\infty(S^{n-1}) \end{array}$$

reveals that it suffices to construct the operator $\gamma : W_\infty(S^{n-1}) \to W_\infty(B^n)$ such that $j^*_{S^{n-1},B^n} \circ \gamma = \mathrm{id}$. (The j^* is analogous to 2.3.8.) To that end define $U := B^n \setminus B_{1/4}(0)$. Notice that $r : U \to S^{n-1}$, $x \mapsto \frac{x}{\|x\|}$, is a smooth retraction. Thus we obtain a bounded linear operator $r^* : W_\infty(S^{n-1}) \to W_\infty(U)$ such that for any $\omega \in W_\infty(S^{n-1})$, we obtain $j^*_{S^{n-1},B^n}(r^*(\omega)) = \omega$. By multiplying $r^*(\omega)$ with a smooth bump function $\psi \in \mathcal{C}^\infty(B^n)$ such that

$$\psi|_{B^n \setminus B_{1/2}} \equiv 1, \qquad\qquad \psi|_{B_{1/4}} \equiv 0,$$

we obtain our desired operator γ.

Step 2: Let $\theta \in S^k_p(L)$ be arbitrary, K^i be the i-skeleton of K and $K_i := L \cup K^i$. We shall establish by induction over i that j^*_{L,K_i} is an epimorphism. For $i = 0$ define

$$\theta_0(\sigma) := \begin{cases} \theta(\sigma), & \sigma \in L \\ 0, & \sigma \in K_0 \setminus L. \end{cases}$$

Clearly $\theta_0 \in S^k_p(K_0)$ and $j^*_{L,K_0}(\theta_0) = \theta$.
Now assume $\theta_{i-1} \in S^k_p(K_{i-1})$ such that $j^*_{L,K_{i-1}}(\theta_{i-1}) = \theta$. Define $\theta_i \in S^k(K_i)$

$$\theta_i(\sigma) := \begin{cases} \theta_{i-1}(\sigma), & \sigma \in K_{i-1} \\ (\gamma(j^*_{\partial\sigma,K_{i-1}}(\theta_{i-1})), & \sigma \in K_i \setminus K_{i-1}. \end{cases}$$

Clearly, this is an S-form, $j^*_{L,K_i}(\theta_i) = \theta$, and if N is the star-bound of K, its norm is given by

$$\begin{aligned} \|\theta_i\|^p_{S^k_p(K_i)} &= \|\theta_{i-1}\|^p_{S^k_p(K_{i-1})} + \sum_{\sigma \in K_i \setminus K_{i-1}} \|\gamma(j^*_{\partial\sigma,K_{i-1}}(\theta_{i-1})\|^p_{W^k_\infty(\sigma)} \\ &\leq \|\theta_{i-1}\|^p_{S^k_p(K_{i-1})} + \|\gamma\|^p \sum_{\sigma \in K_i \setminus K_{i-1}} \sum_{\tau \in \partial\sigma} \|\theta_{i-1}(\tau)\|^p_{W^k_\infty(\tau)} \\ &\leq \|\theta_{i-1}\|^p_{S^k_p(K_{i-1})} + \|\gamma\|^p N \sum_{\tau \in K_{i-1}} \|\theta_{i-1}(\tau)\|^p_{W^k_\infty(\tau)} \\ &= (1 + \|\gamma\|^p N)\|\theta_{i-1}\|^p_{S^k_p(K_{i-1})}. \end{aligned}$$

□

2.3.15 Corollary. By defining $S^k_p(K, L) := \ker j^*_{L,K}$, we obtain a short exact sequence of cochain complexes

$$0 \longrightarrow S^*_p(K, L) \longrightarrow S^*_p(K) \longrightarrow S^*_p(L) \longrightarrow 0$$

inducing a long exact sequence in cohomology.

Proof. This follows directly from the definition and Lemma 2.3.14. □

3 The Isomorphism between the simplicial and the S-form cohomology

In this section we establish an isomorphism $\mathcal{H}_p(K) \to \mathscr{H}_p(K)$. To that end we construct two maps

$$w : C_p^*(K,L) \rightleftarrows S_p^*(K,L) : I$$

on the chain level, which induce two maps, $[I]$,$[w]$ on the cohomology level. We will then prove that these induced maps $[I]$,$[w]$ provide isomorphisms in cohomology (although I,w are usually not isomorphisms):

$$\begin{array}{ccc}
C_p(K,L) & \overset{w}{\underset{I}{\rightleftarrows}} & S_p(K,L) \\
\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\
\mathcal{H}_p(K,L) & \overset{[w]}{\underset{[I]}{\rightleftarrows}} & \mathscr{H}_p(K,L)
\end{array}$$

3.1 The de Rham homomorphism

The map I is well-known: It is the same map that was already used by de Rham to prove the isomorphism between singular cohomology and de Rham cohomology.

3.1.1 Lemma. Let K be a simplicial complex and L be a subcomplex. The map

$$\begin{aligned} I : S^k(K) &\to C^k(K), \\ I(\theta)([\sigma]) &:= \int_\sigma \theta(\sigma), \end{aligned}$$

induces a well-defined chain map $I : S^*(K,L) \to C^*(K,L)$, called *de Rham homomorphism* or just *integration map*.
For any $1 \leq p \leq \infty$, I restricts to a chain map $I : S_p^*(K) \to C_p^*(K)$. We claim in particular:

(i) $I(d\theta) = d(I(\theta))$,

(ii) $I(S_p^k(K)) \subset C_p^k(K)$.

Proof.

(i) By Stokes' theorem, we calculate

$$I(d\theta)(\sigma) = \int_\sigma d\theta(\sigma) = \int_{\partial\sigma} j^*_{\partial\sigma,\sigma}\theta(\sigma) = \int_{\partial\sigma} \theta(\partial\sigma) = I(\theta)(\partial\sigma) = d(I(\theta))(\sigma).$$

One might worry, if Stokes' Theorem is really applicable here. We think of σ as a smooth manifold with corners and although the boundary is not smooth, this theorem still holds, c.f. [16, 14.20]. One might object further that $\theta(\sigma) \in W_\infty(\sigma)$ is not necessarily smooth. Although there are many versions of Stokes' theorem in the literature that require smoothness it is intuitively clear that this theorem should hold under weaker assumptions since it involves integration only. A suitable generalization can be found in [7, Theorem 9].

(ii) Remember that by convention from 2.3.1 K is endowed with the standard S-metric. Thus the Riemannian volume of every simplex $\sigma \in K^{(k)}$ is given by $|\sigma| = |\Delta_k| =: v_k = \frac{1}{k!}$. Let $\theta \in S_p^k(K)$ and calculate for any $1 \leq p < \infty$

$$\begin{aligned}\|I(\theta)\|^p_{C_p^k(K)} &= \sum_{\sigma \in K^{(k)}} |I(\theta)(\sigma)|^p = \sum_{\sigma \in K^{(k)}} \left| \int_\sigma \theta(\sigma) \right|^p \\ &\leq \sum_{\sigma \in K^{(k)}} |\sigma|^p \|\theta(\sigma)\|^p_{W^k_\infty(\sigma)} \leq v_k^p \|\theta\|^p_{S^k_p(K)}.\end{aligned}$$

In case $p = \infty$, we calculate analogously

$$\begin{aligned}\|I(\theta)\|_{C^k_\infty(K)} &= \operatorname*{ess\,sup}_{\sigma \in K^{(k)}} |I(\theta)(\sigma)| = \operatorname*{ess\,sup}_{\sigma \in K^{(k)}} \left| \int_\sigma \theta(\sigma) \right| \\ &\leq \operatorname*{ess\,sup}_{\sigma \in K^{(k)}} |\sigma| \|\theta(\sigma)\|_{W^k_\infty(\sigma)} \leq v_k \|\theta\|_{S^k_\infty(K)}.\end{aligned}$$ □

3.1.2 Theorem (S-form Isomorphism). For any simplicial complex K the transformation $I : S^*(K) \to C^*(K)$ induces an isomorphism in cohomology.

This theorem is discussed in [26]. We just mention it here for reasons of completeness. We will neither prove nor use this theorem and instead focus our attention entirely to the L_p-case.

3.2 Whitney transformation

We now construct the map w, the Whitney transformation, which is much more complicated. Our approach is based on the work of Whitney and Dodziuk ([31, IV.C,§27] and [2, 1]). Although this still seems to be state of the art today, we cannot refer directly to them, because they use finite simplicial complexes. The simplicial as well as the S-form cohomology are defined in terms of a simplicial complex K and do not refer to any manifold themselves. Of course we could restrict our attention to finite simplicial complexes as well. But for our purpose this had most unpleasent consequences: If K is a triangulation of a manifold M, the finiteness of K would force the manifold M to be compact. Since we are particularly interested in the study of noncompact manifolds, this would be fatal. Therefore we will generalize Whitneys approach to infinite simplicial complexes. Some preparatory work for this has already been carried out in 2.2.24 and 2.2.25.

3.2.1 Definition (Whitney transformation). Let K be a simplicial complex, $\sigma \in K$ and $[\sigma] = [x_0, \ldots, x_s]$ be oriented arbitrarily. For each vertex x_i denote by $\beta_i := \beta_i(\sigma) : |\sigma| \to \mathbb{R}$ its barycentric coordinate function (c.f. 2.2.1). Let $c \in C^k(K)$, $k \leq s$, and define

$$w(c)(\sigma) := k! \sum_{0 \leq i_0 < \ldots < i_k \leq s} c([x_{i_0}, \ldots, x_{i_k}]) \sum_{r=0}^{k} (-1)^r \beta_{i_r} d\beta_{i_0} \wedge \ldots \wedge \widehat{d\beta_{i_r}} \wedge \ldots \wedge d\beta_{i_k}.$$

We will see in Lemma 3.2.3 below that this does not depend on the orientation chosen on σ. The resulting map $w : C^k(K) \to S^k(K)$ is called *Whitney transformation*. We employ the convention $w(c) = 0$, if $k > s$. Since this formula is rather cumbersome, we introduce the following abbreviations: For any index $I = (i_0, \ldots, i_k)$, define

$$[x_I] := [x_{i_0}, \ldots, x_{i_k}]$$

and for any $0 \le r \le k$ define

$$B_I^r := B_{i_0,\ldots,i_k}^r(\sigma) := d\beta_{i_0}(\sigma) \wedge \ldots \wedge \widehat{d\beta_{i_r}(\sigma)} \wedge \ldots \wedge d\beta_{i_k}(\sigma),$$
$$B_I := B_{i_0,\ldots,i_k}(\sigma) := d\beta_{i_0}(\sigma) \wedge \ldots \wedge d\beta_{i_k}(\sigma).$$

3.2.2 Remark (factorization). The following factorization will also help to work with the Whitney transformation: Let $k \in \mathbb{N}$, K be a simplicial complex, $\sigma \in K$, $[\sigma] = [x_0, \ldots, x_s]$ and $c \in C^k(K)$. For any increasing multi-index $I = (0 \le i_0 < \ldots < i_k \le s)$ define

$$\omega_I(\sigma) := \sum_{r=0}^{k} (-1)^r \beta_{i_r}(\sigma) B_{i_0,\ldots,i_k}^r(\sigma).$$

Let $N := \binom{s+1}{k+1}$ and $\Lambda = (I_1, \ldots, I_N)$ be an enumeration of all such multi-indices I of length $k+1$. Define the maps

$$w'(\sigma) : C^k(K) \to \mathbb{R}^N, \qquad c \mapsto (c([x_I]))_{I \in \Lambda},$$
$$w''(\sigma) : \mathbb{R}^N \to \Omega^k(\sigma), \qquad (y_I)_{I \in \Lambda} \mapsto k! \sum_{I \in \Lambda} y_I \omega_I(\sigma).$$

Then the Whitney transformation has a factorization

$$w(c)(\sigma) = (w''(\sigma) \circ w'(\sigma))(c).$$

In particular $w(\sigma)(c)$ depends only on σ and the values of c on the k-dimensional faces of σ.

3.2.3 Lemma. For every $k \in \mathbb{N}$ the Whitney transformation is a well-defined map

$$w : C^k(K) \to S^k(K).$$

With the notation above, we claim in particular that $w(c)(\sigma)$ does not depend on the orientation of σ and that the collection $\{w(c)(\sigma)\}_{\sigma \in K}$ is an S-form, i.e.

$$\forall \tau, \sigma \in K : \tau \le \sigma \Rightarrow j_{\tau,\sigma}^*(w(\sigma)) = w(\tau).$$

Proof. Let $c \in C^k(K)$ be arbitrary. It is clear that w is linear in c.

Step 1 (well-defined): The problem is that $c \in C^k(K)$ can only be applied to (equivalence classes of) oriented simplices. But an S-form attatches a differential form to a topolocial simplex $\sigma \in K$. Therefore we have to check that the orientation chosen on $[\sigma]$ does not change the differential form $w(c)(\sigma) \in \Omega^k(\sigma)$.

If $\pi \in \mathfrak{S}_{k+1}$ is any permutation,

$$c([x_{i_{\pi(0)}}, \ldots, x_{i_{\pi(k)}}]) = c(\operatorname{sgn}(\pi)[x_{i_0}, \ldots x_{i_k}]) = \operatorname{sgn}(\pi) c([x_{i_0}, \ldots x_{i_k}]). \tag{3.1}$$

Now we analyse the inner sum on the right hand side. Since any permutation can be decomposed into a finite sequence of transpositions, it suffices to show that the sign changes, whenever two adjacent indices are transposed. In order to avoid complicated notation, we illustrate this for the indices $(0, 1)$. So now let π be the permutation transposing 0 and 1, leaving all the other indices fixed. To shorten the notation further, define

$$B := d\beta_{i_2} \wedge \ldots \wedge d\beta_{i_k}, \qquad \tilde{B}_r := B_{i_2,\ldots,i_k}^r = d\beta_{i_2} \wedge \ldots \wedge \widehat{d\beta_{i_r}} \wedge \ldots \wedge d\beta_{i_k}$$

and calculate:

$$\begin{aligned}
&\sum_{r=0}^{k}(-1)^r \beta_{i_{\pi(r)}} d\beta_{i_{\pi(0)}} \wedge d\beta_{i_{\pi(1)}} \wedge d\beta_{i_{\pi(2)}} \wedge \dots \widehat{d\beta_{i_{\pi(r)}}} \wedge \dots \wedge d\beta_{i_{\pi(k)}} \\
&= \beta_{i_1} d\beta_{i_0} \wedge B - \beta_{i_0} d\beta_{i_1} \wedge B + \sum_{r=2}^{k}(-1)^r \beta_r d\beta_{i_1} \wedge d\beta_{i_0} \wedge \tilde{B}_r \\
&= -(\beta_{i_0} d\beta_{i_1} \wedge B + \beta_{i_1} d\beta_{i_0} \wedge B) - \sum_{r=2}^{k}(-1)^r \beta_{i_r} d\beta_{i_0} \wedge d\beta_{i_1} \wedge \tilde{B}_r \\
&= -\sum_{r=0}^{k}(-1)^r \beta_{i_r} d\beta_{i_0} \wedge \dots \wedge \widehat{d\beta_{i_r}} \wedge \dots \wedge d\beta_{i_k}.
\end{aligned} \tag{3.2}$$

Thus by combining (3.1) and (3.2), we see that any permutation π of the vertices of σ changes the Whitney transformation altogether by a factor $\operatorname{sgn}(\pi)\operatorname{sgn}(\pi) = 1$.

Step 2 (S-form): Let $\sigma = \langle x_0, \dots, x_s \rangle$ be an s-simplex. We first prove the case, where $\tau < \sigma$ is a boundary face. For simplicity let us assume $\tau = \langle x_0, \dots, x_{s-1} \rangle$. Then the inclusion $j_{\tau,\sigma} : \tau \hookrightarrow \sigma$ is given in barycentric coordinates by

$$x = \sum_{i=0}^{s-1} \lambda_i x_i \mapsto \sum_{i=0}^{s-1} \lambda_i x_i + 0 \cdot x_s.$$

Thus

$$\beta_i(\sigma) \circ j_{\tau,\sigma} = \begin{cases} \beta_i(\tau), & 0 \le i \le s-1 \\ 0, & i = s. \end{cases} \tag{3.3}$$

Let us analyze the expression

$$\begin{aligned}
&j^*_{\tau,\sigma}(B^r_{i_0,\dots,i_k}(\sigma)) \\
&\quad = (\beta_{i_r}(\sigma) \circ j_{\tau,\sigma}) d(\beta_{i_0}(\sigma) \circ j_{\tau,\sigma}) \wedge \dots \wedge d(\widehat{\beta_{i_r}(\sigma) \circ} j_{\tau,\sigma}) \wedge \dots \wedge d(\beta_{i_k}(\sigma) \circ j_{\tau,\sigma}),
\end{aligned} \tag{3.4}$$

where $0 \le i_0 < \dots < i_k \le s$, $0 \le r \le k$, using the relations (3.3). We distinguish two cases.

Case 1 ($s \in \{i_0, \dots, i_k\}$): This implies $i_k = s$, since the indices are increasing. Again two subcases may occur: Either $k = r$, which implies $s = i_k = i_r$, which implies that the prefactor

$$\beta_{i_r}(\sigma) \circ j_{\tau,\sigma} = \beta_s(\sigma) \circ j_{\tau,\sigma} = 0$$

vanishes in (3.4). Or $k \neq r$, which implies that the factor

$$d(\beta_{i_k}(\sigma) \circ j_{\tau,\sigma}) = d(\beta_s(\sigma) \circ j_{\tau_\sigma} = 0$$

occurs in the wedge product in (3.4). In both cases $j^*_{\tau,\sigma}(B^r_{i_0,\dots,i_k}(\sigma)) = 0$.

Case 2 ($s \notin \{i_0, \dots, i_k\}$): In that case (3.3) implies that for any $0 \le \nu \le k$, $\beta_{i_\nu}(\sigma) \circ j_{\tau,\sigma} = \beta_{i_\nu}(\tau)$, thus (3.4) equals $B^r_{i_0,\dots,i_k}(\tau)$.

Using these two cases, we obtain

$$\begin{aligned} j^*_{\tau,\sigma} w(\sigma) &= k! \sum_{0\le i_0<\ldots<i_k\le s} c([x_{i_0},\ldots,x_{i_k}]) \sum_{r=0}^{k} (-1)^r j^*_{\tau,\sigma}(B^r_{i_0,\ldots,i_k}(\sigma)) \\ &= k! \sum_{0\le i_0<\ldots<i_k\le s-1} c([x_{i_0},\ldots,x_{i_k}]) \sum_{r=0}^{k} (-1)^r B^r_{i_0,\ldots,i_k}(\tau) = w(\tau). \end{aligned}$$

Applying this argument inductively to the boundary faces of smaller dimensions, we obtain the statement for an arbitrary face $\tau \le \sigma$. □

3.2.4 Lemma. Let K be a simplicial complex, $(x_i)_{i\in\mathbb{N}}$ be an enumeration of the vertices of K, $\sigma \in K$ and let $[\sigma] := [x_{i_0},\ldots,x_{i_s}]$ be an s-simplex such that $i_0 < \ldots < i_s$. Let $\Lambda = (J_1,\ldots,J_N)$ be an enumeration of all increasing multi-indices $J = (j_0,\ldots,j_k)$ such that $\{j_0,\ldots,j_k\} \subset \{i_0,\ldots,i_s\}$. With convention 2.2.24 in power, we may write

$$w\Big(\sum_{\tau\in K^{(k)}} c([\tau])[\tau]^*\Big)(\sigma) = w(c)(\sigma) = k! \sum_{I\in\Lambda} c([x_I]) w_I(\sigma) = \sum_{\tau\in K^{(k)}} c([\tau]) w([\tau]^*)(\sigma).$$

Thus $w(c)(\sigma)$ depends only on σ and the values of c on the k-boundary of σ.

Proof. The first equality holds by convention 2.2.24, the second holds by remark 3.2.2 above. To see the last equality, notice that if τ is not in the k-boundary of σ,

$$w'(\sigma)([\tau]^*) = ([\tau]^*([x_I]))_{I\in\Lambda} = 0,$$

thus $w([\tau]^*)(\sigma) = 0$. Thus the rightmost sum is finite. In case τ is in the k-boundary of σ, there exists exactly one $J \in \Lambda$ and an orientation of τ such that $[\tau] = [x_J]$. In that case

$$w([\tau]^*)(\sigma) = k! \sum_{I\in\Lambda} [\tau]^*([x_I])\omega_I(\sigma) = k!\omega_J(\sigma).$$

Conversely for any $I \in \Lambda$ there exists exactly one k-boundary face τ such that $[x_I] = [\tau]$, which implies the statement. □

3.2.5 Lemma (cochain map properties). For any complex K and any subcomplex L, the Whitney transformation induces a well-defined chain map $w : C^*(K,L) \to S^*(K,L)$, i.e.

$$\forall \sigma \in K : d(w(c))(\sigma) = w(dc)(\sigma). \tag{3.5}$$

If K is star-bounded, w restricts to a map $w : C^*_p(K,L) \to S^*_p(K,L)$, $1 \le p \le \infty$.

Proof.

Step 1 (cochain map): Let $\sigma = \langle x_0,\ldots,x_s\rangle$ be an s-simplex and $\beta_i := \beta_i(\sigma)$ be its barycentric coordinate functions. Define the index set

$$\Lambda^k_s := \{I = (i_0,\ldots,i_k) \mid 0 \le i_0 < \ldots < i_k \le s\}.$$

STEP 1.1 ($d(w(c))(\sigma)$): The product rule for exterior differentiation implies

$$\begin{aligned} & d\Big(\beta_{i_r} d\beta_{i_0} \wedge \ldots \widehat{d\beta_{i_r}} \wedge \ldots \wedge d\beta_{i_k}\Big) \\ & = d\beta_{i_r} \wedge d\beta_{i_0} \wedge \ldots \wedge \widehat{d\beta_{i_r}} \wedge \ldots \wedge d\beta_{i_k} + \beta_{i_r} d\Big(d\beta_{i_0} \wedge \ldots \wedge \widehat{d\beta_{i_r}} \wedge \ldots \wedge d\beta_{i_k}\Big) \\ & = d\beta_{i_r} \wedge d\beta_{i_0} \wedge \ldots \wedge \widehat{d\beta_{i_r}} \wedge \ldots \wedge d\beta_{i_k} = (-1)^r d\beta_{i_0} \wedge \ldots \wedge d\beta_{i_k}. \end{aligned} \tag{3.6}$$

Thus

$$\begin{aligned} d(w(c))(\sigma) & \overset{(3.6)}{=} k! \sum_{I \in \Lambda_s^k} c([x_I]) \sum_{r=0}^{k} d\beta_{i_0} \wedge \ldots \wedge d\beta_{i_k} \\ & = (k+1)! \sum_{I \in \Lambda_s^k} c([x_I]) d\beta_{i_0} \wedge \ldots \wedge d\beta_{i_k}. \end{aligned}$$

Notice that this expression depends only on simplices in the finite complex $\mathrm{cl}(\sigma) \subset K$. Therefore, when calculating $d(w(c))(\sigma)$, we may replace the possibly infinite complex K by $\mathrm{cl}(\sigma)$. In that case, the dualized simplices form a basis of $C^k(\mathrm{cl}(\sigma))$. For any such $[\tau]^* := [x_J]^* := [x_{j_0}, \ldots, x_{j_k}]^*$, this expression simplifies to

$$d(w([\tau]^*))(\sigma) = (k+1)! d\beta_{j_0}(\sigma) \wedge \ldots \wedge d\beta_{j_k}(\sigma). \tag{3.7}$$

STEP 1.2 ($w(dc)(\sigma)$): By Lemma 3.2.4, the form $w(dc)(\sigma)$ also depends only on simplices in $\mathrm{cl}(\sigma)$. Therefore we may assume that $K = \mathrm{cl}(\sigma)$. Thus it suffices to check (3.5) on a basis, i.e. on a single dualized simplex $[\tau]^* = [x_J]^*$.
We employ the notation $I := \{0, \ldots, s\}$, $I' := I \setminus \{j_0, \ldots, j_k\}$ and the abbreviations from Definition 3.2.1, assume that the vertices are enumerated such that $j_0 < \ldots < j_k$ and calculate (see explainations (1),(2) below):

$$\begin{aligned} \frac{w(d([\tau]^*))(\sigma)}{(k+1)!} & = \sum_{I \in \Lambda_s^{k+1}} d([x_J]^*)([x_I]) \sum_{r=0}^{k+1} (-1)^r \beta_{i_r} B_I^r \\ & \overset{2.2.25}{=} \sum_{I \in \Lambda_s^{k+1}} \sum_{i \in I'} [x_i, x_J]^*([x_I]) \sum_{r=0}^{k+1} (-1)^r \beta_{i_r} B_I^r \\ & = \sum_{i \in I'} \sum_{r=0}^{k+1} \sum_{I \in \Lambda_s^{k+1}} [x_i, x_J]^*([x_I]) (-1)^r \beta_{i_r} B_I^r \\ & \overset{(1)}{=} \sum_{i \in I'} \Big(\beta_i B_J + \sum_{r=0}^{k} (-1)^{r+1} \beta_{j_r} d\beta_i \wedge B_J^r\Big) \\ & = B_J \sum_{i \in I'} \beta_i + \sum_{r=0}^{k} (-1)^{r+1} \beta_{j_r} d\Big(\sum_{i \in I'} \beta_i\Big) \wedge B_J^r \\ & \overset{2.2.12}{=} B_J \sum_{i \in I'} \beta_i + \sum_{r=0}^{k} (-1)^{r+1} \beta_{j_r} d\Big(1 - \sum_{i \in \{j_0, \ldots, j_k\}} \beta_i\Big) \wedge B_J^r \\ & = B_J \sum_{i \in I'} \beta_i + \sum_{i \in \{j_0, \ldots, j_k\}} \sum_{r=0}^{k} (-1)^r \beta_{j_r} d\beta_i \wedge B_J^r \end{aligned}$$

$$\begin{aligned}
&\overset{(2)}{=} B_J \sum_{i\in I'} \beta_i + \sum_{r=0}^{k} (-1)^r \beta_{j_r} d\beta_{j_r} \wedge B_J^r \\
&= B_J \sum_{i\in I'} \beta_i + \sum_{i\in\{j_0,\ldots,j_k\}} \beta_i B_J \\
&= \Big(\sum_{i\in I'} \beta_i + \sum_{i\in\{j_0,\ldots,j_k\}} \beta_i \Big) B_J \\
&\overset{2.2.12}{=} d\beta_{j_0} \wedge \ldots \wedge d\beta_{j_k}.
\end{aligned}$$

(1): All summands are zero unless $(i_0, i_1, \ldots, i_{k+1}) = (i, j_0, \ldots, j_k)$.
(2): The summands are zero for all r, except for the one such that $i = j_r$.

Altogether we have shown that w is a cochain map.

Step 2 ([restriction): Let $c \in C_p^k(K, L)$, $\sigma = \langle x_0, \ldots, x_s\rangle \in K$, $\theta := w(c)$. By definition of the Whitney transformation $c(\sigma) = 0 \Longrightarrow \theta(\sigma) = 0$. So the subcomplex poses no problem. Now we consider any standard chart $(\beta_0, \ldots, \widehat{\beta_i}, \ldots, \beta_s) : |\sigma| \to \Delta_s \subset \mathbb{R}^s$ (c.f. 2.2.5) and remember that K is endowed with the standard S-metric (c.f. 2.3.1). We obtain

$$|d\beta_{i_0} \wedge \ldots \widehat{d\beta_{i_r}} \wedge \ldots \wedge d\beta_{i_k}| = \sqrt{\det(\langle d\beta^i, d\beta^j\rangle)} \overset{1.2.1\text{d})}{=} \sqrt{\det(\langle \partial\beta_i, \partial\beta_j\rangle)} = 1.$$

Since $\beta_{i_r} \leq 1$, the definition of the Whitney transformation implies

$$\|\theta(\sigma)\|_{W^k_\infty(\sigma)} \leq (k+1)! \sum_{0\leq i_0<\ldots<i_k\leq s} |c([x_{i_0}, \ldots, x_{i_k}])|. \tag{3.8}$$

Let

$$\alpha(k,s) := \#\{(i_0, \ldots, i_k) \,|\, 0 \leq i_0 < \ldots < i_k \leq s\} = \binom{k+1}{s}, \qquad \alpha := \max_{0\leq s\leq m} \alpha(k,s),$$

and N be the star-bound of K. Denoting by $\sigma = \langle \sigma_{i_0}, \ldots, \sigma_{i_k}\rangle$ the various simplices in K, we obtain

$$\begin{aligned}
\|\theta\|^p_{S^k_p(K)} &= \sum_{\sigma\in K} \|\theta(\sigma)\|^p_{W^k_\infty(\sigma)} \overset{(3.8)}{\leq} \sum_{s=0}^{m} \sum_{\sigma\in K^{(s)}} ((k+1)!)^p \Big(\sum_{0\leq i_0<\ldots<i_k\leq s} |c([\sigma_{i_0}, \ldots, \sigma_{i_k}])| \Big)^p \\
&\leq \sum_{s=0}^{m} 2^{\alpha(k,s)p} ((k+1)!)^p \sum_{\sigma\in K^{(s)}} \sum_{0\leq i_0<\ldots<i_k\leq s} |c([\sigma_{i_0}, \ldots, \sigma_{i_k}])|^p \\
&\leq 2^{\alpha p} ((k+1)!)^p N m \sum_{\tau\in K^{(k)}} |c([\tau])|^p \leq \text{const}\, \|c\|^p_{C^k_p(K)}.
\end{aligned}$$

□

3.2.6 Lemma (I is surjective). The Whitney transformation w is a right-inverse of the map I from 3.1.1, i.e.

$$\forall c \in C^k(K) : \forall \sigma \in K : I(w(c))([\sigma]) = c([\sigma]).$$

Proof.

Step 1: First we check this for a single co-simplex $c = [\tau]^* \in C^k(K)$. Let $(x_i)_{i\in\mathbb{N}}$ be an enumeration of the vertices of K, $\sigma = [x_{i_0}, \ldots, x_{i_k}]$, $i_0 < \ldots < i_k$, $\tau = [x_{j_0}, \ldots, x_{j_k}]$, $j_0 < \ldots < j_k$. By definition

$$(I \circ w)(c)([\sigma]) = \int_\sigma w(c)(\sigma) = k!\, c([x_I]) \sum_{r=0}^{k} (-1)^r \int_\sigma \beta_{i_r} B^r_{i_0,\ldots,i_k}. \tag{3.9}$$

We will prove that this equals $c([\sigma])$ by induction over k.

STEP 1.1 ($k = 0$): In that case, the calculation above simplifies to

$$(I \circ w)(c)([\sigma]) = 0![\tau]^*([\sigma]) \int_{\langle x_{i_0} \rangle} \beta_{j_0} = [\tau]^*([\sigma]) = c([\sigma]).$$

(For the integral remember Convention 2.3.6.)

STEP 1.2 ($(k-1) \to k$): If $\tau \neq \sigma$, then

$$[\tau]^*([\sigma]) = 0 = I(w([\tau]^*))([\sigma])$$

by (3.9). So let $[\tau] = [\sigma]$, $[\rho] := [x_{j_1}, \dots, x_{j_k}] < [x_{j_0}, x_{j_1}, \dots, x_{j_k}] = [\tau]$ be a boundary face and $I' := I \setminus \{j_0, \dots, j_k\}$. By Lemma 2.2.25

$$d([\rho]^*) = [\tau]^* + \sum_{i \in I'} [x_i, x_{j_1}, \dots, x_{j_k}]^*. \tag{3.10}$$

Since

$$\sum_{i \in I'} [x_i, x_{j_1}, \dots, x_{j_k}]^*([\sigma]) = 0, \tag{3.11}$$

and

$$I(w([\rho]^*)) = [\rho]^*, \tag{3.12}$$

by induction hypothesis, we may calculate

$$\begin{aligned} I(w([\tau]^*))([\sigma]) &\overset{(3.10)}{=} I(w(d([\rho]^*) - \sum_{i \in I'} [x_i, x_{j_1}, \dots, x_{j_k}]^*))([\sigma]) \\ &\overset{(3.11)}{=} I(w(d([\rho]^*))([\sigma]) \overset{3.2.5}{=} d(I(w([\rho]^*))([\sigma]) \overset{(3.12)}{=} d([\rho]^*)([\sigma]) \\ &= d([\rho]^*)([\sigma]) - \sum_{i \in I'} [x_i, x_{j_1}, \dots, x_{j_k}]^*([\sigma]) = [\tau]^*([\sigma]). \end{aligned}$$

By Lemma 3.2.4 this is sufficient.

STEP 2: For a general cochain $c \in C^k(K)$, Lemma 3.2.4 again ensures the finiteness of the following sums and allows us to calculate

$$\begin{aligned} I(w(c))([\sigma]) &= \int_\sigma w(c)(\sigma) = \int_\sigma \sum_{\tau \in K^{(k)}} c([\tau]^*) w([\tau]^*)(\sigma) \\ &= \sum_{\tau \in K^{(k)}} c([\tau]^*) I(w([\tau]^*))(\sigma) = \sum_{\tau \in K^{(k)}} c([\tau]^*)[\tau]^*(\sigma) = c([\sigma]). \end{aligned}$$

□

3.2.7 Lemma (Sullivan). Let σ be an i-simplex. There exists an isomorphism of normed vector spaces

$$\mathscr{H}_p^k(\sigma, \partial\sigma) \cong \begin{cases} \mathbb{R}, & k = i \\ 0, & k < i. \end{cases}$$

In the case $k = i$, the isomorphism is given by the map

$$\begin{aligned} [\varphi] : \mathscr{H}_p^k(\sigma, \partial\sigma) &\to \mathbb{R}, \\ [\theta] &\mapsto \int_\sigma \theta(\sigma), \end{aligned}$$

and thus it is induced by I. In particular the spaces $\mathscr{B}_p^i(\sigma, \partial\sigma) \subset S_p(\sigma, \partial\sigma)$ are closed.

Proof (Sketch). This is more or less a variant of de Rhams theorem, which was also discussed by [26, 7]. Nevertheless all arguments except injectivity can be easily seen directly.
Step 1 (construction of φ): First define $\varphi : S_p^k(\sigma, \partial\sigma) \to \mathbb{R}$ by

$$\theta \mapsto \int_\sigma \theta(\sigma).$$

By Stokes' theorem and the fact that any $\omega \in S_p^{k-1}(\sigma, \partial\sigma)$ vanishes on $\partial\sigma$, we obtain

$$\varphi(d\theta) = \int_\sigma d\theta(\sigma) = \int_{\partial\sigma} \theta(\sigma) = 0.$$

Thus φ factors through the quotient and we obtain our map $[\varphi] : \mathscr{H}_p^k(\sigma, \partial\sigma) \to \mathbb{R}$.
Step 2 (surjectivity): The range of φ is one-dimensional. Since $[\varphi] \neq 0$, $[\varphi]$ is surjective.
Step 3 (closedness): The space $\mathscr{H}_p^i(\sigma, \partial\sigma) = \mathscr{Z}_p^i(\sigma, \partial\sigma)/\mathscr{B}_p^i(\sigma, \partial\sigma)$ is a quotient space. By what we have just shown, it is isomorphic to $\{0\}$ or $\mathbb{R}$, i.e. to a Banach space. Therefore $\mathscr{B}_p^i(\sigma, \partial\sigma)$ is closed. □

3.2.8 Main Theorem. Let K be a star-bounded complex and let $L \subset K$ be a subcomplex. For any $1 \leq p \leq \infty$, there are well-defined cochain maps

$$w : C_p^*(K, L) \rightleftarrows S_p^*(K, L) : I$$

inducing topological isomorphisms in cohomology, which are mutually inverse to each other.

Proof. The Lemmata 3.1.1, 3.2.3 and 3.2.5 established the maps.
Step 1 (strategy): We want to show that I induces an isomorphism $[I]$ in cohomology. We already have the commutative diagram:

$$\begin{array}{ccc} \mathscr{Z}_p^k(K, L) & \xrightarrow{I} & \mathcal{Z}_p^k(K, L) \\ \downarrow & & \downarrow \\ \mathscr{H}_p^k(K, L) & \xrightarrow[{[I]}]{} & \mathcal{H}_p^k(K, L), \end{array}$$

where the arrows to the bottom are the canonical projections. The map $I \circ w$ is the identity and this implies that I and $[I]$ are surjective (c.f. 3.2.6). In Step 42, we will show that

$$\forall \theta \in \mathscr{Z}_p^k(K, L) : I(\theta) = 0 \Rightarrow \exists \omega \in S_p^{k-1}(K, L) : d\omega = \theta \tag{3.13}$$

and in Step 42, why this implies the statement.
Step 2 (preparations): Let K^i be the i-skeleton of K and denote by $K_i := L \cup K^i$. We will construct S-forms $\omega_i \in S_p^{k-1}(K_i, L)$, $i \geq k$, such that $d\omega_i = j_{K_i,K}^*(\theta)$ by induction over i. For $i = \dim K$ this implies the claim. The space $\mathscr{B}_p^k(\sigma, \partial\sigma)$ of k-dimensional co-boundaries is closed in $S_p^k(\sigma, \partial\sigma)$ by 3.2.7. Therefore the map $d : S_p^{k-1}(\sigma, \partial\sigma) \to \mathscr{B}_p^k(\sigma, \partial\sigma)$ is an epimorphism of Banach spaces. As a direct consequence of the inverse operator theorem (c.f. [30, IV.5.2] or [21, III.11]), there exists a constant $C(\sigma) > 0$ such that

$$\forall \alpha \in \mathscr{B}_p^k(\sigma, \partial\sigma) : \exists \gamma \in S_p^{k-1}(\sigma, \partial\sigma) : d\gamma = \alpha \text{ and } \|\gamma\|_{S_p^{k-1}(\sigma,\partial\sigma)} \leq C(\sigma)\|\alpha\|_{S_p^k(\sigma,\partial\sigma)}. \tag{3.14}$$

We claim that this constant $C := C(\sigma)$ does in fact not depend on σ. This is due to the fact that any k-simplex σ is isometric to Δ_k. Therefore we obtain a commutative diagram

$$\begin{array}{ccc} S_p^{k-1}(\sigma, \partial\sigma) & \xrightarrow{d} & \mathscr{B}_p^k(\sigma, \partial\sigma) \\ \downarrow & & \downarrow \\ S_p^{k-1}(\Delta_k, \partial\Delta_k) & \longrightarrow & \mathscr{B}_p^k(\Delta_k, \partial\Delta_k), \end{array}$$

where the vertical arrows are induced by isometries. Consequently the constant $C(\Delta_k)$ does the trick.

Step 3 (construction of the form): We will now carry out the details of the construction.

Step 3.1 (induction start $i = k$): We will now construct the S-form $\omega_k \in S_p^{k-1}(K_k, L)$. If $\sigma \in K_{k-1}$, then $\omega_k(\sigma) := 0$. Let $\sigma \in K_k \setminus K_{k-1}$. By hypothesis $d\theta(\sigma) = 0$, so $\theta(\sigma)$ represents a cohomology class, and

$$0 = I(\theta)(\sigma) = \int_\sigma \theta(\sigma) = [\varphi]([\theta]),$$

where φ is as in Lemma 3.2.7. Thus $\theta(\sigma) \in \mathscr{B}_p^k(\sigma, \partial\sigma)$ and (3.14) implies

$$\exists C > 0 : \exists \gamma(\sigma) \in S_p^{k-1}(\sigma, \partial\sigma) : d\gamma(\sigma) = \theta(\sigma) \text{ and } \|\gamma(\sigma)\|_{W_\infty^{k-1}(\sigma)} \leq C\|\theta(\sigma)\|_{W_\infty^k(\sigma)} \quad (3.15)$$

(notice that $\|\_\|_{S_p^{k-1}(\sigma,\partial\sigma)}$ and $\|\_\|_{W_\infty^{k-1}(\sigma)}$ are equivalent). Our convention 2.3.6 ensures that every k-simplex σ is isometric to the standard k-simplex and thus the constant C does not depend on the simplex σ. By defining $w_k(\sigma) := \gamma(\sigma)$, we obtain

$$\|\omega_k(\sigma)\|_{W_\infty^{k-1}(\sigma)} = \|\gamma(\sigma)\|_{W_\infty^{k-1}(\sigma)} \leq C\|\theta(\sigma)\|_{W_\infty^k(\sigma)}$$

and thus

$$\|\omega_k\|^p_{S_p^{k-1}(K_k,L)} = \sum_{\sigma\in K^{(k)}} \|\omega_k(\sigma)\|^p_{W_\infty^{k-1}(\sigma)} \leq C^p \sum_{\sigma\in K^{(k)}} \|\theta(\sigma)\|^p_{W_\infty^{k-1}(\sigma)} = C^p\|\theta\|^p_{S_p^k(K,L)}.$$

Therefore $\omega_k \in S_p^{k-1}(K_k, L)$ and by construction $d\omega_k = j^*_{K_k,L}\theta$.

Step 3.2 (induction step $(i-1) \to i$): Assume now that we have constructed a form $\omega_{i-1} \in S_p^{k-1}(K_{i-1}, L)$, for which $d\omega_{i-1} = j^*_{K_{i-1},K}\theta$. By Lemma 2.3.14, there exists $\omega' \in S_p^{k-1}(K_i)$ such that $j^*_{K_{i-1},K_i}\omega' = \omega_{i-1}$.

If $\sigma \in K_{i-1}$, then set $\omega''(\sigma) := 0$. If $\sigma \in K_i \setminus K_{i-1}$, then

$$\begin{aligned} j^*_{\partial\sigma,K_i}(\theta - d\omega') &= d\omega_{i-1}(\partial\sigma) - d(j^*_{\partial\sigma,K_i}(\omega')) = 0, \\ d(\theta - d\omega') &= d\theta - dd\omega' = 0. \end{aligned}$$

Consequently $\theta - d\omega' \in \mathscr{Z}_p^k(\sigma, \partial\sigma)$. Since $\mathscr{H}_p^k(\sigma, \partial\sigma) = 0$ by 3.2.7, there exists $\gamma(\sigma) \in S_p^{k-1}(\sigma, \partial\sigma)$ such that $\theta(\sigma) - d\omega'(\sigma) = d\gamma(\sigma)$ and $\|\gamma(\sigma)\|_{W_\infty^{k-1}(\sigma)} \leq C\|\theta - d\omega'\|_{W_\infty^k(\sigma)}$. Set $\omega''(\sigma) := \gamma(\sigma)$. If $\tau < \sigma$ is any boundary face

$$j^*_{\tau,\sigma}\omega''(\sigma) = j^*_{\tau,\sigma}\gamma(\sigma) = 0 = \omega''(\tau).$$

Thus the various $\omega''(\sigma)$ assemble to an S-form $\omega'' \in S_p^{k-1}(K_i, L)$. Define $\omega_i := \omega' + \omega''$. Then $\omega_i \in S_p^{k-1}(K_i, L)$ and by construction:

$$\begin{aligned} &\forall \sigma \in K_{i-1} : d\omega_i(\sigma) = d\omega'(\sigma) + d\omega''(\sigma) = d\omega_{i-1}(\sigma) = \theta(\sigma) \\ &\forall \sigma \in K_i : d\omega_i(\sigma) = d\omega'(\sigma) + d\omega''(\sigma) = d\omega'(\sigma) + d\gamma(\sigma) = d\omega'(\sigma) + \theta(\sigma) - d\omega'(\sigma) = \theta(\sigma). \end{aligned}$$

Consequently, $d\omega_i = j^*_{K_i,K}\theta$.

STEP 4 ($[I]$ is a monomorphism): Let $\theta \in \mathscr{Z}_p^k(K,L)$ such that $[\theta] \in \ker[I]$, i.e.

$$0 = [I]([\theta]) = [I(\theta)].$$

By definition there exists $c \in \mathcal{Z}_p^k(K,L)$ such that $I(\theta) = dc$. This implies

$$0 = I(\theta) - dc = I(\theta) - I(w(dc)) = I(\theta - w(dc)).$$

Since in addition

$$d(\theta - w(dc))) = d(\theta) - d(w(d(c))) = 0 - w(d(d(c))) = 0,$$

equation (3.13) implies that there exists $\omega \in S_p^{k-1}(K,L)$ such that $\theta - w(dc) = d\omega$, which means precisely that

$$[\theta] = [w(dc) + d\omega] = [d(w(c) + \omega)] = 0. \qquad \square$$

4 Currents on manifolds

In this section we introduce the basic notions about currents on manifolds. One should think of currents as a generalization of distribution theory: Instead of working with the dual space of smooth functions having compact support in some open subset of $\mathbb{R}^m$, we work with the dual space of smooth differential forms having compact support in a manifold M. The primary reference for currents is [23, III]. For distributions one may consult [9].
In principle the theory of currents has nothing to do with L_p-cohomology. It is a topic of its own. The reason we include it here is that some properties of the regularization operators introduced in section 5 have a nicer and more general form when expressed in terms of currents. This chapter does not involve the Riemannian metric on M, so it suffices that M is a smooth oriented m-manifold without boundary.

4.1 Basic definitions

4.1.1 Definition (test forms). Denote by

$$\mathscr{D}(M) := \Omega_c(M)$$

the *space of compactly supported test forms*, i.e. the space of all smooth differential forms having compact support endowed with the following notion of convergence: A sequence of forms ω_j *converges in* $\mathscr{D}(M)$, if there exists $\omega \in \mathscr{D}(M)$ such that: There exists a compact $K \subset M$ such that

$$\forall j \in \mathbb{N} : \operatorname{supp} \omega_j \subset K$$

and a finite cover of K by charts $\varphi_i : U_i \to V_i \subset \mathbb{R}^m$, $i \in \Lambda \subset \mathbb{N}$ such that for every component function $\omega_{j,I}$ of ω_j with respect to φ_i (i.e. $\omega_j = \sum_I \omega_{j,I} d\varphi^I$)

$$\forall i \in \Lambda : \forall k \in \mathbb{N} : \|\varphi_{i*}\omega_{j,I} - \varphi_{i*}\omega_I\|_{\mathcal{C}^k(V_i)} \to 0.$$

We denote this by

$$\omega_j \xrightarrow[\mathscr{D}(M)]{j\to\infty} \omega \ .$$

We also denote by $\mathscr{D}_k(M) := \Omega_c^k(M)$ the space of compactly supported test forms of degree k (endowed with the same notion of convergence).
Denote by

$$\mathscr{E}(M) := \Omega(M)$$

the *space of test forms*, i.e. the space of all smoth differential forms endowed with the following notion of convergence: A sequence of forms ω_j *converges in* $\mathscr{E}(M)$, if there exists $\omega \in \mathscr{E}(M)$ such that: For every compact subset $K \subset M$ and every chart $\varphi : U \cap K \to V$ and for every component function $\omega_{j,I}$ of ω_j with respect to φ

$$\forall k \in \mathbb{N} : \|\varphi_*\omega_{j,I} - \varphi_*\omega_I\|_{\mathcal{C}^k(\varphi(U\cap K))} \to 0.$$

We will denote this by

$$\omega_j \xrightarrow[\mathscr{E}(M)]{j\to\infty} \omega$$

We also define $\mathscr{E}_k(M) := \Omega^k(M)$ (endowed with the same notion of convergence).

We should convince ourselves that these notions of convergence are independent of the choice of charts. Both cases are proven by the following lemma.

4.1.2 Lemma. Let $\omega_j \in \mathscr{E}(M)$. Let $K \subset U \subset M$ be compact and assume there exists a chart $\varphi : U \to V$ such that

$$\forall k \in \mathbb{N} : \|\varphi_* \omega_{j,I}\|_{\mathcal{C}^k(\varphi(K))} \to 0,$$

where $\omega_{j,I}$ are the components of ω_j with respect to φ. If $\psi : U \to \tilde{V}$ is any other chart and $\tilde{\omega}_{j,J}$ are the components of ω_j w.r.t. ψ, then

$$\forall k \in \mathbb{N} : \|\psi_* \tilde{\omega}_{j,I}\|_{\mathcal{C}^k(\psi(K))} \to 0.$$

as well.

Proof.
STEP 1: For any k-times differentiable function $h : U \subset \mathbb{R}^n \to \mathbb{R}^m$ define

$$\|h\|_{\mathcal{C}^k(U)} := \sup_{x \in U} \max_{\alpha \in \mathbb{N}_0^n : \, |\alpha| \leq k} \max_{1 \leq i \leq m} |\partial^\alpha h^i|(x).$$

Now let $F \in \mathcal{C}_b^k(U \subset \mathbb{R}^n, V \subset \mathbb{R}^m)$, $k \geq 1$, and $g \in \mathcal{C}_b^k(V, \mathbb{R})$. We claim there exists $C > 0$ such that

$$\|g \circ F\|_{\mathcal{C}^k(U)} \leq C_k \|F\|_{\mathcal{C}^k(U)} \|g\|_{\mathcal{C}^k(V)}. \tag{4.1}$$

This can be proven by induction over k using the chain rule and the Leibniz rule: For $k = 1$, this follows from

$$|\partial_j (g \circ F)| \leq \sum_{i=1}^n |\partial_i F^j \, \partial_i g| \leq n \|F\|_{\mathcal{C}^1(U)} \|g\|_{\mathcal{C}^1(V)}.$$

For the induction step, we just notice that for any $\alpha \in \mathbb{N}^n$ such that $|\alpha| = k+1$ there exist $\beta \in \mathbb{N}^n$ and $1 \leq j \leq n$ such that $|\beta| = k$ and $\alpha = \beta + e_j$. Therefore

$$\begin{aligned} |\partial^\alpha (g \circ F)| = |\partial^\beta \partial_j (g \circ F)| &\leq \sum_{i=1}^n |\partial^\beta (\partial_i F^j \partial_i g)| \\ &\leq \sum_{i=1}^n \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} |\partial^\gamma \partial_i F^j \, \partial^{\beta - \gamma} \partial_i g)| \leq C \|F\|_{\mathcal{C}^{k+1}} \|g\|_{\mathcal{C}^{k+1}(V)}. \end{aligned}$$

STEP 2: Let $F := \psi \circ \varphi^{-1} : V \to \tilde{V}$ be the transition map and let $A := \nabla F$ be its Jacobian. The coordinates transform by

$$\tilde{\omega}_J = \omega(\partial \psi_{j_1}, \ldots, \partial \psi_{j_\nu}) = \omega(A_{j_1}^{i_1} \partial \varphi_{i_1}, \ldots, A_{j_\nu}^{i_k} \partial \varphi_{i_\nu}) =: A_J^I \omega_I,$$

where $A_J^I := A_{j_1}^{i_1} \ldots A_{j_k}^{i_k}$. We obtain

$$\begin{aligned} \|\psi_* \tilde{\omega}_{j,J}\|_{\mathcal{C}^k(\psi(K))} &= \|\tilde{\omega}_{j,J} \circ \varphi^{-1} \circ \varphi \circ \psi^{-1}\|_{\mathcal{C}^k(\psi(K))} \\ &\overset{\text{Step 1}}{\leq} C \|F^{-1}\|_{\mathcal{C}^k(\psi(K))} \|\tilde{\omega}_{j,J} \circ \varphi^{-1}\|_{\mathcal{C}^k(\varphi(K))} \\ &= C \|F^{-1}\|_{\mathcal{C}^k(\psi(K))} \|A_J^I \omega_{j,J} \circ \varphi^{-1}\|_{\mathcal{C}^k(\varphi(K))}. \end{aligned}$$

STEP 3: This implies the statement: The $A^I_J \circ \varphi^{-1}$ are a finite number of functions, which are all bounded in $\mathcal{C}^k(\varphi(K))$. Therefore by the Leibniz rule

$$\begin{aligned}\|A^I_J \omega_{j,J} \circ \varphi^{-1}\|_{\mathcal{C}^k(\varphi(K))} &= \|A^I_J \circ \varphi^{-1} \cdot \omega_{j,J} \circ \varphi^{-1}\|_{\mathcal{C}^k(\varphi(K))} \leq C\|\omega_{j,J} \circ \varphi^{-1}\|_{\mathcal{C}^k(\varphi(K))} \\ &= C\|\varphi_* \omega j, J\|_{\mathcal{C}^k(\varphi(K))} \to 0.\end{aligned}$$

□

4.1.3 Definition (current). A linear functional $T : \mathscr{D}(M) \to \mathbb{R}$ is *continuous*, if for any sequence $\{\omega_j\}$ in $\mathscr{D}(M)$

$$\omega_j \xrightarrow[\mathscr{D}(M)]{} \omega \implies T(\omega_j) \xrightarrow[\mathbb{R}]{} T(\omega) .$$

A continuous linear functional $T : \mathscr{D}(M) \to \mathbb{R}$ is called *current*. We denote by

$$\mathscr{D}'(M)$$

the space of all currents on M.

4.1.4 Definition (restriction). If $U \subset M$ is open, any form $\omega \in \mathscr{D}(U)$ can be extended by zero outside U to a form $\tilde{\omega} \in \mathscr{D}(M)$. If $T \in \mathscr{D}(M)$ we say $T|_U : \mathscr{D}(U) \to \mathbb{R}$,

$$\omega \mapsto T(\tilde{\omega}),$$

is the *restriction of* T *to* U. It is clear that $T|_U \in \mathscr{D}'(U)$.

4.1.5 Lemma (first sheaf axiom). For any open set $U \subset M$, any open cover $\{U_i\}_{i\in\mathbb{N}}$ of U and any $T \in \mathscr{D}'(U)$:

$$\forall i \in \mathbb{N} : T|_{U_i} = 0 \implies T = 0.$$

Proof. Take a partition of unity $\{\psi_i\}$ subordinate to the $\{U_i\}$. Let $\omega \in \mathscr{D}(U)$ be arbitrary. Since $\operatorname{supp}\omega$ is compact, it is contained in the union of finitely many U_i. So $\omega = \sum_{i\in\mathbb{N}} \psi_i \omega$, where all terms are zero except for finitely many i. Let $I \subset \mathbb{N}$ be the finite set of those i. We calculate

$$T(\omega) = T\Big(\sum_{i\in\mathbb{N}} \psi_i \omega\Big) = \sum_{i\in I} T|_{U_i}(\psi_i \omega) = 0.$$

□

4.1.6 Remark. It is certainly not necessary to think of currents as a sheaf. Nevertheless it is interesting to know that $\mathscr{D}'$ also satisfies the second sheaf axiom: For every open cover $\{U_i\}_{i\in\mathbb{N}}$ and any given $T_i \in \mathscr{D}'(U_i)$ such that $T_i|_{U_i\cap U_j} = T_j|_{U_i\cap U_j}$ there exists $T \in \mathscr{D}'(U)$ such that for any $i \in \mathbb{N}$, we have $T|_{U_i} = T_i$. For distributions a proof of this can be found in [9, 2.2.4].

4.1.7 Definition (support). Let $T \in \mathscr{D}'(M)$. We say $p \in M$ *is not in the support of* T, if there is an open neighbourhood U of p such that $T|_U = 0$. The complement of the set of all points not in the support of T is the *support of* T and is denoted by $\operatorname{supp} T$.

4.1.8 Definition (currents with compact support). A current $T \in \mathscr{D}'(M)$ has *compact support*, if $\operatorname{supp} T \subset M$ is compact. The set of all such currents is denoted by

$$\mathscr{E}'(M).$$

4.1.9 Remark. This notation is due to the fact that $\mathscr{E}'(M)$ is the topological dual of $\mathscr{E}$(M). We can apply a $T \in \mathscr{E}'(M)$ to a form $\omega \in \mathscr{E}(M)$ as follows: Since $\operatorname{supp} T =: K$ is compact, there exists a smooth bump function $\psi \in \mathcal{C}_c^\infty(M)$ and an open set $K \subset U \subset M$ such that $\psi|_K \equiv 1$, $\operatorname{supp} \psi \subset U$ and $\psi|_{M \setminus U} \equiv 0$. Then $\omega\psi \in \mathscr{D}(M)$ and consequently, we may apply the compactly supported $T \in \mathscr{D}'(M)$ to $\omega\psi$:

$$T(\omega) := T(\omega\psi).$$

This does not depend on the chosen ψ, since if $\tilde{\psi}$ is any other such function $\psi\omega - \tilde{\psi}\omega = 0$. Linearity of T implies $0 = T(\psi\omega - \tilde{\psi}\omega) = T(\psi\omega) - T(\tilde{\psi}\omega)$.

4.1.10 Definition (homogenous). A current $T \in \mathscr{D}'(M)$ is *homogenous of order* k, if

$$\forall \omega \in \mathscr{D}(M) : \deg \omega \neq k \Longrightarrow T(\omega) = 0.$$

Similar, a current $T \in \mathscr{E}'(M)$ is *homogenous of order* k, if

$$\forall \omega \in \mathscr{E}(M) : \deg \omega \neq k \Longrightarrow T(\omega) = 0.$$

In both cases, we call $m - k$ the *degree of* T. We denote all such currents by $\mathscr{D}'_{m-k}(M)$ respectively $\mathscr{E}'_{m-k}(M)$.

4.1.11 Lemma. Any form $\omega \in L^k_{1,\mathrm{loc}}(M)$ defines a current by setting $\langle\omega\rangle : \mathscr{D}_{m-k}(M) \to \mathbb{R}$

$$\eta \mapsto \int_M \omega \wedge \eta,$$

and $\langle\omega\rangle(\eta) := 0$, if $\deg \eta \neq m - k$. Consequently the order of $\langle\omega\rangle$ is $m - k$ and the degree of $\langle\omega\rangle$ is $m - (m - k) = k$. This defines an injective embedding $\langle\ \rangle : L^k_{1,\mathrm{loc}} \to \mathscr{D}'_k(M)$, which allows us to identify the form ω with its generated current $\langle\omega\rangle$.

Proof. This can be proven by the same method as in 2.1.13. □

4.1.12 Definition (convergence of currents). Convergence of currents $\{T_j\}$ in $\mathscr{D}'(M)$ respectively $\mathscr{E}'(M)$ is defined as

$$T_j \xrightarrow[\mathscr{D}'(M)]{} T \quad :\Longleftrightarrow \quad \forall \omega \in \mathscr{D}(M) : \ T_j(\omega) \xrightarrow[\mathbb{R}]{} T(\omega)\ ,$$

resp:

$$T_j \xrightarrow[\mathscr{E}'(M)]{} T \quad :\Longleftrightarrow \quad \forall \omega \in \mathscr{E}(M) : \ T_j(\omega) \xrightarrow[\mathbb{R}]{} T(\omega)\ .$$

This enables us to define continous operators between currents.

4.1.13 Definition (continuous operator). A linear map $\Psi : \mathscr{D}'(M) \to \mathscr{D}'(N)$ is *continuous*, if for any sequence of currents $\{T_j\}, T \in \mathscr{D}'(M)$:

$$T_j \xrightarrow[\mathscr{D}'(M)]{} T \quad \Longrightarrow \quad \Psi(T_j) \xrightarrow[\mathscr{D}'(N)]{} \Psi(T)\ .$$

In that case Ψ is a *continuous operator*.

In 2.1.9 we defined the notion of a weak differential for L_p-forms, which led to our definition of exterior Sobolev spaces in 2.1.10. This construction also works for currents.

4.1.14 Definition (exterior differential). Let $T \in \mathscr{D}'_k(M)$. For any $\eta \in \mathscr{D}_{m-k-1}(M)$ define

$$dT(\eta) := (-1)^{k+1} T(d\eta).$$

This defines a map $dT : \mathscr{D}(M) \to \mathbb{R}$, the *exterior differential of* T.

Exactly as in the case of Sobolev spaces, we have the following Lemma.

4.1.15 Lemma (properties of exterior distributional differential). Let $T \in \mathscr{D}'(M)$.

(i) $d : \mathscr{D}(M) \to \mathscr{D}(M)$ is continuous.

(ii) $dT \in \mathscr{D}'(M)$, i.e. $d : \mathscr{D}'(M) \to \mathscr{D}'(M)$ is a continuous operator.

(iii) $d^2 T = 0$.

(iv) If $T \in \mathscr{D}'_k(M)$, then $dT \in \mathscr{D}'_{k+1}(M)$.

(v) The following diagram commutes (provided $\partial M = \emptyset$):

$$\begin{array}{ccc} W^k_{1,\mathrm{loc}}(M) & \xrightarrow{\langle \_ \rangle} & \mathscr{D}'_k(M) \\ \downarrow d & & \downarrow d \\ L^{k+1}_{1,\mathrm{loc}}(M) & \xrightarrow{\langle \_ \rangle} & \mathscr{D}'_{k+1}(M) \end{array}$$

Proof.

(i) Assume $\omega_j \in \mathscr{D}'(M)$ such that

$$\omega_j \xrightarrow[\mathscr{D}]{j\to\infty} 0 \,.$$

By definition this means that there exists a compact subset $K \subset M$ such that $\operatorname{supp} \omega_j \subset K$ and for any chart $\varphi : U \to V$, $U \cap K \neq \emptyset$,

$$\forall k \in \mathbb{N} : \|\varphi_* \omega_{j,I}\|_{\mathcal{C}^k(\varphi(K\cap U))} \to 0.$$

In other words ω_j already tends to zero with all its derivatives. Since the component functions of $d\omega_j$ are given by the derivatives of the component functions of ω_j, the statement follows.

(ii) Is a direct consequence of (i).

(iii) Clear.

(iv) Clear.

(v) Let $\omega \in W^k_{1,\mathrm{loc}}(M)$ and $\eta \in \mathscr{D}_{m-k-1}(M)$. By definition

$$\langle d\omega \rangle(\eta) = \int_M d\omega \wedge \eta = (-1)^{k+1} \int_M \omega \wedge d\eta = (-1)^{k+1} \langle \omega \rangle (d\eta) = d(\langle \omega \rangle)(\eta). \qquad \square$$

4.1.16 Definition (distributional de-Rham-Complex). Lemma 4.1.14 allows us to define the *distributional de Rham complex* to be the chain complex of vector spaces $\mathscr{D}'_k(M)$ with the distributional exterior differential $d : \mathscr{D}'_k(M) \to \mathscr{D}'_{k+1}(M)$.

4.1.17 Definition (multi-Kronecker delta). Let $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_l)$ be two multi-indices. We denote their *concatination* by

$$IJ := (i_1, \ldots, i_k, j_1, \ldots, j_l).$$

If $\pi \in \mathfrak{S}_k$ is a permutation, we denote by

$$\pi(I) := (i_{\pi(1)}, \dots, i_{\pi(k)}).$$

Furthermore we denote by

$$\delta^I_J := \begin{cases} \operatorname{sgn} \pi, & \text{if neither } I \text{ nor } J \text{ has a repeated index and } J = \pi(I) \text{ for some permutation } \pi, \\ 0, & \text{if } I \text{ or } J \text{ has a repeated index or } J \text{ is not a permutation of } I, \end{cases}$$

the *Kronecker delta for multi-indices.*. Denote

$$E := E_m := (1, \dots, m).$$

and for any I denote by I^c the complementary multi-index to E, i.e. if $I = (i_1, \dots, i_k)$, then $I^c = J = (j_1, \dots, j_{m-k})$ is the multi-index, obtained by taking the increasing order of the set $\{1, \dots, m\} \setminus \{i_1, \dots, i_k\}$.

4.1.18 Definition (exterior product). Let $T \in \mathscr{D}'(M)$ and $\alpha \in \mathscr{E}(M)$. Then we denote by

$$T \wedge \alpha \in \mathscr{D}'(M)$$

the *exterior product between T and α* defined by

$$\forall \omega \in \mathscr{D}'(M) : (T \wedge \alpha)(\omega) := T(\alpha \wedge \omega).$$

If T is homogenous of degree k and $\omega \in \mathscr{E}_l(M)$, we define

$$\alpha \wedge T := (-1)^{kl} T \wedge \alpha.$$

4.1.19 Theorem (local decomposition). Let $\varphi : U \to V$ be a chart and let $T \in \mathscr{D}'_k(U)$. Then there exists a unique decomposition

$$T = \sum_{I \in \mathcal{I}_k} T_I \wedge d\varphi^I, \tag{4.2}$$

where $T_I \in \mathscr{D}'_0(U)$ and as usual $\mathcal{I}_k$ is the set of all increasing multi-indices $I = (i_1, \dots, i_k)$). The currents T_I satisfy

$$T_I(d\varphi^E) = \delta^E_{II^c} T(d\varphi^{I^c}).$$

In particular if $T = \langle \eta \rangle$, $\eta \in L_{1,\mathrm{loc}}(M)$, then $\langle \eta \rangle_I = \langle \eta_I \rangle$.

Proof. By definition, if $\omega \in \mathscr{D}(U)$ is of degree $m - k$

$$(T_I \wedge d\varphi^I)(\omega) = T_I(d\varphi^I \wedge \omega),$$

so the right hand side is indeed a well-defined current of degree k.

Step 1 (uniqueness): Assume T is decomposed as in (4.2). Let $J = (j_1, \dots, j_{m-k})$ be an increasing multi-index. We obtain

$$T(d\varphi^J) = \sum_I (T_I \wedge d\varphi^I)(d\varphi^J) = \sum_I T_I(d\varphi^I \wedge d\varphi^J) = \delta^E_{J^c J} T_{J^c}(d\varphi^E),$$

so the T_I are uniquely determined by T.

STEP 2 (existence): We have no choice, but to define

$$T_I(d\varphi^E) := \delta^E_{II^c} T(d\varphi^{I^c}).$$

We already discussed that the wedge-combination of these expressions is a current. The equality holds by construction.

STEP 3: The general formula for T_I follows already from what we have proven. In case $T = \langle \eta \rangle$, we just verify

$$\begin{aligned}\langle \eta \rangle_I(d\varphi^E) &= \delta^E_{II^c} \langle \eta \rangle (d\varphi^{I^c}) = \delta^E_{II^c} \int_U \eta \wedge d\varphi^{I^c} = \delta^E_{II^c} \int_U \eta_J d\varphi^J \wedge d\varphi^{I^c} \\ &= \delta^E_{II^c} \int_U \eta_I d\varphi^{II^c} = \int_U \eta_I d\varphi^E = \langle \eta_I \rangle (d\varphi^E).\end{aligned}$$ □

In closing we say a word concerning currents on product manifolds. These are systematically studied by de Rham in [23, III.§12], where he introduces them as *double currents*. Even in classical distribution theory the study of distributions on product spaces is a rather involved subject culminating in the celebrated Schwarz Kernel Theorem, c.f. [9, V] We don't want to elaborate on this topic here and merely remind of some rather easy results from distribution theory and reformulate them in terms of currents.

4.1.20 Theorem. Let $T_1 \in \mathscr{D}'(M_1)$, $T_2 \in \mathscr{D}'(M_2)$. There exists a current $T \in \mathscr{D}'(M_1 \times M_2)$ such that

$$\forall \varphi_1 \in \mathscr{D}(M_1) : \forall \varphi_2 \in \mathscr{D}(M_2) : T(\varphi_1 \wedge \varphi_2) = T_1(\varphi_1) T_2(\varphi_2).$$

This current T satisfies

$$\forall \psi \in \mathscr{D}(M_1 \times M_2) : T(\psi) = T_1(x_1 \mapsto T_2(x_2 \mapsto \psi(x_1, x_2))) = T_2(x_2 \mapsto T_1(x_1 \mapsto \psi(x_1, x_2)))$$

and is called the *tensor product of* T_1 *and* T_2. We define

$$T_1 \otimes T_2 := T.$$

Analogous statements hold if $\mathscr{D}'$ is replaced by $\mathscr{E}'$.

5 Regularization operators

In this section we introduce regularization operators. They will be required for the proof of the Main Theorem 6.2.1 and imply some nice relations between L_p-cohomology and classical de Rham cohomology of a manifold. We will basically follow [3] and [23].
The setup for this section is the following: $U \subset \mathbb{R}^m$ is an open set equipped with an arbitrary Riemannian metric g. We will denote the induced norm by $|\_|$. The Euclidean norm is still denoted by $\|\_\|$. We assume that U contains the closed Euclidean unit ball $\overline{B} := \overline{B_1(0)}$.

5.1 Notation and technical preliminaries

Before we start, let us briefly collect some basic calculus facts about regularization techniques in $\mathbb{R}^n$.

5.1.1 Definition (ε-neighbourhood). Let (X, d) be a metric space, $A \subset X$ and $\varepsilon > 0$. Then

$$A^\varepsilon := \mathcal{O}_\varepsilon(A) := \{x \in X \mid \exists a \in A : d(x,a) < \varepsilon\} = \bigcup_{a \in A} B_\varepsilon(a)$$

is the *ε-neighbourhood of A*.

5.1.2 Theorem (standard mollifier). Let $\psi : \mathbb{R} \to \mathbb{R}$ be defined by

$$x \mapsto \begin{cases} \exp(-\frac{1}{x}), & x > 0 \\ 0, & x \leq 0. \end{cases}$$

For any $\varepsilon > 0$, define $\varphi, \varphi_\varepsilon : \mathbb{R}^n \to \mathbb{R}$

$$c^{-1} := \int_{\mathbb{R}^n} \psi(1 - \|x\|^2)dx, \qquad \varphi(x) := c\psi(1 - \|x\|^2), \qquad \varphi_\varepsilon(x) := \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right).$$

Then the following holds:

(i) $\psi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$, $\varphi, \varphi_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}_{\geq 0})$.

(ii) $\int_{\mathbb{R}^n} \varphi_\varepsilon(x)dx = 1$.

(iii) $\operatorname{supp} \varphi_\varepsilon \subset B_\varepsilon(0)$.

(iv) $0 \leq \varphi_\varepsilon \leq \frac{c}{\varepsilon^n}$.

We call φ_ε the *standard mollifier* due to the following theorem.

5.1.3 Theorem. For any $f \in L_1(\mathbb{R}^n)$

(i) $f * \varphi_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n)$,

(ii) $\operatorname{supp} f * \varphi_\varepsilon \subset \mathcal{O}_\varepsilon(\operatorname{supp} f)$,

(iii) $f * \varphi_\varepsilon \xrightarrow[L_1]{\varepsilon \to 0} f$,

(iv) $f * \varphi_\varepsilon \xrightarrow[\mathcal{C}^0(\mathbb{R}^n)]{\varepsilon \to 0} f$, whenever f is uniformly continuous and bounded,

(v) $f * \varphi_\varepsilon \xrightarrow[\mathcal{C}^k(\mathbb{R}^n)]{\varepsilon \to 0} f$, whenever $f \in \mathcal{C}^k_c(\mathbb{R}^n)$.

(vi) $f * g_j \xrightarrow[\mathcal{C}^k]{j\to\infty} f * g$, whenever $g_j \in \mathcal{C}_c^k(\mathbb{R}^n)$ such that $g_j \xrightarrow[\mathcal{C}^k]{j\to\infty} g$.

5.1.4 Definition (interior multiplication). Let M be a manifold, $X \in \mathcal{T}(M)$, $\omega \in \Omega^k(M)$. The map $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$, defined by

$$\iota_X(\omega)(Y_1, \dots, Y_{k-1}) := \omega(X, Y_1, \dots, Y_k),$$

is the *interior multiplication with* X.

5.1.5 Theorem (properties of interior multiplication). The interior multiplication satisfies:

(i) $\iota_X\omega$ is linear in X and ω.

(ii) $\iota_Y^2 = 0$ and therefore $\iota_X \circ \iota_Y = -\iota_Y \circ \iota_X$.

(iii) If $F : M \to N$, $X \in \mathcal{T}(M)$, $Y \in \mathcal{T}(N)$, X and Y are F-related, $\omega \in \Omega(N)$, then

$$\iota_X \circ F^* = F^* \circ \iota_Y.$$

(iv) "Cartans Magic formula"

$$d \circ \iota_Y + \iota \circ d_Y = \mathscr{L}_Y,$$

where $\mathscr{L}$ is the Lie derivative.

(v) Anti-Derivation-Property

$$\forall \omega \in \Omega^k(M) : \forall \eta \in \Omega^l(M) : \forall X \in \mathcal{T}(M) : \iota_X(\omega \wedge \eta) = \iota_X(\omega) \wedge \eta + (-1)^k \omega \wedge \iota_X(\eta).$$

Proof. See [20, p. 379] □

5.1.6 Convention (extensions to products). Let M, N be smooth manifolds and $X \in \mathcal{T}(M)$, $Y \in \mathcal{T}(N)$. Then both fields admit an extension to the product manifold $M \times N$ by defining

$$\forall (p,q) \in M \times N : \tilde{X}_{(p,q)} := X_p + 0 \in T_pM \oplus T_qN \cong T_{(p,q)}(M \times N),$$
$$\forall (p,q) \in M \times N : \tilde{Y}_{(p,q)} := 0 + Y_q \in T_pM \oplus T_qN \cong T_{(p,q)}(M \times N).$$

Therefore we may routinely extend suhc fields to $\tilde{X}, \tilde{Y} \in \mathcal{T}(M \times N)$. If $\pi_M : M \times N \to M$, $\pi_N : M \times N \to N$ are the canonical projections, $\tilde{X}$ is π_M-related to X and $\tilde{Y}$ is π_N-related to Y.

5.2 Localization

Regulariziers may be constructed rather easily when the involved domain of definition is $\mathbb{R}^m$. But we will require them to work on a neighbourhood of the unit ball B, since later, we want to use this domain to regularize forms on manifolds. Therefore we need to establish some tools in order to localize the theory. All these techniques were already developed by de Rham in [23, III,§15] and are admittedly very technical. Nevertheless we will give a revised self-contained treatment of this theory, because [23] is not very detailed, the notation is a bit cumbersome, it contains several typos and this book is no longer printed.

5.2.1 Lemma (localization).

(i) There exists a function $\eta \in \mathcal{C}^\infty(]0,1[,]0,\infty[)$ such that

$$\eta(r) = \begin{cases} r, & r \in]0, \frac{1}{3}[\\ e^{(r-1)^{-2}}, & r \in]\frac{2}{3}, 1[\end{cases},$$

such that $\eta' > 0$. Define $\eta_0 := \eta|_{[\frac{1}{3},\frac{2}{3}]}$. The function η is a smooth diffeomorphism.

(ii) The function $\psi : B \to \mathbb{R}^m$,

$$x \mapsto \begin{cases} \frac{\eta(|x|)}{|x|}x, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is a smooth diffeomorphism with inverse $\psi^{-1} : \mathbb{R}^m \to B$,

$$y \mapsto \begin{cases} \frac{\eta^{-1}(|y|)}{|y|}y, & y \neq 0, \\ 0, & y = 0. \end{cases}$$

(iii) For any $k \in \mathbb{N}$

$$\eta^{(k)}(r) \xrightarrow[r\nearrow 1]{} \infty \,.$$

Define $\theta := 1/\eta$. Then for any $k \in \mathbb{N}$

$$\theta^{(k)}(r) \xrightarrow[r\nearrow 1]{} 0 \,.$$

Proof.

(i) Since

$$\lim_{r\nearrow\frac{1}{3}} r = \frac{1}{3} < e^9 = \lim_{r\swarrow\frac{2}{3}} e^{(r-1)^{-2}},$$

it is clear that η_0 may be chosen such that η is smooth and $\eta' > 0$. Therefore η is a diffeomorphism.

(ii) The smoothness of ψ follows from the fact that if $0 < |x| < \frac{1}{3}$, this implies

$$\psi(x) = \frac{\eta(|x|)}{|x|}x = \frac{|x|}{|x|}x = x.$$

Therefore ψ is smooth even at $x = 0$ and so is ψ^{-1} (by the same reasoning). Since η and η^{-1} are inverse to each other, we obtain

$$\forall x \neq 0 : \psi^{-1}(\psi(x)) = \frac{\eta^{-1}(|\psi(x)|)}{|\psi(x)|}\psi(x) = \frac{\eta^{-1}(\eta(|x|))}{\eta(|x|)}\frac{\eta(|x|)}{|x|}x = x,$$

and an analogous calculation holds for $\psi \circ \psi^{-1}$. Therefore ψ and ψ^{-1} are inverse to each other.

(iii) First we show that for any $k \in \mathbb{N}$ there are polynomials p_{3k} of degree $3k$ such that for any $r > 2/3$

$$\eta^{(k)}(r) = p_{3k}((r-1)^{-1})e^{(r-1)^{-2}}.$$

This can be seen by induction: In case $k = 0$, the statement is trivial. For the induction step $k \to k+1$, we calculate

$$\eta^{(k+1)}(r) = \left(p_{3k}((r-1)^{-1})e^{(r-1)^{-2}}\right)'(r)$$
$$= \underbrace{-(p'_{3k}((r-1)^{-1})(r-1)^{-2} + 2(r-1)^{-3}p_{3k}((r-1)^{-1}))}_{=:p_{3(k+1)}((r-1)^{-1})} e^{(r-1)^{-2}}.$$

This immediately implies the first statement. The second follows from the generalized reciprocal rule (taken from an exercise in [32, (0.4)]) for the k-th derivative of the function $\theta = 1/\eta$:

$$\theta^{(k)}(r) = \sum_{\beta_1+\ldots\beta_k=k} C_{\beta_1,\ldots,\beta_k} \frac{\eta^{(\beta_1)}(r)\ldots\eta^{(\beta_k)}(r)}{\eta^{k+1}(r)}$$
$$= \sum_{\beta_1+\ldots\beta_k=k} C_{\beta_1,\ldots,\beta_k} \frac{\overbrace{p_{3\beta_1}((r-1)^{-1})\ldots p_{3\beta_k}((r-1)^{-1})}^{=:P_{3k}(r-1)^{-1}} e^{k(r-1)^{-2}}}{e^{(k+1)(r-1)^{-2}}}$$
$$= \sum_{\beta_1+\ldots\beta_k=k} C_{\beta_1,\ldots,\beta_k} P_{3k}((r-1)^{-1})e^{-(r-1)^{-2}} \to 0,$$

as $r \to 1$. □

5.2.2 Theorem. For any $y \in \mathbb{R}^n$, let $\tau_y : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x+y$, be the translation. Define the map $s : \mathbb{R}^m \times U \to U$ by

$$(y,x) \mapsto \begin{cases} (\psi^{-1} \circ \tau_y \circ \psi)(x), & x \in B \\ x, & x \notin B. \end{cases}$$

We call s the *localized translation group.*

(i) For any $y \in \mathbb{R}^m$, the map $s_y : U \to U$, $x \mapsto s(y,x)$, is a smooth diffeomorphism with inverse

$$s_y^{-1}(x) = \begin{cases} (\psi^{-1} \circ \tau_{-y} \circ \psi)(x), & x \in B \\ x, & x \notin B \end{cases}.$$

Clearly $s_y|_{U \setminus B} = \mathrm{id}$.

(ii) For any point $x \in B$, the map $\alpha_x : \mathbb{R}^m \to B$, $y \mapsto s(y,x)$, is a diffeomorphism and $\alpha_x^{-1} : B \to \mathbb{R}^m$ is given by

$$w \mapsto (\tau_{-\psi(x)} \circ \psi)(w).$$

(iii) The map s is smooth.

Proof.

STEP 1: Since

$$\forall y \neq 0 : |\psi^{-1}(y)| = \eta^{-1}(|y|) \in]0,1[,$$

we obtain

$$\forall y \in \mathbb{R}^n : \forall x \in U : s(y,x) = \begin{cases} \psi^{-1}(\tau_y(\psi(x))), & x \in B \\ x, & x \in U \setminus B \end{cases} \in U.$$

Thus s has the correct range. This implies that all the s_y have the correct range as well and that $s_y|_{U\setminus B} = \mathrm{id}$. For $x \in B$, we calculate

$$z = s(y,x) = \psi^{-1}(\tau_y(\psi(x))) = \psi^{-1}(\psi(x)+y) \Leftrightarrow \psi(z) = \psi(x)+y,$$

which implies

$$s_y^{-1} = \psi^{-1} \circ \tau_{-y} \circ \psi, \qquad \alpha_x^{-1} = \tau_{-\psi(x)} \circ \psi.$$

STEP 2 (smoothness): We have to show that $s \in \mathcal{C}^\infty(\mathbb{R}^m \times U, U)$, i.e. we have to check that $(y,x) \mapsto s(y,x)$ is smooth in every component x_i, y_i, $1 \le i \le m$. Clearly, for any fixed $x \in U$, the map $\mathbb{R}^m \to U$

$$y \mapsto \alpha_x(y) = s(y,x) = \begin{cases} (\psi^{-1} \circ \tau_y \circ \psi)(x), & x \in B \\ x, & x \notin B \end{cases},$$

is smooth in both cases. The crucial problem is to show that for any fixed $y \in \mathbb{R}^m$, the map $s_y : U \to U$ is smooth. We will obtain this result in several substeps.

STEP 2.1 (flow): For any $1 \le i \le m$ define $\beta_i : \mathbb{R} \times U \to U$, $(t,x) \mapsto s(te_i, x)$. Fix any $1 \le i \le m$. We claim that β_i is a flow, i.e. it satisfies for any $x \in U$ and any $t_1, t_2 \in \mathbb{R}$

$$\beta_i(0,x) = x, \qquad \beta_i(t_1+t_2, x) = \beta_i(t_1, \beta_i(t_2, x)).$$

This follows from the following calculations (let $y, y' \in \mathbb{R}^m$):

$$s(0,x) = \begin{cases} \psi^{-1}(\tau_0(\psi(x))), & x \in B \\ x, & x \notin B \end{cases} = x \tag{5.1}$$

$$\begin{aligned} s(y, s(y',x)) &= \begin{cases} s(y, (\psi^{-1} \circ \tau_{y'} \circ \psi)(x)), & x \in B \\ s(y,x), & x \notin B \end{cases} \\ &\overset{(*)}{=} \begin{cases} (\psi^{-1} \circ \tau_y \circ \psi \circ \psi^{-1} \circ \tau_{y'} \circ \psi)(x), & x \in B \\ x, & x \notin B \end{cases} &(5.2) \\ &= s(y+y', x) &(5.3) \end{aligned}$$

(*): Notice that $s_y|_{U\setminus B} = \mathrm{id} : U \setminus B \to U \setminus B$ is bijective and consequently s_y restrict to a map $s_y|_B : B \to B$. Thus if $x \in B$, so is $s_y(x)$ and we do not need another two case differentiations.

This implies that

$$X_i(x) := \partial_t(\beta_i(t,x))|_{t=0} = \begin{cases} (\nabla\psi^{-1}(\psi(x)))^i, & x \in B \\ x, & 0 \notin B \end{cases}$$

defines a vector field on U.

STEP 2.2 (X_i is smooth): Let $y \in \mathbb{R}^m$ and define

$$\Phi := \psi^{-1}, \qquad x := \Phi(y), \qquad r := \eta^{-1}(|y|), \qquad \sigma := |y|. \tag{5.4}$$

We obtain the equations

$$x^i = \Phi^i(y) = \frac{\eta^{-1}(|y|)}{|y|} y^i = \frac{r}{\sigma} y^i, \tag{5.5}$$

$$\partial_{y_j}(|y|) = \frac{y^j}{\sigma} \overset{(5.5)}{=} \frac{x^j}{r}, \tag{5.6}$$

$$\partial_{y_j}(\eta^{-1}(|y|)) = \frac{1}{\eta'(\eta^{-1}(|y|))} \frac{y^j}{|y|} \overset{(5.5)}{=} \frac{1}{\eta'(r)} \frac{x^j}{r}. \tag{5.7}$$

Having this in mind, we calculate

$$\begin{aligned}
\partial_{y_j}(\psi^{-1 i})(y) = \partial_{y_j}(\Phi^i)(y) &\overset{(5.5)}{=} \partial_{y_j}\Big(\frac{\eta^{-1}(|y|)}{|y|} y^i\Big) = \frac{\partial_{y_j}(\eta^{-1}(|y|)y^i)|y| - \partial_{y_j}(|y|)\eta^{-1}(|y|)y^i}{|y|^2} \\
&= \frac{\partial_{y_j}(\eta^{-1}(|y|)y^i|y|}{|y|^2} + \frac{\eta^{-1}(|y|)\partial_{y_j}(y^i)|y|}{|y|^2} - \frac{\partial_{y_j}(|y|)\eta^{-1}(|y|)y^i}{|y|^2} \\
&\overset{(5.5),(5.6),(5.7)}{=} \frac{x^i x^j}{\eta'(r)r^2} + \delta^i_j \frac{r}{\eta(r)} - \frac{x^i x^j}{r\eta(r)}.
\end{aligned} \tag{5.8}$$

Now consider $x \in B$, $|x| \to 1$. Then $y := \psi(x) \to \infty$. This implies that

$$r = r(y) = \eta^{-1}(|\psi(x)|) = |x| \to 1, \qquad \eta(r) = \eta(|x|) \to \infty, \qquad \eta'(r) \to \infty.$$

Consequently

$$\partial_{y_j}(\psi^{-1})^i(\psi(x)) \xrightarrow[|x|\to 1]{} 0 \,.$$

Though intuitively clear, one has to show that this convergence holds for all the derivatives of (5.8) as well: Notice that if $r(x) = |x| \to 1$, then r is bounded by 5.2.3 proven below. Remember that $\|f \circ g\|_{\mathcal{C}^k} \le C\|f\|_{\mathcal{C}^k}\|g\|_{\mathcal{C}^k}$ (this can be proven by induction using the Leibniz and the chain rule).
Consequently there are generic constants C such that

$$\begin{aligned}
\Big\|\frac{x^i x^j}{\eta' \circ r r^2}\Big\|_{\mathcal{C}^k} &\le C\|x^i\|_{\mathcal{C}^k}\|\|x^j\|_{\mathcal{C}^k}\|r^{-1}\|^2_{\mathcal{C}^k}\|1/\eta'\|_{\mathcal{C}^k}\|r\|_{\mathcal{C}^k} \to 0, \\
\Big\|\delta^i_j \frac{r}{\eta \circ r}\Big\|_{\mathcal{C}^k} &\le C\|r\|^2_{\mathcal{C}^k}\|1/\eta\|_{\mathcal{C}^k} \to 0, \\
\Big\|\frac{x^i x^j}{r\eta \circ r}\Big\|_{\mathcal{C}^k} &\le C\|x^i\|_{\mathcal{C}^k}\|\|x^j\|_{\mathcal{C}^k}\|r^{-1}\|_{\mathcal{C}^k}\|r\|_{\mathcal{C}^k}\|1/\eta\|_{\mathcal{C}^k} \to 0,
\end{aligned}$$

as $|x| \to 1$ using 5.2.1(iii).

STEP 2.3 (β_i is smooth): Since X_i is a smooth vector field, X_i generates a unique maximal smooth flow θ_i (c.f. [16, 17.8]). In particular its integral curves vary smoothly with the initial data. Since the generated flow is unique, we obtain $\beta_i = \theta_i$ by construction of X_i. Consequently β_i is smooth.

STEP 2.4 (s is smooth): By (5.2), we obtain

$$\begin{aligned}
s(y, x) = s\Big(\sum_{i=1}^{m} y_i e_i, x\Big) &= s(y_1 e_1, s(y_2 e_2, (\ldots (s(y_m e_m, x)) \ldots))) \\
&= \beta_1(y_1(\beta_2(y_2(\ldots(\beta_m(y_m, x))\ldots)))),
\end{aligned}$$

thus s is smooth as a finite composition of smooth maps. □

5.2.3 Auxiliary Lemma. Let $0 < r < R < \infty$ and

$$K_{r,R} := B_R(0) \setminus B_r(0) \subset \mathbb{R}^m.$$

Then for any multi-index α, $|\alpha| = k$, there exist a constants C_α, $\tilde{C}_\alpha$ such that for any $x \in K_{r,R}$

$$|\partial^\alpha(\|x\|)| \leq C_\alpha, \qquad \left|\partial^\alpha\left(\frac{1}{\|x\|}\right)\right| \leq \tilde{C}_\alpha.$$

Proof. We will prove this statement by induction over k. The case $k = 0$ holds by construction. For the induction step consider any $\alpha = \beta + e_i$, $|\alpha| = k+1$, $|\beta| = k$, $1 \leq i \leq m$, $x \in K_{r,R}$ and calculate

$$\partial^\alpha\|x\| = \partial^\beta\partial_i\|x\| = \partial^\beta\left(\frac{x_i}{\|x\|}\right) = \sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\partial^{\beta-\gamma}(x_i)\partial^\gamma\left(\frac{1}{\|x\|}\right). \tag{5.9}$$

By hypothesis $|\gamma| \leq |\beta| \leq k$ and therefore this expression is bounded by some constant C_α. Now consider

$$\partial^\alpha\left(\frac{1}{\|x\|}\right) = \partial^\beta\left(\frac{-1}{\|x\|^2}\partial_i(\|x\|)\right) = -\partial^\beta\left(\frac{1}{\|x\|}\frac{1}{\|x\|}\partial_i(\|x\|)\right).$$

Now we can apply the Leibniz rule twice to these product of three functions. All occuring expressions are bounded by hypothesis or by what we have just proven. □

We need to establish some Lipschitz properties and therefore introduce the following notational conventions.

5.2.4 Definition (Lipschitz). A function $f : D \subset (X, \|\_\|) \to (Y, \|\_\|)$ is *Lipschitz continuous* or just *"is Lipschitz" with constant* $L(f) > 0$, if

$$\forall x_1, x_2 \in D : \|f(x_2) - f(x_1)\| \leq L(f)\|x_2 - x_1\|.$$

We denote by $L(f)$ a Lipschitz constant of f, although this constant does not have to be optimal (and is therefore not unique).

5.2.5 Theorem (properties of Lipschitz functions). Let $(X, \|\_\|)$, $(Y, \|\_\|)$, $(Z, \|\_\|)$ be normed spaces.

(i) Assume $f : X \to Y$ and $g : Y \to Z$ are Lipschitz with constants $L(f)$, $L(g)$. Then $g \circ f$ is Lipschitz with constant $L(g \circ f) \leq L(f)L(g)$.

(ii) Assume $f, g : D \to \mathbb{C}$ are Lipschitz with constants $L(f)$,$L(g)$ and globally bounded with constants $B(f), B(g)$. Then $f \cdot g : D \to \mathbb{C}$ is Lipschitz with constant $L(fg) \leq \max(B(g)L(f), B(f)L(g))$.

(iii) Assume $X_1, X_2 \subset D \subset X$, where $D \subset X$ is convex and X_1, X_2 are not empty, closed, and satisfy $X_1^\circ \cap X_2^\circ = \emptyset$, $D = X_1 \cup X_2$. In other words D is split up into two closed subset X_1 and X_2 wich meet at a common nonemtpy boundary. Assume $f : D \to Y$ is continuous, $f|_{X_1}$ is Lipschitz with constant $L_1(f)$, $f|_{X_2}$ is Lipschitz with constant $L_2(f)$. Then f is Lipschitz with constant $L(f) \leq \max(L_1(f), L_2(f))$.

Proof. Let $x_1, x_2 \in D$.

(i)

$$\|g(f(x_2)) - g(f(x_1))\| \leq L(g)\|f(x_2) - f(x_1)\| \leq L(g)L(f)\|x_2 - x_1\|.$$

(ii)

$$\|f(x_2)g(x_2) - f(x_1)g(x_1)\| \leq \|f(x_2)g(x_2) - f(x_2)g(x_1)\| + \|f(x_2)g(x_1) - f(x_1)g(x_1)\|$$
$$\leq B(f)L(g)\|x_2 - x_1\| + B(g)L(f)\|x_2 - x_1\|.$$

(iii) In case $x_1, x_2 \in X_1$ or $x_1, x_2 \in X_2$, this is clear. So let $x_1 \in X_1^\circ$ and $x_2 \in X_2^\circ$. Then there exists a $t \in [0,1]$ such that $x' := x_1 + t(x_2 - x_1) \in X_1 \cap X_2$. We calculate

$$\begin{aligned}\|f(x_2) - f(x_1)\| &\leq \|f(x_2) - f(x')\| + \|f(x') - f(x_1)\| \\ &= \|f_2(x_2) - f_2(x')\| + \|f_1(x') - f_1(x_1)\| \\ &\leq L_1(f)\|x_2 - x'\| + L_2(f)\|x' - x_1\| \\ &\leq \max(L_1(f), L_2(f)(\|x_2 - x'\| + \|x' - x_1\|)\end{aligned}$$

and

$$\|x_2 - x'\| + \|x' - x_1\| = \|(1-t)x_2 - (1-t)x_1\| + \|t(x_1 - x_2\|$$
$$= ((1-t) + t)\|x_2 - x_1\| = \|x_2 - x_1\|.$$ □

5.2.6 Theorem. With the notation from 5.2.1:

(i) η^{-1} is Lipschitz with constant $L(\eta^{-1}) \leq \max(1, L(\eta_0)^{-1})$, $L(\eta_0^{-1}) := \max_{r \in [\frac{1}{3}, \frac{2}{3}]} |\eta_0'(r)|$.

(ii) The map $\psi^{-1} : \mathbb{R}^m \to \mathbb{B}^m$ is Lipschitz continuous with constant $L(\psi^{-1}) \leq \max(6, L(\eta_0^{-1}))$.

Proof.

(i) We already established that η and η^{-1} are globally smooth. We analyse the derivative of η: Clearly

$$\forall r \in]0, \frac{1}{3}[: \eta'(r) = 1.$$

We calculate

$$\forall r \in]\frac{2}{3}, 1[: \Big(e^{(r-1)^{-2}}\Big)' = -2(r-1)^{-3}e^{(r-1)^{-2}},$$
$$\Big(e^{(r-1)^{-2}}\Big)'' = \Big(6(r-1)^{-4} + 4(r-1)^{-6}\Big)e^{(r-1)^{-2}} > 0.$$

The second equation tells us that the first derivative is monotonously increasing. Therefore

$$\forall r \in]\frac{2}{3}, 1[: \eta'(r) \geq \Big(e^{(r-1)^{-2}}\Big)'\Big(\frac{2}{3}\Big) = 54e^{\frac{1}{9}} > 1 > 0.$$

Since η_0 is strictly monotonously increasing, η_0 is invertible on its image. Since $[\frac{1}{3}, \frac{2}{3}]$ is compact, we obtain some bound

$$L(\eta_0^{-1}) := \max_{r \in [\frac{1}{3}, \frac{2}{3}]} |\eta_0'(r)| < \infty.$$

Using the fact that $\forall r \in]0,1[: (\eta^{-1})'(r) = \frac{1}{\eta'(\eta^{-1}(r))}$, we obtain that

$$\forall r \in]0,1[: |\eta'^{-1}(r)| \leq \max(1, 54e^{-\frac{1}{9}}, L(\eta_0^{-1})) = \max(1, L(\eta_0^{-1})) =: L(\eta^{-1}).$$

Now the itermediate value theorem tells us that

$$\forall r_1, r_2 \in]0,1[: \exists \xi \in [r_1, r_2] : |\eta^{-1}(r_2) - \eta^{-1}(r_1)| = |(\eta^{-1})'(\xi)||r_2 - r_1| \leq L(\eta^{-1})|r_2 - r_1|.$$

(ii) Now we use Theorem 5.2.5 to obtain the following results: The function $\mathbb{R}^m \to \mathbb{R}$, $\eta^{-1} \circ |\_|$, is Lipschitz with constant $\leq L(\eta^{-1})$.
Furthermore we claim that $\mathbb{R}^m \setminus B_{1/3}(0) \to \mathbb{S}^m$, $y \mapsto \frac{y}{|y|}$, is Lipschitz continuous with constant ≤ 6:

$$\forall x, y \in \mathbb{R}^m \setminus B_{1/3}(0) : \Big|\frac{y}{|y|} - \frac{x}{|x|}\Big| = \Big|\frac{y}{|y|} - \frac{x}{|y|} + \frac{x}{|y|} - \frac{x}{|x|}\Big| \leq \frac{|y-x|}{|y|} + \Big|\frac{x|x| - x|y|}{|y||x|}\Big|$$
$$\leq 3|y-x| + \frac{|x|||x|-|y||}{|y||x|} \leq 3|y-x| + \frac{|x-y|}{|y|} \leq 6|y-x|.$$

Now
$$\forall y \in B_{1/3}(0) : \psi^{-1}(y) = \frac{\eta^{-1}(|y|)}{|y|} y = y,$$
thus ψ^{-1} is Lipschitz on $B_{1/3}(0)$ with constant 1. Outside it satisfies
$$\forall y \in \mathbb{R}^m \setminus B_{1/3}(0) : \psi^{-1}(y) = \eta^{-1}(|y|) \frac{y}{|y|}$$
and is therefore a product of two bounded Lipschitz functions. Using 5.2.5 and the first part, we calculate

$$L(\psi^{-1}) \leq \max\Big(L(\psi^{-1}|_{B_{1/3}(0)}), L(\psi^{-1}|_{\mathbb{R}^m \setminus B_{1/3}(0)})\Big)$$
$$\leq \max\Big(1, \max\Big(B(\eta^{-1}(|y|))L(\frac{y}{|y|}|_{\mathbb{R}^m \setminus B_{1/3}(0)}), L(\eta^{-1}(|y|))B(\frac{y}{|y|}|_{\mathbb{R}^m \setminus B_{1/3}(0)})\Big)\Big)$$
$$\leq \max(1, 1 \cdot 6, L(\eta^{-1}) \cdot 1) \leq \max(6, L(\eta_0^{-1})).$$
□

5.2.7 Corollary. Let $y \in \mathbb{R}^m$ and let $s_y : U \to U$ be a localized translation. Then
$$\forall x \in U : |s_y(x) - x| \leq L(\psi^{-1})|y|.$$

Proof. Since ψ^{-1} is Lipschitz with some constant L by 5.2.6, we obtain
$$\forall x \in U : |s_y(x) - x| = |\psi^{-1}(\tau_y(\psi(x))) - \psi^{-1}(\psi(x))| \leq L|\tau_y(\psi(x))) - \psi(x)| = L|y|,$$
$$\forall x \in B : |s_y(x) - x| = |x - x| = 0 \leq L|y|.$$
□

5.3 Construction of the operators

Before we can start with the construction of the regularization operators in detail, we need to remind of some results concerning the classical de Rham theory of smooth differential forms (c.f. [16, 15]).

5.3.1 Definition (notation). Let M, N be smooth manifolds, $I := [0, 1]$ and let $H : M \times I \to N$ be a homotopy between $F, G : M \to N$. For any $t \in I$ define $i_t : M \to M \times I$, $p \mapsto (p, t)$.

5.3.2 Definition (homotopy Operator). The map $h : \Omega^k(M \times I) \to \Omega^{k-1}(M)$,
$$\omega \mapsto \int_0^1 \iota_{\partial_t} \omega dt,$$

is the *homotopy operator*. Somewhat more explicity, this means that for any $X_1, \dots, X_{k-1} \in \mathcal{T}(M)$

$$\forall p \in M : h(\omega)|_p(X_1, \dots, X_{k-1}) = \int_0^1 \omega_{(p,t)}(\tilde{\partial}_t, \tilde{X}_1, \dots, \tilde{X}_{k-1})dt,$$

where $\tilde{\partial}_t, \tilde{X}_i \in \mathcal{T}(M \times I)$ are the vector fields on $M \times I$ that are defined by 5.1.6.

5.3.3 Theorem. The operator h is a cochain-homotopy between i_0^* and i_1^*, i.e.

$$h \circ d + d \circ h = i_1^* - i_0^*.$$

Proof. See [16, 15.4]. □

5.3.4 Corollary. The operator $\tilde{h} := h \circ H^*$ is a cochain homotopy between F^* and G^*.

Proof. Since $H \circ i_0 = F$ and $H \circ i_1 = G$,

$$\tilde{h} \circ d + d \circ \tilde{h} = h \circ H^* \circ d + d \circ h \circ H^* = (i_1^* - i_0^*) \circ H^* = G^* - F^*.$$ □

5.3.1 Regularization on R^n

Before we proceed, we will outline how to define regularization operators on $\mathbb{R}^n$ as a motivation for the localized version in the next section. These operators were introduced by de Rham in [23, III.§15].

5.3.5 Lemma (special Case). Let $y \in \mathbb{R}^m$ and define $S_y : \mathbb{R}^m \times I \to \mathbb{R}^m$, $(x,t) \mapsto x+ty$. Obviously S_y is a homotopy between $\tau_y : \mathbb{R}^m \to \mathbb{R}^m$, $x \mapsto x+y$, and $\mathrm{id} : \mathbb{R}^m \to \mathbb{R}^m$, $x \mapsto x$. We claim that $\tilde{h} = h \circ S_y^* : \Omega^{k+1}(\mathbb{R}^m) \to \Omega^k(\mathbb{R}^m)$, h as in 5.3.2, is given by

$$\tilde{h}(\omega)|_x = \int_0^1 \iota_Y(\omega)|_{x+ty} dt,$$

where $Y \in \mathcal{T}(\mathbb{R}^m)$ is the vector field with constant coefficients y.

Proof. In that case $M = N = \mathbb{R}^m$. Consider $X_1, \dots, X_k \in \mathcal{T}(\mathbb{R}^m)$ and observe

$$\tilde{X}_i|_{(x,t)} = \tilde{X}_i^j|_{(x,t)} \tilde{\partial} x_j|_{(x,t)} = X_i^j|_x \tilde{\partial} x_j|_{(x,t)}.$$

To calculate the push-forward, notice that in classical notation

$$\nabla S_y(x,t) = \begin{pmatrix} E_m & y \end{pmatrix} \in \mathbb{R}^{m \times (m+1)}.$$

Combining these insights, we conclude

$$\begin{aligned} S_{y*}|_{(x,t)}(\tilde{X}_i) &= X_i^j|_{x+ty} S_{y*}|_{(x,t)}(\tilde{\partial} x_j|_{(x,t)}) && (5.10) \\ &= X_i^j|_{x+ty} (\nabla S_y(x,t))_j^k \partial x_k|_x = X_i^j|_{x+ty} \partial x_j|_x = X_i|_{x+ty}, \\ S_{y*}|_{(x,t)}(\tilde{\partial}_t) &= (\nabla S_y(x,t))_{m+1}^i \partial x_i|_{x+ty} = y^i \partial x_i|_{x+ty} = Y|_{x+ty}. && (5.11) \end{aligned}$$

This allows us to clalculate for $x \in \mathbb{R}^m$

$$\begin{aligned} \tilde{h}(\omega)|_x(X_1, \dots, X_k) = h(S_y^*(\omega))|_x(X_1, \dots, X_k) &= \int_0^1 S_y^*(\omega)|_{(x,t)}(\tilde{\partial}_t, \tilde{X}_1, \dots, \tilde{X}_k)dt \\ &= \int_0^1 \omega|_{S_y(x,t)}(S_{y*}|_{(x,t)}\tilde{\partial}_t, S_{y*}|_{(x,t)}\tilde{X}_1, \dots, S_{y*}|_{(x,t)}\tilde{X}_k)dt \\ &\overset{(5.10)}{=} \int_0^1 \omega|_{x+ty}(Y, X_1, \dots, X_k)dt = \int_0^1 \iota_Y(\omega)|_{x+ty}(X_1, \dots, X_k)dt. \end{aligned}$$ □

The operator $\tilde{h}$ from Lemma 5.3.5 above can be used directly to define regularization operators in $\mathbb{R}^m$.

5.3.6 Definition (regularization in $\mathbb{R}^m$). Denote by $s_y : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x + y$, $S_y : \mathbb{R}^n \times I$, $(x,t) \mapsto x + ty$, the standard translations and let $\varepsilon > 0$. Define $R^k_\varepsilon : \mathscr{D}_k(\mathbb{R}^n) \to \mathscr{D}_k(\mathbb{R}^n)$ by

$$\forall x \in U : R^k_\varepsilon(\omega)|_x := \int_{\mathbb{R}^m} (s^*_y\omega)|_x \varphi_\varepsilon(y)dy := \sum_{I \in \mathcal{I}_k} \Big(\int_{\mathbb{R}^m} (s^*_y\omega)_I(x)\varphi_\varepsilon(y)dy \Big) dx^I,$$

where $\mathcal{I}_k$ is the set of all increasing multi-indices of length k and φ_ε is the standard mollifier from 5.1.2. Similar, define $A^k_\varepsilon : \mathscr{D}_k(U) \to \mathscr{D}_{k-1}(U)$ by

$$\begin{aligned}\forall x \in U : A^k_\varepsilon(\omega)|_x &:= \int_{\mathbb{R}^m} \tilde{h}(\omega)\, \varphi_\varepsilon(y)dy \\ &:= \sum_{I \in \mathcal{I}_{k-1}} \Big(\int_{\mathbb{R}^m} \int_0^1 \iota_{\partial_t}(S^*_y(\omega))_I(x,t)\varphi_\varepsilon(y)dtdy \Big) dx^I,\end{aligned}$$

where $\tilde{h}$ is taken from Lemma 5.3.5 w.r.t. the homotopy S_y. Denote by R'_ε, A'_ε the corresponding operators on currents $\mathscr{D}'(\mathbb{R}^m)$.

These operators have numerous interesting properties. We just collect a few, since we will study the localized versions in more detail.

5.3.7 Theorem (properties of regularization operators on $\mathbb{R}^m$).

(i) The operator R_ε is a cochain map, i.e.

$$R_\varepsilon \circ d = d \circ R_\varepsilon.$$

The analogous result holds for R'_ε.

(ii) The operator A_ε is a cochain homotopy from R_ε to the identity, i.e.

$$A_\varepsilon \circ d + d \circ A_\varepsilon = R_\varepsilon - \mathrm{id}\,.$$

The analogous result holds for A'_ε.

(iii) ThmRegOpsRegRn For any current $T \in \mathscr{D}'_k(\mathbb{R}^m)$ there exists a smooth form $\omega \in \Omega^k(\mathbb{R}^m)$ such that $R'_\varepsilon(T) = \langle\omega\rangle$, i.e.

$$\forall \eta \in \mathscr{D}_{m-k} : R'_\varepsilon(T)(\eta) = \langle\omega\rangle(\eta) = \int_{\mathbb{R}^m} \omega \wedge \eta.$$

Proof. The proof for properties (i) and (ii) is almost identical to the proof of the localized versions in Theorem 5.3.11. Therefore we will skip them. Let us prove property (iii), the reason why R_ε is called "regularization operator":

STEP 1 (preparations): For any $x \in \mathbb{R}^m$ and any function $F \in \mathscr{D}(U)$

$$\begin{aligned}R_\varepsilon(F)(x) &= \int_{\mathbb{R}^m} (s^*_y F)(x)\varphi_\varepsilon(y)dy = \int_{\mathbb{R}^m} F(x+y)\varphi_\varepsilon(y)dy = \int_{\mathbb{R}^m} F(z) \underbrace{\varphi_\varepsilon(z-x)}_{=:\psi_x(z)} dz \\ &= \int_{\mathbb{R}^m} F\psi_x d\mu = \langle F\rangle(\psi_x d\mu) = (\langle F\rangle \wedge d\mu)(\psi_x). \qquad (5.12)\end{aligned}$$

We consider ψ as a smooth function $\mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, $(x,z) \mapsto \psi(x,z)$. By fixing x or z we obtain two families of functions ψ_x, ψ_z on $\mathbb{R}^m$ and any ψ_x has compact support.
Now since

$$\forall 1 \leq i \leq m : \forall x, y \in \mathbb{R}^m : s_y^*(dx^i)|_x = d(x^i \circ s_y)|_x = ds_y^i|_x = \partial_{x_j}(x^i + y^i)|_x dx^j|_x = dx^i|_x,$$

the Euclidean volume element $d\mu = dx^1 \wedge \ldots dx^m$ is invariant under all translations. Therefore

$$\forall f \in \mathscr{D}_0(\mathbb{R}^m) : R_\varepsilon(f d\mu) = R_\varepsilon(f) d\mu. \tag{5.13}$$

Step 2 (regularity): The strategy is to use the local decomposition theorem 4.1.19.
Step 2.1: Let us first consider the case where $T \in \mathscr{D}_0'(\mathbb{R}^m)$. Any form $\omega \in \mathscr{D}_m(\mathbb{R}^m)$ may be written as $\omega = f d\mu$, where $f \in \mathscr{D}_0(\mathbb{R}^m)$. We calculate

$$\begin{aligned}(R_\varepsilon' T)(\omega) &= (R_\varepsilon' T)(f d\mu) = T(R_\varepsilon(f d\mu)) \overset{(5.13)}{=} T(R_\varepsilon(f) d\mu) = T(d\mu \wedge R_\varepsilon(f)) \\ &= (T \wedge d\mu)(R_\varepsilon(f)) \overset{(5.12)}{=} (T \wedge d\mu)(x \mapsto (\langle f \rangle \wedge d\mu)(\psi_x)) \\ &\overset{4.1.20}{=} (\langle f \rangle \wedge d\mu)(z \mapsto \underbrace{(T \wedge d\mu)(\psi_z)}_{=:\theta_T(z)}) = \int_U f \theta d\mu \\ &= \int_U \theta f d\mu = \langle \theta \rangle (f d\mu) = \langle \theta \rangle(\omega),\end{aligned} \tag{5.14}$$

where

$$\theta := \theta_T := (T \wedge d\mu)(\psi\_) \in \Omega^k(\mathbb{R}^m). \tag{5.15}$$

Step 2.2: For the general case, notice that we may choose a global coordinate system on $\mathbb{R}^m$ (the identity) and decompose $T \in \mathscr{D}_k'(\mathbb{R}^m)$ into

$$T = \sum_I T_I \wedge dx^I, T_I \in \mathscr{D}_0'(\mathbb{R}^m)$$

by Theorem 4.1.19. For any I let $R_\varepsilon'(T_I) = \langle \theta_I \rangle$, where $\theta_I := \theta_{T_I} \in \Omega_k(\mathbb{R}^m)$.
Using the translation invariance of $d\mu$ again, we obtain for k-form $\omega \in \mathscr{D}(\mathbb{R}^m)$

$$R_\varepsilon(dx^I \wedge \omega) = dx^I \wedge R_\varepsilon(\omega). \tag{5.16}$$

Therefore

$$\begin{aligned}R_\varepsilon' T(\omega) &= \sum_I R_\varepsilon'(T_I \wedge dx^I)(\omega) = \sum_I (T_I \wedge dx^I)(R_\varepsilon \omega) \\ &= \sum_I T_I(dx^I \wedge R_\varepsilon \omega) \overset{(5.16)}{=} \sum_I T_I(R_\varepsilon(dx^I \wedge \omega)) \\ &= \sum_I R_\varepsilon' T_I(dx^I \wedge \omega) = \sum_I \langle \theta_I \rangle (dx^I \wedge \omega) = \sum_I \langle \theta_I \wedge dx^I \rangle(\omega)\end{aligned} \tag{5.17}$$

is smoothly generated. □

5.3.2 Localized regularization

If we exchange the translation with the localized translation in the definition of R_ε and A_ε on $\mathbb{R}^m$, we obtain the localized versions on U.

5.3.8 Definition (regularization operators). Let $s : \mathbb{R}^m \times U \to U$ be the localized translation group from 5.2.2. Let L be the Lipschitz constant of ψ calculated in 5.2.6. For any $y \in \mathbb{R}^m$ define $S_y : U \times I \to U$, $(x,t) \mapsto s_{ty}(x)$. We call S_y a *localized translation homotopy*. For any $1 > \varepsilon > 0$ such that $\overline{\mathcal{O}_\varepsilon(B)} \subset U$, define[4] $R^k_\varepsilon : \Omega^k_c(U) \to \Omega^k_c(U)$ by

$$\forall x \in U : R^k_\varepsilon(\omega)|_x := \int_{\mathbb{R}^m} (s_y^*\omega)|_x \varphi_{\varepsilon/L}(y)dy := \sum_{I \in \mathcal{I}_k} \Big(\int_{\mathbb{R}^m} (s_y^*\omega)_I(x)\varphi_{\varepsilon/L}(y)dy \Big) dx^I,$$

where $\mathcal{I}_k$ is the set of all increasing multi-indices of length k and φ_ε is the standard mollifier from 5.1.2. It will follow from a more detailed analysis in 5.3.9 that $R_\varepsilon(\omega) \in \Omega^k_c(U)$ as claimed. Notice that for each $y \in \mathbb{R}^m$, $s_y(x) \in U$, so this integral is well-defined. We call R_ε the *regularizer*.
Similar, define $A^k_\varepsilon : \Omega^k_c(U) \to \Omega^{k-1}_c(U)$ by

$$\begin{aligned}\forall x \in U : A^k_\varepsilon(\omega)|_x &:= \int_{\mathbb{R}^m} \int_0^1 \iota_{\partial_t}(S_y^*(\omega))|_{(x,t)} dt\, \varphi_{\varepsilon/L}(y)dy \\ &:= \sum_{I \in \mathcal{I}_{k-1}} \Big(\int_{\mathbb{R}^m} \int_0^1 \iota_{\partial_t}(S_y^*(\omega))_I(x,t)\varphi_{\varepsilon/L}(y)dtdy \Big) dx^I.\end{aligned}$$

We employ the convention $A^0_\varepsilon := 0$ as well as 5.1.6.

5.3.9 Theorem (basic properties). Let $\omega \in \Omega^k_c(U)$.

(i) For any $x \in U \setminus B$

$$R_\varepsilon(\omega)|_x = \omega|_x, \qquad A_\varepsilon(\omega)|_x = 0.$$

(ii) If $K := \operatorname{supp}\omega$, we obtain

$$\begin{aligned}\operatorname{supp} R_\varepsilon(\omega) &\subset \overline{\mathcal{O}_\varepsilon(K \cap B)} \cup K \setminus B \Subset U, \\ \operatorname{supp} A_\varepsilon(\omega) &\subset \overline{\mathcal{O}_\varepsilon(K \cap B)} \Subset U,\end{aligned}$$

and therefore in particular

$$\operatorname{supp} R_\varepsilon(\omega) \subset \mathcal{O}_\varepsilon(\operatorname{supp}\omega), \qquad \operatorname{supp} A_\varepsilon(\omega) \subset \mathcal{O}_\varepsilon(\operatorname{supp}\omega).$$

(iii) R_ε and A_ε define continuous operators $R_\varepsilon, A_\varepsilon : \mathscr{D}(U) \to \mathscr{D}(U)$.

Proof. It is clear that $R_\varepsilon(\omega), A_\varepsilon(\omega) \in \Omega^k(U)$. Also notice that since $\operatorname{supp}\varphi_{\varepsilon/L} \subset B_{\varepsilon/L}(0)$ by 5.1.2, one may always express these operators by

$$R_\varepsilon(\omega) = \int_{B_{\varepsilon/L}(0)} (s_y^*\omega)\varphi_{\varepsilon/L}(y)dy, \qquad A_\varepsilon(\omega) = \int_{B_{\varepsilon/L}(0)} \int_0^1 \iota_{\partial_t}(S_y^*(\omega))dt\, \varphi_{\varepsilon/L}(y)dy.$$

[4] Of course these definitions make sense for $\varepsilon \geq 1$ as well. But we need this operators only for small values of ε and this avoids technical issues.

(i) Remember that

$$\forall x \in U \setminus B : \forall y \in \mathbb{R}^m : s_y(x) = x$$

by Theorem 5.2.2. Therefore for any $X_1, \ldots, X_k \in \mathcal{T}(U)$, $x \in U \setminus B$, we obtain

$$\begin{aligned} R_\varepsilon(\omega)|_x(X_1, \ldots, X_k) &= \int_{\mathbb{R}^m} s_y^*(\omega)|_x(X_1, \ldots, X_k)\varphi_{\varepsilon/L}(y)dy \\ &= \int_{\mathbb{R}^m} \omega|_{s_y(x)}(s_{y*}|_x X_1, \ldots, s_{y*}|_x X_k)\varphi_{\varepsilon/L}(y)dy \\ &= \int_{\mathbb{R}^m} \omega|_x(X_1, \ldots, X_k)\varphi_{\varepsilon/L}(y)dy \\ &= \omega|_x(X_1, \ldots, X_k) \int_{\mathbb{R}^m} \varphi_{\varepsilon/L}(y)dy = \omega|_x(X_1, \ldots, X_k). \end{aligned}$$

By definition, for any $y \in \mathbb{R}^m$, we have $S_y : U \times I \to U$, $(x, t) \mapsto s_{ty}(x)$. Thus again by 5.2.2, this implies

$$\forall x \in U \setminus B : \forall y \in \mathbb{R}^m : \forall t \in I : S_y(x, t) = s_{ty}(x) = x.$$

Thus S_y does not depend on t here and consequently $S_{y*}|_{(x,t)}(\tilde{\partial}_t) = 0$. This implies

$$\begin{aligned} A_\varepsilon(\omega)|_x(X_1, \ldots, X_{k-1}) &= \int_{\mathbb{R}^m} \int_0^1 \iota_{\partial_t}(S_y^*(\omega))|_{(x,t)}(\tilde{X}_1, \ldots, \tilde{X}_{k-1})dt\varphi_{\varepsilon/L}(y)dy \\ &= \int_{\mathbb{R}^m} \int_0^1 S_y^*(\omega)|_{(x,t)}(\tilde{\partial}_t, \tilde{X}_1, \ldots, \tilde{X}_{k-1})dt\varphi_{\varepsilon/L}(y)dy \\ &= \int_{\mathbb{R}^m} \int_0^1 \omega_{S_y(x,t)}(S_{y*}|_{(x,t)}\tilde{\partial}_t, S_{y*}|_{(x,t)}\tilde{X}_1, \ldots, S_{y*}|_{(x,t)}\tilde{X}_{k-1})dt\varphi_{\varepsilon/L}(y)dy = 0. \end{aligned}$$

(ii) Assume $x \in U$ and $R_\varepsilon(\omega)|_x \neq 0$. We conclude

$$\begin{aligned} &\Rightarrow 0 \neq \int_{B_{\varepsilon/L}(0)} (s_y^*\omega)|_x \varphi_{\varepsilon/L}(y)dy \\ &\Rightarrow \exists y \in B_{\varepsilon/L}(0) : \exists X_1, \ldots, X_k \in \mathcal{T}(U) : \\ &\qquad 0 \neq s_y^*(\omega)|_x(X_1, \ldots, X_k) = \omega_{s_y(x)}(s_{y*}X_1, \ldots, s_{y*}X_k) \\ &\Rightarrow \omega_{s_y(x)} \neq 0 \Rightarrow s_y(x) \in \operatorname{supp}\omega. \end{aligned}$$

Now by 5.2.7, we obtain

$$|s_y(x) - x| \leq L|y| < \varepsilon$$

and therefore $x \in \mathcal{O}_\varepsilon(\operatorname{supp}\omega)$.
With almost the same proof, we obtain the same statement for A_ε: Let $x \in U$ and assume $A_\varepsilon(\omega)|_x \neq 0$. We conclude

$$\begin{aligned} &\Rightarrow 0 \neq \int_{B_{\varepsilon/L}(0)} \int_0^1 \iota_{\partial_t}(S_y^*(\omega))|_{(x,t)}dt\, \varphi_{\varepsilon/L}(y)dy \\ &\Rightarrow \exists y \in B_{\varepsilon/L}(0) : \exists t \in [0,1] : \exists X_1, \ldots, X_{k-1} : 0 \neq \iota_{\partial_t} S_y^*(\omega)|_{(x,t)}(\tilde{X}_1, \ldots, \tilde{X}_{k-1}) \\ &\qquad = \omega_{S_y(x,t)}(S_{y*}\tilde{\partial}_t, S_{y*}\tilde{X}_1, \ldots, S_{y*}\tilde{X}_{k-1}) \\ &\Rightarrow \omega_{S_y(x,t)} \neq 0 \Rightarrow s_{ty}(x) \in \operatorname{supp}\omega. \end{aligned}$$

Again by 5.2.7, we obtain

$$|s_{ty}(x) - x| \leq L|ty| \leq L|y| < \varepsilon$$

and therefore $x \in \mathcal{O}_\varepsilon(\operatorname{supp}\omega)$.

(iii) Now we carefully combine the two statements above to see that $R_\varepsilon, A_\varepsilon$ define maps $\Omega_c(U) \to \Omega_c(U)$: Assume $\omega \in \Omega_c(U)$ and $\operatorname{supp}\omega \subset K \subset U$ and K is compact. By (i), (ii) and our choice of ε

$$\begin{aligned} \operatorname{supp} R_\varepsilon(\omega) &\subset \overline{\mathcal{O}_\varepsilon(K \cap B)} \cup K \setminus B \Subset U. \\ \operatorname{supp} A_\varepsilon(\omega) &\subset \overline{\mathcal{O}_\varepsilon(K \cap B)} \Subset U. \end{aligned}$$

Thus $R_\varepsilon(\omega), A_\varepsilon(\omega) \in \Omega_c(U)$.
Now we check continuity: Let $\omega_j \in \mathscr{D}(U)$ such that

$$\omega_j \xrightarrow[\mathscr{D}]{j\to\infty} 0 \,.$$

Let $K \subset U$ be compact such that

$$\forall j \in \mathbb{N} : \operatorname{supp}\omega_j \subset K \subset U.$$

By what we have already proven and by choice of ε in Definition 5.3.8, we obtain

$$\forall j \in \mathbb{N} : \operatorname{supp} A_\varepsilon(\omega_j), \operatorname{supp} R_\varepsilon(\omega_j) \subset \underbrace{\overline{\mathcal{O}_\varepsilon(K \cap B)} \cup K \setminus B}_{=:K_\varepsilon} \Subset U.$$

Therefore the supports of all the $R_\varepsilon(\omega_j), A_\varepsilon(\omega_j)$ are contained in the compact set K_ε.
Now for any $x \in U$, $\alpha \in \mathbb{N}^m$, $|\alpha| \leq l$, and any $I \in \mathcal{I}_k$, we obtain

$$\begin{aligned} \left|\partial_x^\alpha (R_\varepsilon(\omega_j)|_I\right| &\leq \int_{\mathbb{R}^m} |\partial_x^\alpha (s_y^*\omega_j)_I(x)\varphi_{\varepsilon/L}(y)|dy = \int_{K_\varepsilon \cap B_{\varepsilon/L}(0)} |\partial_x^\alpha (s_y^*\omega_j)_I(x)||\varphi_{\varepsilon/L}(y)|dy \\ &\leq \|\varphi_{\varepsilon/L}\|_{\mathcal{C}^0(K_\varepsilon)} \operatorname{vol}(K_\varepsilon) \max_{y\in K_\varepsilon} |\partial_x^\alpha (s_y^*\omega_j)_I(x)|. \end{aligned}$$

Now assume $\omega_j = f_j dx^I = f_j dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ (which is no loss of generality, since all operations are linear). We obtain

$$s_y^*(\omega_j)|_x = f_j \circ s_y d(x^{i_1} \circ s_y) \wedge \ldots \wedge d(x^{i_k} \circ s_y) = f_j \circ s_y \, \partial_{j_1} s_y^{i_1} \ldots \partial_{j_k} s_y^{i_k} dx^{j_1} \wedge \ldots \wedge dx^{j_k}.$$

By the Leibniz rule, the expression

$$\partial_x^\alpha (f_j \circ s_y \, \partial_{j_1} s_y^{i_1} \ldots \partial_{j_k} s_y^{i_k}),$$

is bounded in terms of $\|f_j \circ s_y\|_{\mathcal{C}^l(K_\varepsilon)}$ and $\|\partial_{j_1} s_y^{i_1} \ldots \partial_{j_k} s_y^{i_k}\|_{\mathcal{C}^l(K_\varepsilon)}$. The latter one is a constant due to compactness of K_ε. The first one is bounded in terms of $\|f_j\|_{\mathcal{C}^l(K_\varepsilon)}$ and $\|s_y\|_{\mathcal{C}^l(K_\varepsilon)}$ by combining the chain rule with the Leibniz rule and again due to compactness of K_ε. Altogether this implies that

$$R_\varepsilon(\omega_j) \xrightarrow[\mathscr{D}]{j\to\infty} 0 \,.$$

In a similar fashion, we estimate

$$\left|\partial_x^\alpha \left(A_\varepsilon(\omega)_I\right)\right| \leq \int_{K_\varepsilon \cap B_{\varepsilon/L}(0)} \int_0^1 |\partial_x^\alpha \left(\iota_{\partial_t}(S_y^*(\omega))_I\right)(x,t)|dt|\varphi_{\varepsilon/L}(y)|dy$$
$$\leq \|\varphi_{\varepsilon/L}\|_{\mathcal{C}^0(K_\varepsilon)} \operatorname{vol}(K_\varepsilon \times I) \max_{t\in[0,1]} \max_{y\in K_\varepsilon} |\partial_x^\alpha \iota_{\partial_t} S_y^*(\omega)_I(x,t)|$$

and the convergence

$$A_\varepsilon(\omega_j) \xrightarrow[\mathscr{D}]{j\to\infty} 0$$

follows analogously. □

5.3.10 Definition. Define $R'_\varepsilon : \mathscr{D}'_k(U) \to \mathscr{D}'_k(U)$ by

$$\forall T \in \mathscr{D}'_k(U) : \forall \eta \in \mathscr{D}_{m-k}(U) : R'_\varepsilon(T)(\eta) := T(R_\varepsilon \eta)$$

and $A'_\varepsilon : \mathscr{D}'_k(U) \to \mathscr{D}'_{k-1}(U)$ by

$$\forall T \in \mathscr{D}'_k(U) : \forall \eta \in \mathscr{D}_{m-k+1} : A'_\varepsilon(T)(\eta) := (-1)^k T(A_\varepsilon \eta).$$

Notice that by Theorem 5.3.9 these maps extend to bounded linear operators $R'_\varepsilon, A'_\varepsilon : \mathscr{D}'(U) \to \mathscr{D}'(U)$. The reason for the sign will become apparent in the proof of Theorem 5.3.11 and 5.3.13.

5.3.11 Theorem (homological properties). The operator $R_\varepsilon : \mathscr{D}(U) \to \mathscr{D}(U)$ is a cochain map, i.e.

$$R_\varepsilon \circ d = d \circ R_\varepsilon,$$

and $A_\varepsilon : \mathscr{D}(U) \to \mathscr{D}(U)$ is a cochain homotopy from R_ε to the identiy, i.e.

$$d \circ A_\varepsilon + A_\varepsilon \circ d = R_\varepsilon - \operatorname{id}.$$

The analogous statements are true for $R'_\varepsilon, A'_\varepsilon : \mathscr{D}'(U) \to \mathscr{D}'(U)$.

Proof.

Step 1 (R is cochain map): Since pullbacks are natural (c.f. [16, 12.16]) and $\omega \in \Omega_c^k(U)$, we obtain

$$\forall x \in U : R_\varepsilon(d\omega)|_x = \int_{\mathbb{R}^m} s_y^*(d\omega)|_x \varphi_{\varepsilon/L}(y)dy = \int_{\mathbb{R}^m} d(s_y^*\omega)|_x \varphi_{\varepsilon/L}(y)dy = dR_\varepsilon(\omega)|_x.$$

Step 2 (A is homotopy): Obviously S_y is a homotopy from $x \mapsto S_y(x,0) = s_0(x) = \operatorname{id}(x) = x$ to $x \mapsto S_y(x,1) = s_y^*(x)$. Consequently 5.3.4 implies that for any $y \in \mathbb{R}^m$, the operator $\tilde{h} = h \circ S_y^*$ satisfies

$$\forall y \in \mathbb{R}^m : d \circ \tilde{h} + \tilde{h} \circ d = s_y^* - \operatorname{id}, \qquad \tilde{h}(\omega) = \int_0^1 \iota_{\partial_t}(s_y^*(\omega))dt.$$

Thus for any $\omega \in \mathscr{D}(U)$ we obtain

$$\int_{\mathbb{R}^m} (d \circ \tilde{h} + \tilde{h} \circ d)(\omega)\varphi_\varepsilon(y)dy = \int_{\mathbb{R}^m} (s_y^* - \operatorname{id})(\omega)\varphi_\varepsilon(y)dy$$
$$\Rightarrow \int_{\mathbb{R}^m} d(\tilde{h}(\omega))\varphi_\varepsilon(y)dy + \int_{\mathbb{R}^m} \tilde{h}(d(\omega))\varphi_\varepsilon(y)dy = \int_{\mathbb{R}^m} s_y^*(\omega)dy - \int_{\mathbb{R}^m} \omega\varphi_\varepsilon(y)dy$$
$$\Rightarrow dA_\varepsilon(\omega) + A_\varepsilon(d\omega) = R_\varepsilon(\omega) - \omega.$$

By dualization we obtain the corresponding statements for $R'_\varepsilon, A'_\varepsilon$: For any $T \in \mathscr{D}'_k(U)$ and $\eta \in \mathscr{D}_{m-k-1}(U)$, we calculate

$$(R'_\varepsilon \circ d)(T)(\eta) = d(T)(R_\varepsilon \eta) = (-1)^{m-k} T(dR_\varepsilon \eta) = (-1)^{m-k} T(R_\varepsilon d\eta) = (d \circ R'_\varepsilon)(T)(\eta)$$

and

$$\begin{aligned}(d \circ A'_\varepsilon + A'_\varepsilon \circ d)(T)(\eta) &= (-1)^{k-1+1} A'_\varepsilon(T)(d\eta) + (-1)^{k+1} d(T)(A_\varepsilon \eta) \\ &= (-1)^{k+k} T(A_\varepsilon d\eta) + (-1)^{k+1+k+1} T(dA_\varepsilon \eta) \\ &= T(A_\varepsilon d\eta + dA_\varepsilon \eta) \\ &= T(R_\varepsilon \eta - \eta) = (R'_\varepsilon(T) - T)(\eta). \qquad \square\end{aligned}$$

5.3.12 Convention. Since $L_{1,\mathrm{loc}}$ may be embedded into $\mathscr{D}'$, we could regard $R_\varepsilon, A_\varepsilon$ as operators on the spaces $L_{1,\mathrm{loc}}$, $W_{1,\mathrm{loc}}$, L_p, W_p and so forth. On the other hand, we could define R_ε, A_ε on $L_{1,\mathrm{loc}}(U)$ directly by the same formlae 5.3.8 as in the smooth case. The next theorem will ensure in particular that both definitions agree. Therefore Theorem 5.3.11 is also valid for the operators defined directly on $W_{1,\mathrm{loc}}$. In the following we may therefore always choose the definition that is the most convenient.

5.3.13 Theorem (regularity).

(i) If $\omega \in L^k_{1,\mathrm{loc}}(U)$, then
$$R'_\varepsilon(\langle \omega \rangle) = \langle R_\varepsilon \omega \rangle$$
and $R_\varepsilon(\omega)|_B \in \Omega^k(B)$. We say R'_ε is *regularizing in* B.

(ii) If $\omega \in L^k_{1,\mathrm{loc}}(U)$, then
$$A'_\varepsilon(\langle \omega \rangle) = \langle A_\varepsilon \omega \rangle$$
and if in addition $\omega \in \mathcal{C}^r(U)$, then $A_\varepsilon(\omega) \in \mathcal{C}^r(U)$. We say A'_ε is *nowhere deregularizing.*

In particluar the following diagram commutes:

$$\begin{array}{ccc} L_{1,\mathrm{loc}}(M) & \xrightarrow{R_\varepsilon, A_\varepsilon} & L_{1,\mathrm{loc}}(M) \\ {\scriptstyle \langle \_ \rangle}\downarrow & & {\scriptstyle \langle \_ \rangle}\downarrow \\ \mathscr{D}'(M) & \xrightarrow{R'_\varepsilon, A'_\varepsilon} & \mathscr{D}'(M). \end{array}$$

Proof. The idea is to modify the proof of Theorem 5.3.7(iii). Since the volume element is not invariant under localized translations s_y^*, equations (5.13) and (5.16) unfortunately do not hold in this setting. Therefore we cannot establish the claim step by step. Consequently we will prove the claim directly with a similar approach as in (5.12).

Step 1 (regularity of R_ε): Let $y \in \mathbb{R}^m$, $x \in U$ and $I = (i_1, \ldots, i_k)$ be any multi-index. We calculate

$$\begin{aligned} s_y^*(dx^I)|_x &= s_y^*(dx^{i_1} \wedge \ldots \wedge dx^{i_k})|_x \\ &= d(x^{i_1} \circ s_y) \wedge \ldots \wedge d(x^{i_k} \circ s_y)|_x = ds_y^{i_1} \wedge \ldots \wedge ds_y^{i_k}|_x \\ &= \sum_{J=(j_1,\ldots,j_k)} \partial_{x_{j_1}} s_y^{i_1}(x) \ldots \partial_{x_{j_k}}(x) dx^{j_1} \wedge \ldots \wedge dx^{j_k}|_x \\ &=: \sum_J \partial_{x_J}(s_y^I)(x) dx^J. \end{aligned} \tag{5.18}$$

Remember from 5.2.2 that for any $x \in B : s_y(x) = s(y,x) = \alpha_x(y)$ and that $\alpha_x : \mathbb{R}^m \to B$ is a diffeomorphism. Thus we obtain

$$\forall x \in B : s_y(x) = z \Leftrightarrow \alpha_x^{-1}(z) = y.$$

This implies that for any $\omega \in L^k_{1,\mathrm{loc}}(U)$ and any $x \in B$

$$\begin{aligned}
R_\varepsilon(\omega)|_x &= \int_{\mathbb{R}^m} s_y^*(\omega)\varphi_{\varepsilon/L}(y)dy = \int_{\mathbb{R}^m} s_y^*\Big(\sum_{I\in\mathcal{I}_k}\omega_I dx^I\Big)|_x\varphi_{\varepsilon/L}(y)dy \\
&= \sum_{I\in\mathcal{I}_k}\int_{\mathbb{R}^m}\omega_I(s_y(x))s_y^*(dx^I)|_x\varphi_{\varepsilon/L}(y)dy \\
&\overset{(5.18)}{=} \sum_{I\in\mathcal{I}_k}\sum_J\int_{\mathbb{R}^m}\omega_I(s_y(x))\partial_{x_J}(s_y^I)(x)\varphi_{\varepsilon/L}(y)dydx^J \\
&= \sum_{I\in\mathcal{I}_k}\sum_J\int_{B_{\varepsilon/L}}\omega_I(\alpha_x(y))\partial_{x_J}(\alpha_x^I)(y)\varphi_{\varepsilon/L}(y)dydx^J \\
&= \sum_{I\in\mathcal{I}_k}\sum_J\int_{\alpha_x(B_{\varepsilon/L})}\omega_I(z)\ \underbrace{\partial_{x_J}(\alpha_x^I)(\alpha_x^{-1}(z))\varphi_{\varepsilon/L}(\alpha_x^{-1}(z))|\det(\nabla\alpha_x(\alpha_x^{-1}(z)))|^{-1}}_{=:\theta_J(z,x)}dzdx^J \\
&= \sum_{I\in\mathcal{I}_k}\sum_J\int_{\mathbb{R}^m}\omega_I(z)\theta_J(z,x)dzdx^J. \qquad (5.19)
\end{aligned}$$

Since $\theta_J : \mathbb{R}^m \times \bar{B} \to \mathbb{R}$ is a smooth function compactly supported in x, this implies $R_\varepsilon(\omega)|_B \in \Omega^k(B)$.

Step 2 (commutativity of R_ε): This part of the proof is inspired by a remark in [3, p.255]. Let $\omega \in L^k_{1,\mathrm{loc}}(U)$ and $\eta \in \mathscr{D}_{m-k}(U)$. Notice that for any $y \in \mathbb{R}^m$, the map $s_y : U \to U$ is a diffeomorphism. Therefore by the diffeomorphism invariance of the integral, the fact that $\varphi_{\varepsilon/L}(-y) = \varphi_{\varepsilon/L}(y)$ and $(s_y^*)^{-1} = s_{-y}^*$, we obtain

$$\begin{aligned}
\langle R_\varepsilon\omega\rangle(\eta) &= \int_U R_\varepsilon(\omega)|_x \wedge \eta|_x dx = \int_U\int_{\mathbb{R}^m} s_y^*(\omega)|_x\varphi_{\varepsilon/L}(y)dy\wedge\eta|_x dx \\
&= \int_U\int_{\mathbb{R}^m} s_y^*(\omega)|_x\wedge\eta|_x\ \varphi_{\varepsilon/L}(y)dydx = \int_{\mathbb{R}^m}\int_U s_y^*(\omega)|_x\wedge\eta|_x dx\varphi_{\varepsilon/L}(y)dy \\
&= \int_{\mathbb{R}^m}\int_U s_y^*(\omega\wedge s_{-y}^*(\eta))|_x dx\varphi_{\varepsilon/L}(y)dy = \int_{\mathbb{R}^m}\int_U \omega|_x\wedge s_{-y}^*(\eta)|_x dx\varphi_{\varepsilon/L}(y)dy \\
&= \int_U\int_{\mathbb{R}^m}\omega|_x\wedge s_{-y}^*(\eta)|_x\varphi_{\varepsilon/L}(y)dydx = \int_U\omega|_x\wedge\int_{\mathbb{R}^m}s_{-y}^*(\eta)|_x\varphi_{\varepsilon/L}(y)dydx \\
&= \int_U\omega|_x\wedge R_\varepsilon(\eta)|_x dx = \langle\omega\rangle(R_\varepsilon(\eta)) = R'_\varepsilon(\langle\omega\rangle)(\eta).
\end{aligned}$$

Step 3 (regularity of A_ε): The theorem on differentiable parameter dependence of integrals ensures that if ω is $\mathcal{C}^r$, then for any $|\alpha| \le r$

$$\partial_x^\alpha((A_\varepsilon(\omega))_I)(x) = \int_U\int_0^1\partial_x^\alpha\iota_{\partial_t}(S_y^*(\omega))_I(x,t)\varphi_\varepsilon(y)dtdy.$$

Since ι_{∂_t}, S_y^* are smooth, the statement follows.

STEP 4 (commutativity of A_ε): Let $\omega \in L^k_{1,\mathrm{loc}}(U)$ and $\eta \in \mathscr{D}_{m-k+1}(U)$.

$$\begin{aligned}
\langle A_\varepsilon(\omega)\rangle(\eta) &= \int_U A_\varepsilon(\omega)|_x \wedge \eta|_x dx = \int_U \int_{\mathbb{R}^m} \int_0^1 \iota_{\partial_t}(S_y^*(\omega))|_{(x,t)} dt\, \varphi_{\varepsilon/L}(y)\, dy \wedge \eta|_x dx \\
&= \int_{\mathbb{R}^m} \int_U \int_0^1 \iota_{\partial_t}(S_y^*(\omega))|_{(x,t)} dt \wedge \eta|_x dx\, \varphi_{\varepsilon/L}(y)\, dy \\
&= \int_{\mathbb{R}^m} \int_0^1 \int_U \iota_{\partial_t}(S_y^*(\omega))|_{(x,t)} \wedge \pi_U^*(\eta)|_{(x,t)} dx\, dt\, \varphi_{\varepsilon/L}(y)\, dy \\
&\overset{5.1.5(v)}{=} \int_{\mathbb{R}^m} \int_0^1 \int_U \iota_{\partial_t}\left((S_y^*(\omega)) \wedge \pi_U^*(\eta)\right)|_{(x,t)} dx\, dt\, \varphi_{\varepsilon/L}(y)\, dy \\
&\qquad + (-1)^{k+1} \int_{\mathbb{R}^m} \int_0^1 \int_U (S_y^*(\omega))|_{(x,t)} \wedge \underbrace{\iota_{\partial_t}(\pi_U^*(\eta))|_{(x,t)}}_{=0} dx\, dt\, \varphi_{\varepsilon/L}(y)\, dy \\
&= \int_{\mathbb{R}^m} \int_0^1 \iota_{\partial_t}\left(\int_U \left(s_{ty}^*(\omega) \wedge \pi_U^*(\eta)\right)|_{(x,t)}\, dx\right) dt\, \varphi_{\varepsilon/L}(y)\, dy \\
&= \int_{\mathbb{R}^m} \int_0^1 \iota_{\partial_t}\left(\int_U s_{ty}^*\left(\omega \wedge s_{-ty}^*(\pi_U^*(\eta))\right)|_{(x,t)}\, dx\right) dt\, \varphi_{\varepsilon/L}(y)\, dy \\
&= \int_{\mathbb{R}^m} \int_0^1 \iota_{\partial_t}\left(\int_U \pi_U^*(\omega)|_{(x,t)} \wedge S_{-y}^*(\eta)|_{(x,t)}\, dx\right) dt\, \varphi_{\varepsilon/L}(y)\, dy \\
&= \int_{\mathbb{R}^m} \int_0^1 \int_U \iota_{\partial_t}\left(\pi_U^*(\omega)|_{(x,t)} \wedge S_y^*(\eta)|_{(x,t)}\right) dx\, dt\, \varphi_{\varepsilon/L}(-y)\, dy \\
&= \int_U \int_{\mathbb{R}^m} \int_0^1 \iota_{\partial_t}\left(\pi_U^*(\omega)|_{(x,t)} \wedge S_y^*(\eta)|_{(x,t)}\right) \varphi_{\varepsilon/L}(y)\, dt\, dy\, dx
\end{aligned} \tag{5.20}$$

Now on the other hand

$$A'_\varepsilon(\langle\omega\rangle)(\eta) = (-1)^k \langle\omega\rangle(A_\varepsilon\eta) = (-1)^k \int_U \omega|_x \wedge A_\varepsilon(\eta)|_x dx. \tag{5.21}$$

This integral can be transformed by the same methods as above. The prefactor $(-1)^k$ fits in perfectly because of 5.1.5(v): In (5.20) we used this theorem in the form of

$$\iota_X(\alpha) \wedge \beta = \iota_X(\alpha \wedge \beta) - (-1)^k \underbrace{\alpha \wedge \iota_X(\beta)}_{=0}$$

and in (5.21) we have to use it in the form of

$$(-1)^k \alpha \wedge \iota_X(\beta) = \iota_X(\alpha \wedge \beta) - \underbrace{\iota_X(\alpha) \wedge \beta}_{=0}.$$

This shows that (5.21) equals (5.20). □

5.3.14 Theorem (integrability). The linear operators $R_\varepsilon : L_p^k(U) \to L_p^k(U)$ and $A_\varepsilon : L_p^k(U) \to L_p^{k-1}(U)$ are bounded. (We remind that L_p is taken with respect to an arbitrary Riemannian metric on U).

Proof.

STEP 1 (preliminaries): Before we can start the estimates, we need some preliminary inequalities.

STEP 1.1: By Theorem 5.2.2, the map s is smooth. Therefore, the map $\bar{B} \times \bar{B} \to \mathbb{R}$, $(x, y) \mapsto \|(s_y)_*|_x\|$, is continuous (here $\|(s_y)_*|_x\|$ is the operator norm of s_{y*} at x) . We obtain

$$C_1 := \sup_{(x,y)\in\bar{B}\times\bar{B}} \|(s_y)_*|_x\| < \infty,$$

since this is a supremum of a continuous function over a a compact set. Again by 5.2.2, we know that for all $x \in U \setminus B$ and any $y \in \mathbb{R}^m$, we obtain $s_y(x) = x$, which implies

$$\forall x \in U : \forall y \in \bar{B} : \|(s_y)_*|_x\| \le \max(C_1, 1) =: C_2'.$$

By Corollary 1.2.10 this implies that the operator norm of the induced pull-back on k-forms satisfies

$$\forall x \in U : \forall y \in \bar{B} : \|s_y^*|_x\| \le \binom{m}{k} m C_2'^k =: C_2. \tag{5.22}$$

Also notice that by 5.2.6 the Lipschitz constant L satisfies $L \ge 1$ and since $\varepsilon < 1$ by definition 5.3.8, this implies that the Euclidean ball $B_{\varepsilon/L}(0)$ satisfies

$$B_{\varepsilon/L}(0) \subset \bar{B} \subset U.$$

STEP 1.2: In a similar fashion consider $S_y : U \times I \to U$. We obtain

$$C_4' := \sup_{(y,x,t)\in\bar{B}\times\bar{B}\times I} \|S_{y*}|_{(x,t)}\| < \infty$$

and since for any $x \in U \setminus B$, $S_y(x, t) = s_{ty}(x) = x$, we obtain

$$\sup_{(y,x,t)\in\bar{B}\times U\times I} \|S_{y*}|_{(x,t)}\| \le \max(C_4', 1) =: C_4''.$$

Again by 1.2.10, this implies that norm of the corresponding operator on pull-backs of k-forms satisfies

$$\forall x \in U : \forall y \in \bar{B} : \forall t \in I : \|S_y^*|_{(x,t)}\| \le \binom{m}{k} C_4''^k =: C_4. \tag{5.23}$$

STEP 1.3: The interior multiplication $\iota_{\partial_t} : \Omega^k(U \times I) \to \Omega^{k-1}(U \times I)$, $\omega \mapsto \iota_{\partial_t}(\omega)$, is smooth and $\overline{\mathcal{O}_\varepsilon(B)} \subset U$. Therefore

$$\sup_{(x,t)\in\overline{\mathcal{O}_\varepsilon(B)}\times I} \|\iota_{\partial_t}|_{(x,t)}\| =: C_5 < \infty, \tag{5.24}$$

since we again take the supremum of a continuous function over a compact domain.

STEP 2: For any $1 \le p < \infty$, we calculate

$$\begin{aligned}\|R_\varepsilon(\omega)\|^p_{L^k_p(U)} &= \int_U |R_\varepsilon(\omega)(x)|^p dx = \int_U \Big| \int_{\mathbb{R}^m} s_y^*(\omega)|_x \varphi_{\varepsilon/L}(y) dy\Big|^p dx \\ &\le \int_U \Big(\int_{B_{\varepsilon/L}(0)} |s_y^*(\omega)|_x| |\varphi_{\varepsilon/L}(y)| dy\Big)^p dx\end{aligned}$$

$$
\begin{aligned}
&\overset{5.1.2}{\leq} \Big(\frac{c}{L^{-m}\varepsilon^m}\Big)^p \int_U \Big(\int_{B_{\varepsilon/L}(0)} |s_y^*(\omega)|_x|dy\Big)^p dx \\
&\overset{(5.22)}{\leq} \Big(\frac{c}{L^{-m}\varepsilon^m}\Big)^p \int_U \Big(\int_{B_{\varepsilon/L}(0)} C_2|\omega|_x|dy\Big)^p dx \\
&= \Big(\frac{cC_2}{L^{-m}\varepsilon^m}\Big)^p \int_U |\omega(x)|^p \operatorname{vol}(B_{\varepsilon/L}(0))^p dx \\
&= \Big(\frac{cC_2 \operatorname{vol}(B)\varepsilon^m L^{-m}}{L^{-m}\varepsilon^m}\Big)^p \int_U |\omega(x)|^p dx \\
&= \underbrace{(cC_2 \operatorname{vol}(B))^p}_{=:C_3^p} \|\omega\|^p_{L^k_p(U)}.
\end{aligned}
$$

In case $p = \infty$, the calculation is very similar:

$$
\begin{aligned}
\|R_\varepsilon(\omega)\|_{L^k_\infty(U)} &= \operatorname*{ess\,sup}_{x\in U}|R_\varepsilon(\omega)(x)| = \operatorname*{ess\,sup}_{x\in U}\Big|\int_{\mathbb{R}^m} s_y^*(\omega)|_x \varphi_{\varepsilon/L}(y)dy\Big| \\
&\leq \operatorname*{ess\,sup}_{x\in U}\int_{B_{\varepsilon/L}(0)} |s_y^*(\omega)|_x||\varphi_{\varepsilon/L}(y)|dy \\
&\overset{5.1.2}{\leq} \frac{c}{L^{-m}\varepsilon^m} \operatorname*{ess\,sup}_{x\in U}\int_{B_{\varepsilon/L}(0)} |s_y^*(\omega)|_x|dy \\
&\overset{(5.22)}{\leq} \frac{cC_2}{L^{-m}\varepsilon^m} \operatorname{vol}(B_{\varepsilon/L}(0)) \operatorname*{ess\,sup}_{x\in U}|\omega(x)| = C_3\|\omega\|_{L^k_\infty(U)}.
\end{aligned}
$$

STEP 3: Remember that $\operatorname{supp} A_\varepsilon(\omega) \subset \overline{\mathcal{O}_\varepsilon(B)}$. Therefore we calculate in a similar fashion

$$
\begin{aligned}
\|A_\varepsilon(\omega)\|^p_{L^{k-1}_p(U)} &= \int_{\overline{\mathcal{O}_\varepsilon(B)}} |A_\varepsilon(\omega)(x)|^p dx = \int_{\overline{\mathcal{O}_\varepsilon(B)}} \Big|\int_{\mathbb{R}^m}\int_0^1 \iota_{\partial_t}(S_y^*(\omega))|_{(x,t)}dt\, \varphi_{\varepsilon/L}(y)dy\Big|^p dx \\
&\leq \int_{\overline{\mathcal{O}_\varepsilon(B)}} \Big(\int_{\mathbb{R}^m}\int_0^1 |\iota_{\partial_t}(S_y^*(\omega))|_{(x,t)}|dt|\varphi_{\varepsilon/L}(y)|dy\Big)^p dx \\
&\overset{5.1.2}{\leq} \Big(\frac{c}{L^{-m}\varepsilon^m}\Big)^p \int_{\overline{\mathcal{O}_\varepsilon(B)}} \Big(\int_{B_{\varepsilon/L(0)}}\int_0^1 |\iota_{\partial_t}(S_y^*(\omega))|_{(x,t)}|dtdy\Big)^p dx \\
&\overset{(5.24)}{\leq} \Big(\frac{cC_5}{L^{-m}\varepsilon^m}\Big)^p \int_{\overline{\mathcal{O}_\varepsilon(B)}} \Big(\int_{B_{\varepsilon/L(0)}}\int_0^1 |S_y^*(\omega)|_{(x,t)}|dtdy\Big)^p dx \\
&\overset{(5.23)}{\leq} \Big(\frac{cC_5C_4}{L^{-m}\varepsilon^m}\Big)^p \int_{\overline{\mathcal{O}_\varepsilon(B)}} \Big(\int_{B_{\varepsilon/L(0)}}\int_0^1 |\omega|_x|dtdy\Big)^p dx \\
&\leq \Big(\frac{cC_5C_4 \operatorname{vol}(B_{\varepsilon/L})}{L^{-m}\varepsilon^m}\Big)^p \int_U |\omega|_x|^p dx = \underbrace{(cC_5C_4\operatorname{vol}(B))^p}_{=:C_6^p} \|\omega\|^p_{L^k_p(U)}.
\end{aligned}
$$

Again, in case $p = \infty$ this simplifies to

$$
\begin{aligned}
\|A_\varepsilon(\omega)\|_{L^{k-1}_\infty(U)} &= \operatorname*{ess\,sup}_{x\in\overline{\mathcal{O}_\varepsilon(B)}}|A_\varepsilon(\omega)|_x| \leq \operatorname*{ess\,sup}_{x\in\overline{\mathcal{O}_\varepsilon(B)}}\int_{\mathbb{R}^m}\int_0^1 |\iota_{\partial_t}(S_y^*(\omega))|_{(x,t)}|dt|\varphi_{\varepsilon/L}(y)|dy \\
&\overset{5.1.2}{\leq} \frac{c}{L^{-m}\varepsilon^m}\operatorname*{ess\,sup}_{x\in\overline{\mathcal{O}_\varepsilon(B)}}\int_{B_{\varepsilon/L(0)}}\int_0^1 |\iota_{\partial_t}(S_y^*(\omega))|_{(x,t)}|dtdy \\
&\overset{(5.24)}{\leq} \frac{cC_5}{L^{-m}\varepsilon^m}\operatorname*{ess\,sup}_{x\in\overline{\mathcal{O}_\varepsilon(B)}}\int_{B_{\varepsilon/L(0)}}\int_0^1 |S_y^*(\omega)|_{(x,t)}|dtdy
\end{aligned}
$$

$$\overset{(5.23)}{\leq} \frac{cC_5C_4}{L^{-m}\varepsilon^m} \operatorname*{ess\,sup}_{x\in\mathcal{O}_\varepsilon(B)} \int_{B_{\varepsilon/L(0)}} \int_0^1 |\omega|_x| dt dy$$
$$\leq \frac{cC_5C_4 \operatorname{vol}(B_{\varepsilon/L})}{L^{-m}\varepsilon^m} \operatorname*{ess\,sup}_{x\in U} |\omega|_x| = C_6 \|\omega\|_{L^k_\infty(U)}. \qquad \square$$

5.3.15 Corollary. The operators R_ε, A_ε may be regarded as linear maps

$$R_\varepsilon, A_\varepsilon : L_{p,\mathrm{loc}}(U) \to L_{p,\mathrm{loc}}(U), \qquad R_\varepsilon, A_\varepsilon : W_{p,\mathrm{loc}}(U) \to W_{p,\mathrm{loc}}(U),$$
$$R_\varepsilon, A_\varepsilon : L_p(U) \to L_p(U), \qquad R_\varepsilon, A_\varepsilon : W_p(U) \to W_p(U).$$

As such they define bounded linear operators on $L_p(U)$ and $W_p(U)$.

Proof.

STEP 1 (L-spaces): We already know from Theorem 5.3.14 above that $R_\varepsilon, A_\varepsilon : L_p(U) \to L_p(U)$ are continuous linear maps. Now let $\omega \in L_{p,\mathrm{loc}}(U)$. Let $K \subset U$ be compact and define

$$\tilde{\omega} := \begin{cases} \omega, & \text{on } K \\ 0, & \text{on } U \setminus K \end{cases} \in L_p(U).$$

This implies

$$\|R_\varepsilon(\omega)\|_{L_p(K)} = \|R_\varepsilon(\tilde{\omega})\|_{L_p(U)} \leq C\|\tilde{\omega}\|_{L_p(U)} = C\|\omega\|_{L_p(K)} < \infty.$$

The same argument holds for A_ε.

STEP 2 (W-spaces): For any $1 \leq p < \infty$ and any $\omega \in W^k_p(U)$ we obtain

$$\|R_\varepsilon(\omega)\|^p_{W^k_p(U)} = \|R_\varepsilon(\omega)\|^p_{L^k_p(U)} + \|dR_\varepsilon(\omega)\|^p_{L^{k+1}_p(U)}$$
$$\overset{5.3.11}{=} \|R_\varepsilon(\omega)\|^p_{L^k_p(U)} + \|R_\varepsilon(d\omega)\|^p_{L^{k+1}_p(U)} \leq C\|\omega\|^p_{W^k_p(U)},$$

by Theorem 5.3.14. And similar

$$\|A_\varepsilon(\omega)\|^p_{W^k_p(U)} = \|A_\varepsilon(\omega)\|^p_{L^{k-1}_p(U)} + \|dA_\varepsilon(\omega)\|^p_{L^k_p(U)}$$
$$\overset{5.3.11}{=} \|A_\varepsilon(\omega)\|^p_{L^{k-1}_p(U)} + \|R_\varepsilon(\omega) - \omega - A_\varepsilon(d\omega)\|^p_{L^k_p(U)}$$
$$\leq \|A_\varepsilon(\omega)\|^p_{L^{k-1}_p(U)} + \Big(\|R_\varepsilon(\omega)\|_{L^k_p(U)} + \|\omega\|_{L^k_p(U)} + \|A_\varepsilon(d\omega)\|_{L^k_p(U)}\Big)^p$$
$$\leq C\|\omega\|^p_{W^k_p(U)},$$

by what we have proven so far. By the same token if $\omega \in W_\infty(U)$

$$\|R_\varepsilon(\omega)\|_{W^k_\infty(U)} = \max\{\|R_\varepsilon(\omega)\|_{L^k_\infty(U)}, \|dR_\varepsilon(\omega)\|_{L^{k+1}_\infty(U)}\}$$
$$\overset{5.3.11}{=} \max\{\|R_\varepsilon(\omega)\|_{L^k_\infty(U)}, \|R_\varepsilon(d\omega)\|_{L^{k+1}_\infty(U)}\} \leq C\|\omega\|_{W^k_\infty(U)},$$

as well as

$$\|A_\varepsilon(\omega)\|_{W^k_\infty(U)} = \max\{\|A_\varepsilon(\omega)\|_{L^{k-1}_\infty(U)}, \|dA_\varepsilon(\omega)\|_{L^k_\infty(U)}\}$$
$$\overset{5.3.11}{=} \max\{\|A_\varepsilon(\omega)\|_{L^{k-1}_\infty(U)}, \|R_\varepsilon(\omega) - \omega - A_\varepsilon(d\omega)\|_{L^k_\infty(U)}\}$$
$$\leq \max\{\|A_\varepsilon(\omega)\|_{L^{k-1}_\infty(U)}, \|R_\varepsilon(\omega)\|_{L^k_\infty(U)} + \|\omega\|_{L^k_\infty(U)} + \|A_\varepsilon(d\omega)\|_{L^k_\infty(U)}\}$$
$$\leq C\|\omega\|_{W^k_\infty(U)}.$$

The statement for $W_{p,\mathrm{loc}}$ follows as in the first step. $\square$

If we collect all the facts we have proven so far, we obtain a rigorous proof of the first existence theorem for regularization operators, which is stated by Gol'dshtein, Kuz'minov and Shvedov in [5, Lemma 3] and is also discussed by de Rham in [23, III.§15].

5.3.16 Theorem (regularization operators I). Let $U \subset \mathbb{R}^m$ be a bounded open set, endowed whith an arbitrary Riemannian metric, containing a closed Euclidean ball $\bar{B}$ of radius one. Then the maps $R_\varepsilon, A_\varepsilon$ specified in 5.3.8, 5.3.9, 5.3.12 satisfy the following properties for any $1 \le p \le \infty$:

(i) These maps restrict to bounded linear operators

$$R_\varepsilon : L_p^k(U) \to L_p^k(U), \qquad A_\varepsilon : L_p^k(U) \to L_p^{k-1}(U).$$

(ii) They also restrict to maps

$$R_\varepsilon : W_{p,\mathrm{loc}}^k(U) \to W_{p,\mathrm{loc}}^k(U), \qquad A_\varepsilon : W_{p,\mathrm{loc}}^k(U) \to W_{p,\mathrm{loc}}^{k-1}(U)$$

satisfying the relations

$$R_\varepsilon \circ d = d \circ R_\varepsilon, \qquad d \circ A_\varepsilon + A_\varepsilon \circ d = R_\varepsilon - \mathrm{id}\,.$$

So R_ε is a cochain map and A_ε is a cohomotopy between R_ε and id.

(iii) These maps restrict to bounded linear operators

$$R_\varepsilon : W_p^k(U) \to W_p^k(U), \qquad A_\varepsilon : W_p^k(U) \to W_p^{k-1}(U).$$

(iv) For any compactum $F \subset \operatorname{Int} B$ and any $\omega \in L_{p,\mathrm{loc}}^k(U)$ the form $R_\varepsilon(\omega)|_F$ is essentially bounded. R_ε restricts to a bounded linear operator

$$R_\varepsilon : L_p^k(U) \to L_\infty^k(F).$$

(v) For all $\omega \in L_{p,\mathrm{loc}}(U)$

$$(R_\varepsilon\omega)|_{U\setminus B} = \omega, \qquad (A_\varepsilon\omega)|_{U\setminus B} = 0.$$

(vi) For any $\omega \in L_{p,\mathrm{loc}}(U)$

$$\begin{aligned}&\operatorname{supp} R_\varepsilon\omega \subset \operatorname{supp}\omega \setminus B \cup \mathcal{O}_\varepsilon(\operatorname{supp}\omega \cap B) \subset \mathcal{O}_\varepsilon(\operatorname{supp}\omega),\\ &\operatorname{supp} A_\varepsilon\omega \subset B \cap \operatorname{supp}\omega \subset \mathcal{O}_\varepsilon(\operatorname{supp}\omega).\end{aligned}$$

Proof. Everything but (iv) has already been proven in 5.3.9, 5.3.11, 5.3.14. To see (iv) notice that for any $\omega \in L_{p,\mathrm{loc}}^k(U)$, the form $R_\varepsilon(\omega)|_B$ is smooth by 5.3.13 and therefore bounded on any compact $F \subset \operatorname{Int} B$. But we now estimate the operator norm $R_\varepsilon : L_p(U) \to L_\infty(F)$ and obtain this statement independently of 5.3.13. The case $p = \infty$ already follow from (i), therefore assume $1 \le p < \infty$.
The idea of this proof is to generalize the proof of the well-known inequality

$$\|f * g\|_{L_\infty} \le \|f\|_{L_1}\|g\|_{L_\infty},$$

where $f \in L_1(\mathbb{R}^m)$ and $g \in L_\infty(\mathbb{R}^m)$.

Step 1 (preparations): First of all notice that by our choice of ε in 5.3.8 and by hypothesis

$$F, B_{\varepsilon/L}(0) \subset \bar{B} \subset U.$$

Remember that the localized translation group is a smooth map $s : U \times \mathbb{R}^m \to U$, c.f. Theorem 5.2.2. There we also defined the maps $\alpha_x(y) = s(y,x) = s_y(x)$. The set

$$K := s(F, \bar{B}) \subset U$$

is compact as well.

Step 1.1 (metric): We recall again that U is equipped with an arbitrary Riemannian metric g. This will turn out to be inconvenient, but fortunately on a compact manifold any two Riemannian metrics are equivalent. Temporarily denote by $|\_|$ the norm induced by the Riemannian metric, by $\|\_\|$ the Euclidean norm (canonically extended to $\Lambda^k(U)$ by Theorem 1.2.4 as well) and by $|\_|_{L_p}$, $\|\_\|_{L_p}$ the induced L_p-norms. There exist constants $c, C > 0$ such that

$$\forall z \in \bar{B} : \forall V \in T_z\bar{B} : c\|V\| \leq |V| \leq C\|V\|. \tag{5.25}$$

Step 1.2 (pullback): Let $\omega = f dx^I = f dx^{i_1} \wedge \ldots \wedge dx^{i_k} \in L_p^k(U)$ and $x \in F$ be arbitrary. For any $y \in \mathbb{R}^m$, the pullback is given by

$$\begin{aligned} s_y^*(\omega)|_x &= (f \circ s_y)(x) d(x^{i_1} \circ s_y) \wedge \ldots \wedge d(x^{i_k} \circ s_y)|_x = (f \circ s_y)(x) ds_y^{i_1} \wedge \ldots \wedge ds_y^{i_k}|_x \\ &= (f \circ s_y)(x) \partial_{j_1} s_y^{i_1}(x) \ldots \partial_{j_k} s_y^{i_k}(x) dx^{j_1} \wedge \ldots \wedge dx^{j_k}|_x. \end{aligned}$$

Now define

$$C_1 := \max_{1 \leq \nu,\mu \leq k} \max_{(x,y) \in F \times \bar{B}} |\partial_\nu s^\mu(x)|, \qquad C_2 := m^k C_1^k. \tag{5.26}$$

Step 1.3: Finally, we bound the expression

$$C_3 := \max_{z = s(x,y) \in K} \varphi_{\varepsilon/L}(\alpha_x^{-1}(z)) |\det(\nabla \alpha_x(\alpha^{-1}(z)))|^{-1} < \infty. \tag{5.27}$$

Step 2: Remember that the dx^ν are a Euclidean orthonormal basis. Therefore by definition of the extended metric from 1.2.4, we calculate for any $x \in F$

$$\begin{aligned} \|R_\varepsilon(\omega)|_x\| &= \left\| \int_{\mathbb{R}^n} s_y^*(\omega)|_x \varphi_{\varepsilon/L}(y) dy \right\| \\ &= \left\| \int_{\mathbb{R}^n} (f \circ s_y)(x) \partial_{j_1} s_y^{i_1}(x) \ldots \partial_{j_k} s_y^{i_k}(x) \varphi_{\varepsilon/L}(y) dy \; dx^{j_1} \wedge \ldots \wedge dx^{j_k}|_x \right\| \\ &\leq \sum_J \int_{\mathbb{R}^n} |(f \circ s_y)(x) \partial_{j_1} s_y^{i_1}(x) \ldots \partial_{j_k} s_y^{i_k}(x) \varphi_{\varepsilon/L}(y)| dy \; \|dx^{j_1} \wedge \ldots \wedge dx^{j_k}|_x\| \\ &\overset{(5.26)}{\leq} C_1^k \sum_J \int_{\mathbb{R}^n} |(f \circ s_y)(x) \varphi_{\varepsilon/L}(y)| dy \overset{(5.26)}{=} C_2 \int_{B_{\varepsilon/L}(0)} |(f(\alpha_x(y))| \varphi_{\varepsilon/L}(y) dy \\ &= C_2 \int_{\alpha_x(\bar{B})} |f(z)| \varphi_{\varepsilon/L}(\alpha_x^{-1}(z)) |\det(\nabla \alpha_x(\alpha_x^{-1}(z)))|^{-1} dz \\ &\overset{(5.27)}{\leq} C_2 C_3 \int_K |f(z)| dz \leq C_2 C_3 \mu(K)^{\frac{1}{p'}} \|\omega\|_{L_p(K)}, \end{aligned} \tag{5.28}$$

where p' is Hölder conjugate to p. Notice that the last expression is independent of x. This implies the statement via

$$\begin{aligned}|R_\varepsilon(\omega)|_{L_\infty(F)} \le C\|R_\varepsilon(\omega)\|_{L_\infty(F)} &\overset{(5.28)}{\le} CC_2C_3\mu(K)^{\frac{1}{p'}}\|\omega\|_{L_p(K)} \\ &\overset{(5.25)}{\le} cCC_2C_3\mu(K)^{\frac{1}{p'}}|\omega|_{L_p(K)} \le cCC_2C_3\mu(K)^{\frac{1}{p'}}|\omega|_{L_p(U)}.\end{aligned}$$ □

6 The isomorphism between the L_p-cohomology of forms and S-forms

6.1 Preliminaries

Let us recall our setup: M is an oriented Riemannian m-manifold without boundary, K is a simplicial complex in some $\mathbb{R}^n$, $\{x_i\}_{i\in\mathbb{N}}$ is a numbering of its vertices and $h : |K| \to M$ is a smooth triangulation. The aim of this section is to prove that the L_p-cohomology $H^*_p(M) = H^*(W_p(M))$ of M is isomorphic to the L_p-cohomology $\mathscr{H}_p(K) = H^*(S^*_p(K))$ of S-forms on K.

6.1.1 Remark. Before we delve into all the technicalities (some of which we already carried out in the last chapter), it might be useful to take a look at the big picture first. The overall strategy outlined by Gol'dshtein, Kuz'minov and Shvedov in [5] is the following:

$$
\begin{array}{ccccc}
 & & (S_p(M), \|\_\|_{W_p(M)}) & & \\
 & \overset{\varphi_h^{-1}}{\nearrow} & & \overset{\iota}{\searrow} & \\
(S_p(K), \|\_\|_{S_p(K)}) & & & & (W_p(M), \|\_\|_{W_p(M)}) \\
 & \underset{\varphi_h}{\nwarrow} & & \underset{\mathscr{R}}{\swarrow} & \\
 & & (S_p(M), \|\_\|_{S_p(M)}) & &
\end{array}
\tag{6.1}
$$

In Lemma 2.3.13, we already established that $\varphi_h : W_{\infty,\mathrm{loc}}(M) \to S(K)$ is an isomorphism of cochain complexes. Therefore its inverse map is an isomorphism as well. In Lemma 6.1.6 we will show that this map satisfies $\varphi_h^{-1}(S_p(K)) =: S_p(M) \subset W_p(M)$ and is continuous, if $S_p(M)$ is endowed with the subspace topology. By restricting, we obtain a map $\varphi_h : S_p(M) \to S_p(K)$ again. In 6.1.7, we will force this map to be continuous by endowing $S_p(M)$ with another norm $\|\_\|_{S_p(M)}$. The operators $\mathscr{R}, \mathscr{A} : (W_p(M), \|\_\|_{W_p(M)}) \to (S_p(M), \|\_\|_{S_p(M)})$ will be constructed in 6.2.1. In 6.2.2 we will show that $\varphi_h \circ \mathscr{R}$ is an inverse to $\iota \circ \varphi_h^{-1}$ up to the cochain homotopy $\mathscr{A}$.

It turns out that this strategy works only under certain restrictions on the triangulation (c.f. 1.3.6).

6.1.2 Definition (GKS-condition). A smooth triangulation $h : |K| \to M$ *satisfies the Gol'dshtein-Kuz'minov-Shvedov condition* (or just "*is GKS*"), if

(i) The simplicial complex K is star-bounded with star-bound N.

(ii) There are constants $C_1, C_2 > 0$ such that for every simplex $\sigma \in K$ the push-forward of the map $h : |\sigma| \to M$, seen as a smooth map between manifolds with corners, satisfies

$$\sup_{x\in\sigma} \|h_*|_x\| \le C_1. \qquad\qquad \sup_{x\in\sigma} \|h_*^{-1}|_{h(x)}\| \le C_2.$$

Here $\|\_\|$ denotes the operator norm, which is induced by the Riemannian metric on M and the S-metric on K. It is of the utmost importance that Convention 2.3.6 is in power, i.e. K is endowed with the standard S-metric g_S.

A Riemannian manifold M *is GKS* if there exists a triangulation $h : |K| \to M$ that is GKS.

6.1.3 Convention. For purposes of integration, we may think of $|K|$ as a smooth Riemannian manifold: Certainly the union of interior of the simplices of top dimension is a smooth manifold and the rest is a set of measure zero anyway. It is also useful to think of h as a (C_1, C_2)-bounded diffeomorphism, c.f. 1.3.6.
In particular, we may think of $|K|$ as a metric space: For any two points $x, y \in K$ we define $d(x, y)$ as the infimum of all piecewise smooth curves connecting x and y. The notion of piecewise smooth curves is still senseful and the curve length is calculated using the S-metric on K.

6.1.4 Lemma. Let $h : (|K|, g_S) \to (M, g)$ be GKS.

(i) The metrics $(h^* g_S, g)$ are (C_1^{-1}, C_2)-equivalent.

(ii) The induced metrics d_g and d_S on M respectively $|K|$ satisfy

$$\forall x, y \in |K| : C_1^{-1} d_S(x, y) \le d_g(h(x), h(y)) \le C_2 d_S(x, y).$$

In particular, any GKS manifold is complete.

Proof.

(i) This is a direct consequence of Theorem 1.3.7.

(ii) Assume $\gamma : [0, 1] \to |K|$ connects x and y (for simplicity assume γ to be smooth). Then $h \circ \gamma$ connects $h(x)$ and $h(y)$. We calculate

$$L_g(h \circ \gamma) = \int_0^1 |\partial_t (h \circ \gamma)(t)| dt = \int_0^1 |h_*|_{\gamma(t)} \dot{\gamma}(t)| dt \le C_1 \int_0^1 |\dot{\gamma}(t)| dt = C_1 L_S(\gamma).$$

Since any curve connecting $h(x)$ and $h(y)$ is of that form, the first inequality follows. The second one is proven analogously.
This estimate shows that for any d_g-Cauchy sequence (p_j) in M, the sequence $(x_j := h^{-1}(p_j))$ is an S-Cauchy sequence in $|K|$. Since $|K|$ is complete, it has some limit $x \in |K|$. This forces (p_j) to converge to $p := h(x)$. □

6.1.5 Lemma. The map h induces bounded operators $h^* : L_p(M) \to L_p(|K|)$, $(h^{-1})^* : L_p(|K|) \to L_p(M)$.

Proof. This follows from Theorem 2.1.28. □

6.1.6 Lemma. If h is GKS, the isomorphism $\varphi_h : W^k_{\infty,\text{loc}}(M) \to S^k(K)$ from Lemma 2.3.13 satisfies $\varphi_h^{-1}(S^k_p(K)) \subset W^k_p(M)$, $1 \le p \le \infty$, and the map $\varphi_h^{-1} : (S^k_p(K), \|\_\|_{S_p(K)}) \hookrightarrow (W^k_p(M), \|\_\|_{W^k_p(M)})$ is continuous.

Proof. The fact that φ_h is an isomorphism was already established in Lemma 2.3.13. So φ_h^{-1} is a well-defined map and its restriction clearly remains injective. We have to check that $\varphi_h^{-1} : S^k_p(K) \to W^k_p(M)$ is continuous. Therefore let $\theta \in S^k_p(K)$ and let

$\omega := \varphi_h^{-1}(\theta) \in W^k_{\infty,\mathrm{loc}}(M)$. Define $\tilde{C}_2 := \binom{m}{k} C_2^k$ and calculate for any $1 \leq p < \infty$

$$\begin{aligned}\|\omega\|^p_{L^k_p(M)} &= \int_M |\omega|^p dV = \sum_{\sigma \in K^{(m)}} \int_{h(\sigma)} |\omega|^p dV = \sum_{\sigma \in K^{(m)}} \int_{h(\sigma)} |(h^{-1})^*(\theta(\sigma))|^p dV \\ &\overset{1.2.10}{\leq} \tilde{C}_2^p \sum_{\sigma \in K^{(m)}} \int_{h(\sigma)} (h^{-1})^*(|\theta(\sigma)|^p) dV \leq \tilde{C}_2^p \sum_{\sigma \in K^{(m)}} \|\theta(\sigma)\|^p_{W^k_\infty(\sigma)} \operatorname{vol}_g(h(\sigma)) \\ &\overset{1.3.4}{\leq} \tilde{C}_2^p C_1^m \sum_{\sigma \in K^{(m)}} \|\theta(\sigma)\|^p_{W^k_\infty(\sigma)} \operatorname{vol}(\sigma) \leq \underbrace{\tilde{C}_2^p C_1^m v_m}_{=:C^p} \|\theta\|^p_{S^k_p(K)}.\end{aligned}$$

In case $p = \infty$, we calculate analogously:

$$\begin{aligned}\|\omega\|_{L^k_\infty(M)} &= \operatorname*{ess\,sup}_{x \in M} |\omega|(x) = \operatorname*{ess\,sup}_{\sigma \in K, x \in \sigma} |\omega|(h(x)) = \operatorname*{ess\,sup}_{\sigma \in K, x \in \sigma} |(h^{-1})^*(\theta(\sigma))|(h(x)) \\ &\overset{1.2.10}{\leq} \tilde{C}_2 \operatorname*{ess\,sup}_{\sigma \in K, x \in \sigma} |\theta(\sigma)|)(x) \leq \tilde{C}_2 \operatorname*{ess\,sup}_{\sigma \in K} \|\theta(\sigma)\|_{W^k_\infty(\sigma)} = \tilde{C}_2 \|\theta\|_{S^k_\infty(K)}.\end{aligned}$$

Since d commutes with the pull-back of h^{-1} by Theorem 2.1.29, and since d is bounded, we obtain

$$\|d\omega\|_{L^{k+1}_p(M)} \leq C\|d\|\|\theta\|_{S^k_p(K)},$$

which implies the statement. □

6.1.7 Lemma (φ_h and $S_p(M)$). Let $\varphi_h^{-1} : (S^k_p(K), \|\_\|_{S_p(K)}) \hookrightarrow (W^k_p(M), \|\_\|_{W^k_p(M)})$ be as in 6.1.6. Define $S_p(M) := \varphi_h^{-1}(S_p(K))$ and

$$\forall \omega \in S^k_p(M) : \|\omega\|_{S^k_p(M)} := \begin{cases} \left(\sum_{\sigma \in K} \|\omega\|^p_{W^k_\infty(h(\sigma))}\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \sup_{\sigma \in K} \|\omega\|_{W^k_\infty(h(\sigma))}, & p = \infty. \end{cases}$$

(i) Then $\varphi_h : (S_p(M), \|\_\|_{S_p(M)}) \to (S_p(K), \|\_\|_{S_p(K)})$ is continuous.

(ii) If $\omega \in W_p(M)$ and $\|\omega\|_{S_p(M)} < \infty$, then $\omega \in S_p(M)$.

(iii) The inclusion $\iota : (S_p(M), \|\_\|_{S_p(M)}) \to (W_p(M), \|\_\|_{W_p(M)})$ is continuous.

Proof.

(i) First assume $1 \leq p < \infty$. Let $\omega \in S^k_p(M)$ be arbitrary. For any $\sigma \in K$, we calculate

$$\sup_{x \in \sigma} |h|^*_\sigma \omega|_x| \overset{1.2.8}{\leq} \binom{m}{k} C_1^k \sup_{x \in \sigma} |\omega|_{h(x)}| = \binom{m}{k} C_1^k \|\omega\|_{L_\infty(h(\sigma))}.$$

The same can be done with $d\omega$ and since d commutes with h^*, there exists a constant $C > 0$ such that

$$\|\varphi_h(\omega)\|^p_{S_p(K)} = \sum_{\sigma \in K} \|h|^*_\sigma \omega\|^p_{W^k_\infty(\sigma)} \leq C^p \sum_{\sigma \in K} \|\omega\|^p_{W^k_\infty(h(\sigma))} = C^p \|\omega\|^p_{S_p(M)}.$$

The statement for $p = \infty$ follows analogously.

(ii) Assume $\omega \in W^k_p(M)$ and $\|\omega\|_{S_p(M)} < \infty$. Clearly $\omega \in W_{\infty,\mathrm{loc}}(M)$. Therefore $\varphi_h(\omega) \in S^k(K)$. By hypothesis

$$\|\varphi_h(\omega)\|_{S_p(K)} \leq C\|\omega\|_{S_p(M)} < \infty,$$

thus $\varphi_h(\omega) \in S^k_p(K)$. Consequently $\omega = \varphi_h^{-1}(\varphi_h(\omega)) \in S^k_p(M)$.

(iii) We calculate for any $\omega \in S_p(M)$, $1 \leq p < \infty$,

$$\|\omega\|^p_{L_p(M)} = \sum_{\sigma \in K^{(m)}} \int_{h(\sigma)} |\omega|^p \leq \sum_{\sigma \in K^{(m)}} \operatorname{vol}(h(\sigma)) \|\omega\|_{L_\infty(h(\sigma))}$$
$$\overset{1.3.4}{\leq} C_1^m v_m \sum_{\sigma \in K} \|\omega\|_{L_\infty(h(\sigma))}.$$

Applying this to $d\omega$ as well yields the statement. The case $p = \infty$ is similar. □

6.1.8 Definition. Let $i \in \mathbb{N}$ be arbitrary.

(i) Let $\langle x_i \rangle \in K$ be a vertex, let $\operatorname{st}_K(x_i)$ be the star of x_i in K and $\operatorname{st}_{B(K)}(x_i)$ be the star of x_i in the first barycentric subdivision of K. A triple of closed sets

$$X_i \subset Y_i \subset Z_i \subset M$$

such that

$$\begin{aligned} &\Sigma_i' := h(\operatorname{st}_{B(K)}(x_i)) \subset \operatorname{Int} X_i, \\ X_i \subset \operatorname{Int} Y_i, \quad &Y_i \subset \operatorname{Int} Z_i, \quad Z_i \subset \operatorname{Int} \Sigma_i, \\ &\Sigma_i := h(\operatorname{st}_K(x_i)) \end{aligned}$$

is a *star triple* (see figure 6.1).

(ii) A homeomorphism $\varphi_i : \Sigma_i \to \bar{B}_2(0)$, which restricts to a diffeomorphism on each simplex of Σ_i such that

$$\varphi_i(Y_i) \subset B_{1/2}(0) \subset B_1(0) \subset \varphi_i(Z_i)$$

is a *star chart* (see figure 6.1).

(iii) We say that $\tilde{\psi}_{ij} : \Sigma_i \to \Sigma_j$ is a *simplicial isomorphism*, if there exists a simplicial isomorphism $\psi_{ij} : \operatorname{st}_K(x_i) \to \operatorname{st}_K(x_j)$ such that $\tilde{\psi}_{ij} = h \circ \psi_{ij} \circ h^{-1}$.

(iv) A choice $\{X_i, Y_i, Z_i, \varphi_i\}_{i \in \mathbb{N}}$ of star triples and charts is *galacticly compatible*, if whenever $\tilde{\psi}_{ij} : \Sigma_i \to \Sigma_j$ is a simplicial isomorphism, then

$$\varphi_i = \varphi_j \circ \tilde{\psi}_{ij}, \qquad X_j = \tilde{\psi}_{ij}(X_i), \qquad Y_j = \tilde{\psi}_{ij}(Y_i), \qquad Z_j = \tilde{\psi}_{ij}(Z_i). \tag{6.2}$$

(v) Let $1 > \varepsilon > 0$. Define *star regularizers* $R_i : L^k_{1,\text{loc}}(M) \to L^k_{1,\text{loc}}(M)$, $A_i : L^k_{1,\text{loc}}(M) \to L^{k-1}_{1,\text{loc}}(M)$ by

$$R_i(\omega) := \begin{cases} \varphi_i^*(R_\varepsilon((\varphi_i^{-1})^*(\omega))) & \text{, on } \Sigma_i \\ \omega & \text{, outside } \Sigma_i, \end{cases}$$

$$A_i(\omega) := \begin{cases} \varphi_i^*(A_\varepsilon((\varphi_i^{-1})^*(\omega))) & \text{, on } \Sigma_i \\ 0 & \text{, outside } \Sigma_i. \end{cases}$$

Notice that R_i, A_i depend on ε.

(vi) For any closed $F \subset M$ and any $\delta > 0$ denote

$$F_\delta := F \cup \mathcal{O}_\delta(F \cap \Sigma_i).$$

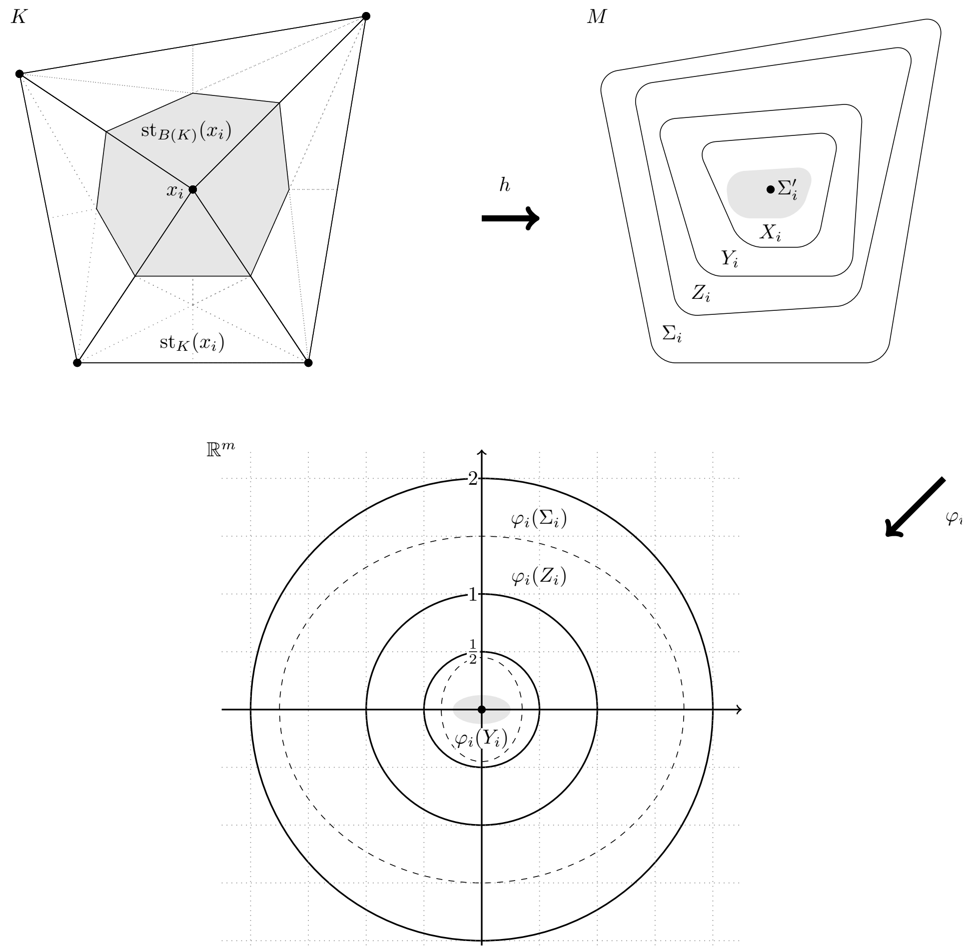


Figure 6.1: Star triple and chart

6.1.9 Lemma. There exists a galactically compatible choice of star charts and triples. In the following we will assume that such a choice has been made.

Proof. The reason for that is the star-boundedness of K: By 2.2.20 there exists only a finite number of isomorphism classes of stars. Therefore we may choose one representative for each of them together with an arbitrary star triple. Since any other star must be simplicially isomorphic to one of them, we just define the other star triples (X_j, Y_j, Z_j) as well as the star charts φ_j by the equations (6.2). □

6.1.10 Theorem (regularization operators II). For any $1 \leq p \leq \infty$ and any $i \in \mathbb{N}$ the following hold.

(i) The star regularizers restrict to bounded linear operators

$$R_i : L_p^k(M) \to L_p^k(M), \qquad A_i : L_p^k(M) \to L_p^{k-1}(M).$$

(ii) They restrict to maps

$$R_i : W_{p,\mathrm{loc}}^k(M) \to W_{p,\mathrm{loc}}^k(M), \qquad A_i : W_{p,\mathrm{loc}}^k(M) \to W_{p,\mathrm{loc}}^{k-1}(M),$$

which satisfy

$$R_i \circ d = d \circ R_i, \qquad d \circ A_i + A_i \circ d = R_i - \mathrm{id}\,.$$

Furthermore there are constants λ_p, λ'_p such that

$$\|R_i(\omega)\|_{W^k_p(\Sigma_i)} \leq \lambda_p \|\omega\|_{W^k_p(\Sigma_i)}, \qquad \|A_i(\omega)\|_{W^{k-1}_p(\Sigma_i)} \leq \lambda'_p \|\omega\|_{W^k_p(\Sigma_i)},$$

and the same equations hold for the L_p-norms (with the same constants).

(iii) There exists a constant λ''_p such that for any ω satisfying $\omega = 0$ outside Y_i, there exists a small $\varepsilon > 0$ such that

$$\|R_i(\omega)\|_{L^k_\infty(\Sigma'_i)} \leq \lambda''_p \|\omega\|_{L^k_p(\Sigma_i)}.$$

(iv) If $F \subset M$ is closed and $\omega|_{M\setminus F} = 0$, then

$$R_i(\omega)|_{M\setminus F_\delta} = 0 = A_i(\omega)|_{M\setminus F_\delta}$$

for some $\delta = \delta(\varepsilon)$ such that $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$. In addition $A_i(\omega)|_{M\setminus Z_i} = 0$, if $\varepsilon > 0$ is sufficiently small.

We may choose $\lambda_p, \lambda'_p, \lambda''_p \geq 1$.

Proof. We want to apply Theorem 5.3.16 of course. Thus we choose $U = B_2(0) \subset \mathbb{R}^m$ and recall that although this is a subset of Euclidean space, we allowed U to be equipped with an arbitrary Riemannian metric. We choose $(\varphi_i^{-1})^* g$, where g is the Riemannian metric on M and therefore declare φ_i to be an isometry. We already noticed in Theorem 2.1.30 that L_p-spaces are preserved by isometries.

(i) Assume $1 \leq p < \infty$. There exists a constant $C > 0$ such that

$$\begin{aligned}\|R_i(\omega)\|^p_{L_p(\Sigma_i)} &= \|\varphi_i^*(R_\varepsilon((\varphi_i^{-1})^*(\omega)))\|^p_{L_p(\Sigma_i)} = \|R_\varepsilon((\varphi_i^{-1})^*(\omega))\|^p_{L_p(\mathrm{st}_K(x_i))}\\ &\leq C\|(\varphi_i^{-1})^*(\omega)\|^p_{L_p(\mathrm{st}_K(x_i))} = C\|\omega\|^p_{L_p(\Sigma_i)}.\end{aligned}$$

Therefore

$$\begin{aligned}\|R_i(\omega)\|^p_{L_p(M)} &= \|R_i(\omega)\|^p_{L_p(\Sigma_i)} + \|R_i(\omega)\|^p_{L_p(M\setminus\Sigma_i)}\\ &\leq C\|\omega\|^p_{L_p(\Sigma_i)} + \|\omega\|^p_{L_p(M\setminus\Sigma_i)} \leq (C+1)\|\omega\|^p_{L_p(M)}.\end{aligned}$$

With exactly the same argumentation, we obtain the statement for A_i. The proof for $p = \infty$ is analogous.

(ii) This follows directly from (i), 5.3.16 and the fact that pull-backs commute with d.

(iii) Choose $\varepsilon > 0$ sufficiently small such that $\mathcal{O}_\varepsilon(Y_i) \subset Z_i$. By hypothesis $\omega|_{M\setminus Y_i} = 0$, thus by construction of φ_i,

$$\begin{aligned}&\mathrm{supp}(\varphi_i^{-1})^*(\omega) \subset \varphi_i(Y_i)\\ \Rightarrow\ &\mathrm{supp}\, R_i((\varphi_i^{-1})^*\omega) \subset \mathcal{O}_\varepsilon(\varphi_i(Y_i)) \subset \varphi_i(Z_i)\\ \Rightarrow\ &\mathrm{supp}\, \varphi_i^*(R_i(\varphi_i^{-1})^*)\omega \subset Z_i \subset \Sigma_i\end{aligned}$$

Therefore we can apply 5.3.16(iv) to $F := \varphi_i(\Sigma'_i) \subset B_{1/2}(0) \subset B_2(0) =: U$ and obtain a constant $\lambda''_p > 0$ such that

$$\begin{aligned}&\|R_i(\omega)\|_{L_\infty(\Sigma'_i)} = \|\varphi_i^*(R_\varepsilon((\varphi_i^{-1})^*(\omega)))\|_{L_\infty(\Sigma'_i)} = \|R_\varepsilon((\varphi_i^{-1})^*(\omega))\|_{L_\infty(F)}\\ &\leq \lambda''_p\|(\varphi_i^{-1})^*(\omega)\|_{L_p(U)} = \lambda''_p\|\omega\|_{L_p(\Sigma_i)}.\end{aligned}$$

(iv) This follows more or less directly from 5.3.16 as well. The only problem is that the ε-neighbourhoods there are taken with respect to the Euclidean metric. If φ_i were an isometry when taking the Euclidean metric in $\mathbb{R}^m$, we could take $\delta(\varepsilon) = \varepsilon$. But since any two Riemannian metrics are equivalent on a compact set, the statement follows.
To see the equation $A_i(\omega)|_{M \setminus Z_i} = 0$, notice that by construction of A_i, we obtain $A_i(\omega)|_{M \setminus \Sigma_i} = 0$ anymway. Now consider a form ω on Σ_i.

$$(\varphi_i^{-1})^* \omega \in L_{1,\mathrm{loc}}(B_2(0)), \quad \Rightarrow \operatorname{supp} A_\varepsilon((\varphi_i^{-1})^* \omega) \subset \mathcal{O}_\varepsilon(\operatorname{supp}(\varphi_i^* \omega) \cap B) \subset \varphi_i(Z_i)$$

, provided $\varepsilon > 0$ is sufficiently small. This implies $\operatorname{supp} A_i \omega \subset Z_i$. □

6.1.11 Lemma. The constants $\lambda_p, \lambda'_p, \lambda''_p$ and δ from 6.1.10 above can be chosen independently of i.

Proof. We discuss the constant λ_p in detail.

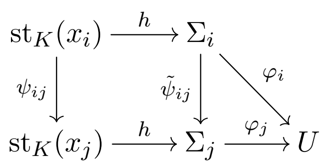
Step 1 (strategy): Let $i, j \in \mathbb{N}$ be arbitrary. Theorem 6.1.10 ensures that such a constant λ_p exists for R_i. We have to show that λ_p can be chosen for i such that it does the job for j as well. The problem here is of course the fact that in general K contains infinitely many vertices. But since K is star-bounded, the galactic cover of $|K|$ is finite (c.f. 2.2.21). Therefore we may at least assume that $i, j \in \mathbb{N}$ belong to the same galaxy (which may contain infinitely many stars).
Now fix i and take again a look at 6.1.8 and the proof of 6.1.10(i). We see that the constant λ_p was obtained by

$$\|R_i \omega\|_{L_p(\Sigma_i)} = \|\varphi_i^*(R_\varepsilon((\varphi_i^{-1})^*))\| \leq \|\varphi_i^*\| \|R_\varepsilon\|_i \|(\varphi_i^{-1})^*\| \|\omega\|_{L_p(\Sigma_i)} = \|R_\varepsilon\|_i \|\omega\|_{L_p(\Sigma_i)},$$

where $\|R_\varepsilon\|_i$ denotes the L_p-operator norm. Remember that $\varphi_i : \Sigma_i \to \bar{B}_2(0) =: U_i$ and that U_i was endowed with the metric $g_i := (\varphi_i^{-1})^* g$, which induces the L_p- norm $|\_|_i$. In other words one may choose $\lambda_p = \|R_\varepsilon\|_i$. The problem is that the operator norm $\|R_\varepsilon\|_i$ depends on the metric g_i. Thus our goal is to prove that there are constants $\tilde{C}_3, \tilde{C}_4 > 0$ such that $(|\_|_i, |\_|_j)$ are $(\tilde{C}_3, \tilde{C}_4)$-equivalent. These constants must not depend on j. If we have archieved this, we are done, because in that case the operator norms transform by

$$\|R_\varepsilon\|_j = \sup_{\omega \neq 0} \frac{|R_\varepsilon \omega|_j}{|\omega|_j} \leq \frac{\tilde{C}_4}{\tilde{C}_3} \sup_{\omega \neq 0} \frac{|R_\varepsilon \omega|_i}{|\omega|_i} = \frac{\tilde{C}_4}{\tilde{C}_3} \|R_\varepsilon\|_i. \tag{6.3}$$

Step 2 (details): In order to find these constants $\tilde{C}_3, \tilde{C}_4$, assume that galactically compatible star triples and charts are chosen as in 6.1.9 and (6.2). Take any j in the galaxy of i. By definition (see 2.2.21) there exists a simplicial isomorphism $\psi_{ij} : \mathrm{st}_K(x_i) \to \mathrm{st}_K(x_j)$. By construction the diagram



commutes. Finally, we are able to establish the result using Theorem 1.3.7: By Theorem 2.3.5(ii) the map ψ_{ij} is and isometry, hence $(1,1)$-bounded. By hypothesis the map h is

(C_1, C_2)-bounded. This implies

$$\|\tilde{\psi}_{ij*}\| = \|h_* \circ \psi_{ij*} \circ h_*^{-1}\| \leq \|h_*\|\|\psi_{ij*}\|\|h_*^{-1}\| \leq C_1C_2$$
$$\|\tilde{\psi}_{ij}^{-1}{}_*\| = \|h_* \circ \psi_{ij}^{-1}{}_* \circ h_*^{-1}\| \leq \|h_*\|\|\psi_{ij*}{}^{-1}\|\|h_*^{-1}\| \leq C_1C_2,$$

thus $\tilde{\psi}_{ij}$ is (C_1C_2, C_1C_2)-bounded. Consequently $((\tilde{\psi}_{ij}^{-1})^* g|_{\Sigma_i}, g|_{\Sigma_j})$ are $(C_1^{-1}C_2^{-1}, C_1C_2)$-equivalent. Finally

$$((\varphi_j^{-1})^*(\tilde{\psi}_{ij}^{-1})^* g|_{\Sigma_i}, (\varphi_j^{-1})^* g|_{\Sigma_j}) = ((\tilde{\psi}_{ij}^{-1} \circ \varphi_j^{-1})^* g|_{\Sigma_i}, g_j) = ((\varphi_i^{-1})^* g|_{\Sigma_i}, g_j) = (g_i, g_j)$$

are $(C_3, C_4) := (C_1^{-1}C_2^{-1}, C_1C_2)$-equivalent. Therefore the induced L_p-norms are $(\tilde{C}_3, \tilde{C}_4)$-equivalent as well by Theorem 1.3.8(iv).

The statement for the other constants follow in the same fashion: The crucial equation (6.3) can also be written down for the operator norms of A_ε to obtain λ'_p and the norms for λ''_p. The second step remains unchanged. This also implies the statement concerning δ. □

6.1.12 Theorem (regularization operators III). For any $i \in \mathbb{N}$, let R_i,A_i be as in Definition 6.1.8. Now define $\mathscr{R}_i, \mathscr{A}_i : L_{1,\mathrm{loc}}(M) \to L_{1,\mathrm{loc}}(M)$ by

$$\mathscr{R}_i := R_1 \circ \ldots \circ R_i, \qquad \mathscr{A}_i := \mathscr{R}_{i-1} \circ A_i.$$

We employ the convention that $\mathscr{R}_0 := \mathrm{id}$, $\mathscr{A}_1 := A_1$. Define $\mathscr{R} : L^k_{1,\mathrm{loc}}(M) \to L^k_{1,\mathrm{loc}}(M)$, $\mathscr{A} : L^k_{1,\mathrm{loc}}(M) \to L^{k-1}_{1,\mathrm{loc}}(M)$,

$$\mathscr{R} := \lim_{i\to\infty} \mathscr{R}_i = \lim_{i\to\infty} R_1 \circ \ldots \circ R_i, \qquad \mathscr{A} := \sum_{i=1}^{\infty} \mathscr{A}_i = \sum_{i=1}^{\infty} R_1 \circ \ldots \circ R_{i-1} \circ A_i. \tag{6.4}$$

We claim that these operators restrict to maps

$$\mathscr{R} : W^k_{1,\mathrm{loc}}(M) \to W^k_{1,\mathrm{loc}}(M), \qquad \mathscr{A} : W^k_{1,\mathrm{loc}}(M) \to W^{k-1}_{1,\mathrm{loc}}(M),$$

such that $\mathscr{R}$ is a cochain map and $\mathscr{A}$ is a co-homotopy from $\mathscr{R}$ to the identity, i.e.

$$\mathscr{R} \circ d = d \circ \mathscr{R}, \qquad d \circ \mathscr{A} + \mathscr{A} \circ d = \mathscr{R} - \mathrm{id}\,.$$

furthermore for any $\omega \in L_{1,\mathrm{loc}}(M)$, the form $\mathscr{R}(\omega)$ is smooth. If $\omega \in \Omega(M)$ is smooth, then $\mathscr{A}(\omega)$ is smooth.

Proof. On any compact subset $K \subset M$ only finitely many R_i are distinct from the identity and finitely many A_i are distinct from zero. So strictly speaking we first use (6.4) to define $\mathscr{R}(\omega)|_K$, $\mathscr{A}(\omega)|_K$ by employing the convention that we ignore the infinitely many identities respectively zeros. This defines a form on any compact subset. But this in turn globally defines forms $\mathscr{R}(\omega)$, $\mathscr{A}(\omega)$. By successively applying Theorem 6.1.10, we obtain that $\mathscr{R}, \mathscr{A}$ are operators $W_{1,\mathrm{loc}}(M) \to W_{1,\mathrm{loc}}(M)$. furthermore we obtain from 6.1.10 that

$$\forall i \in \mathbb{N} : \mathscr{R}_i \circ d = ((R_1 \circ \ldots \circ R_i) \circ d) = d \circ (R_1 \circ \ldots \circ R_i) = d \circ \mathscr{R}_i$$

and therefore $\mathscr{R} \circ d = d \circ \mathscr{R}$. Similar

$$\begin{aligned} d \circ \mathscr{A} + \mathscr{A} \circ d &= \lim_{j\to\infty} \sum_{i=1}^{j} d \circ \mathscr{R}_{i-1} \circ A_i + \mathscr{R}_{i-1} \circ A_i \circ d \\ &= \lim_{j\to\infty} \sum_{i=1}^{j} \mathscr{R}_{i-1}(d \circ A_i + A_i \circ d) = \lim_{j\to\infty} \sum_{i=1}^{j} \mathscr{R}_{i-1}(R_i - \mathrm{id}) \\ &= \lim_{j\to\infty} \sum_{i=1}^{j} \mathscr{R}_i - \mathscr{R}_{i-1} = \mathscr{R} - \mathrm{id}\,. \end{aligned}$$

By construction, the manifold M is covered by the Σ_i. Let $i \in \mathbb{N}$ be arbitrary. By Theorem 5.3.13, the form $R_\varepsilon({\varphi_i^{-1}}^*(\omega))$ is smooth. Therefore $R_i(\omega)|_{\Sigma_i}$ is smooth. Since smoothness is a local property, $\mathscr{R}(\omega)$ is smooth. The statement about $\mathscr{A}$ follows analogously. □

6.1.13 Corollary. Let M be any compact manifold. Then for any $1 \leq p \leq \infty$, the inclusion $\iota : \Omega(M) \hookrightarrow W_p(M)$ induces an isomorphism $H_{\mathrm{dR}}(M) \to H_p(M)$.

Proof.

Step 1: We do not assume M to be triangulated. Simply cover M by finitely many charts $\varphi_i : \bar{U}_i \to \bar{B}_2(0)$, where $U_i \subset M$ are open such that $\{\varphi_i^{-1}(B_1(0))\}$ is still an open cover of M. Define R_i, A_i analogously to 6.1.8(v) (just forget about the stars) and use these operators to define $\mathscr{R}_i$, $\mathscr{A}_i$ and $\mathscr{R}$, $\mathscr{A}$ as in Theorem 6.1.12 above.

Step 2: Since M is compact, any form $\omega \in \Omega^k(M)$ is also an element of $W_p^k(M)$ for any $1 \leq p \leq \infty$. Therefore the inclusion $\iota : \Omega(M) \hookrightarrow W_p(M)$ is well-defined. Choose some $1 > \varepsilon > 0$. By Theorem 6.1.12 for any $\omega \in W_p(M)$, the form $\mathscr{R}(\omega)$ is smooth. Therefore $\mathscr{R}$ can also be seen as a map $\mathscr{R} : W_p(M) \to \Omega(M)$. By Corollary 2.1.14, we may regard $\Omega(M)$ as a subcomplex of $W_p(M)$. Now the statement follows from the fact that the identity

$$\mathscr{R} - \mathrm{id} = d \circ \mathscr{A} + \mathscr{A} \circ d$$

is satisfied on the larger complex $W_p(M)$: Let us temporarily denote by $[\_]_{\mathrm{dR}}$ the de Rahm cohomology class and by $[\_]_p$ the L_p-cohomology class. Then

$$[\mathscr{R} \circ \iota]([\omega]_{\mathrm{dR}}) = [\mathscr{R}(\iota(\omega))]_{\mathrm{dR}} = [\omega]_{\mathrm{dR}}, \qquad [\iota \circ \mathscr{R}]([\omega]_p) = [\mathscr{R}(\omega)]_p = [\omega]_p.$$ □

6.2 Main results

We are finally in a position to prove our second Main Theorem.

6.2.1 Main Theorem. Let $h : |K| \to M$ be a smooth triangulation satisfying the GKS-condition (c.f. Definition 6.1.2) and let $\mathscr{R}$, $\mathscr{A}$ be as in 6.1.12. For any $1 \leq p \leq \infty$, these maps satisfy

(i) $\mathscr{R}(W_p^k(M)) \subset S_p^k(M)$ and the operators

$$\begin{aligned} \mathscr{R} &: (W_p^k(M), \|\_\|_{W_p(M)}) \to (S_p^k(M), \|\_\|_{S_p(M)}), \\ \mathscr{R} &: (W_p^k(M), \|\_\|_{W_p(M)}) \to (W_p^k(M), \|\_\|_{W_p(M)}) \end{aligned}$$

are bounded.

(ii) $\mathscr{A}(W_p^k(M)) \subset W_p^{k-1}(M)$ and the operator

$$\mathscr{A} : (W_p^k(M), \|\_\|_{W_p(M)}) \to (W_p^{k-1}(M), \|\_\|_{W_p(M)})$$

is bounded.

(iii) $\mathscr{A}(S_p^k(M)) \subset S_p^{k-1}(M)$ and the operator

$$\mathscr{A} : (S_p^k(M), \|\_\|_{S_p(M)}) \to (S_p^{k-1}(M), \|\_\|_{S_p(M)})$$

is bounded.

Proof. For simplicity assume that M is connected. In this proof we will employ the notation

$$I(\sigma) := \{i \in \mathbb{N} \mid \langle x_i \rangle \leq \sigma\}$$

for an m-simplex $\sigma \in K$.
We will always discuss the case $1 \leq p < \infty$ in detail (the case $p = \infty$ follows similarly, but easier.)

(i) Step 1: First we show that

$$\mathscr{R} : (W_p^k(M), \|\_\|_{W_p(M)}) \to (S_p^k(M), \|\_\|_{S_p(M)})$$

is bounded.

Step 1.1 (preparations): Let $\omega \in W_p^k(M)$ and $i \in \mathbb{N}$ be arbitrary. Let $x_i \in K$ be a vertex and define

$$\omega_1 := \begin{cases} \omega, & \text{on } X_i \\ 0, & \text{outside } X_i, \end{cases} \qquad \omega_2 := \omega - \omega_1.$$

Assume that $j \in \mathbb{N}$ is sufficiently large that $\{x_1, \dots, x_j\}$ contains all the vertices $x_{j_1}, \dots, x_{j_r}$, $r \leq N$, of the star $\mathrm{st}_K(x_i)$. Choose ε and δ in the definition 6.1.8(vi) (respectively in Theorem 6.1.10(iv)) of R_i and A_i such that

$$\mathcal{O}_{N\delta}(M \setminus X_i) \cap \Sigma_i' = \emptyset. \tag{6.5}$$

Step 1.2 (estimates for a single $\sigma \in K^{(m)}$): We notice that

$$\forall t \in \mathbb{N} : \forall 1 \leq \nu \leq r : t \neq j_\nu \Rightarrow R_t|_{\Sigma_i'} = \mathrm{id}_{\Sigma_i'}$$

by construction. This implies $(R_1 \circ \dots \circ R_j)(\omega_2)|_{\Sigma_i'} = 0$ by Theorem 6.1.10(iv) (applied to $F = \overline{M \setminus X_i}$) and (6.5). Consequently

$$\begin{aligned} \mathscr{R}(\omega)|_{\Sigma_i'} = \mathscr{R}(\omega_1)|_{\Sigma_i'} = \mathscr{R}_j(\omega_1)|_{\Sigma_i'} &= (R_1 \circ \dots \circ R_j)|_{\Sigma_i'}(\omega_1) \\ &= (R_{j_1} \circ \dots \circ R_{j_r})|_{\Sigma_i'}(\omega_1). \end{aligned} \tag{6.6}$$

There exists an $1 \leq s \leq r$ such that $j_s = i$. By Theorem 6.1.10(i)(iii), we obtain

$$\begin{aligned} \|R_{j_1} \circ \dots \circ R_{j_r}(\omega_1)\|_{L_\infty(\Sigma_i')} &\leq \lambda_\infty^{s-1} \|R_{j_s} \circ \dots \circ R_{j_r}(\omega_1)\|_{L_\infty(\Sigma_i')} \\ &\leq \lambda_\infty^{s-1} \lambda_p'' \|R_{j_{s+1}} \circ \dots \circ R_{j_r}(\omega_1)\|_{L_p(\Sigma_i)} \\ &\leq \lambda_\infty^{s-1} \lambda_p'' \lambda_p^{r-s} \|\omega_1\|_{L_p(\Sigma_i)} \\ &\leq \underbrace{\lambda_\infty^{N-1} \lambda_p'' \lambda_p^{N-s}}_{=:C} \|\omega_1\|_{L_p(\Sigma_i)}. \end{aligned} \tag{6.7}$$

Recall from Theorem 6.1.11 that this constant C does not depend on i. Altogether this implies

$$\|\mathscr{R}(\omega)\|_{L^k_\infty(\Sigma_i')} \overset{(6.6)}{=} \|(R_{j_1}\circ\ldots\circ R_{j_r})(\omega_1)\|_{L^k_\infty(\Sigma_i')} \overset{(6.7)}{\leq} C\|\omega\|_{L^k_p(\Sigma_i)}. \tag{6.8}$$

For any $\sigma \in K^{(m)}$, the inclusion $\sigma \subset \bigcup_{i\in I(\sigma)} \mathrm{st}_{B(K)}(x_i)$ implies

$$\|\mathscr{R}(\omega)\|^p_{L_\infty(h(\sigma))} \leq \sum_{i\in I(\sigma)} \|\mathscr{R}(\omega)\|^p_{L_\infty(\Sigma_i')} \overset{(6.8)}{\leq} C^p \sum_{i\in I(\sigma)} \|\omega\|^p_{L_p(\Sigma_i)}. \tag{6.9}$$

Step 1.3 (globalizing estimates): In order to globalize the estimate (6.9) obtained on $h(\sigma)$ to M, we have to sum over all $\sigma \in K^{(m)}$. Therefore we analyse the sum on the right hand side of (6.9) (see explainations (1),(2),(3) below):

$$\begin{aligned}\sum_{\sigma\in K^{(m)}}\sum_{i\in I(\sigma)} \|\omega\|^p_{L_p(\Sigma_i)} &\overset{(1)}{=} \sum_{i\in\mathbb{N}} \sum_{\substack{\sigma\in K^{(m)}\\ \langle x_i\rangle\leq\sigma}} \|\omega\|^p_{L_p(\Sigma_i)} \overset{(2)}{\leq} N\sum_{i\in\mathbb{N}} \|\omega\|^p_{L_p(\Sigma_i)}\\ &= N\sum_{i\in\mathbb{N}}\sum_{\sigma\in\mathrm{st}_K(x_i)^{(m)}} \|\omega\|^p_{L_p(h(\sigma))}\\ &\overset{(3)}{\leq} N(m+1)\sum_{\sigma\in K^{(m)}} \|\omega\|^p_{L_p(h(\sigma))}\\ &= N(m+1)\|\omega\|^p_{L_p(M)}.\end{aligned} \tag{6.10}$$

(1): Here we just swapped the index sets: On the left hand side we sum over all m-simplices σ and then for any such σ over all its vertices. On the right hand side we sum over all vertices x_i of K and then for any such vertex over all m-simplices attached to x_i.
(2): This is due to the fact that N is the star-bound of K.
(3): We again swapped the index sets: On the left hand side we sum over all vertices x_i and then over all the m-simplices σ in its star. This is the same as summing over all m-simplices σ and then over all its $(m+1)$ vertices.
Step 1.4 (estimates for $K^{(m)}$): This yields a constant $C' > 0$ such that

$$\begin{aligned}\sum_{\sigma\in K^{(m)}} \|\mathscr{R}(\omega)\|^p_{L_\infty(h(\sigma))} &\overset{(6.9)}{\leq} C^p\sum_{\sigma\in K^{(m)}}\sum_{i\in I(\sigma)} \|\omega\|^p_{L_p(\Sigma_i)}\\ &\overset{(6.10)}{\leq} N(m+1)C^p\|\omega\|^p_{L_p(M)}.\end{aligned} \tag{6.11}$$

We may apply equation (6.11) above to $d\omega$ instead of ω as well. Since $\mathscr{R}(d\omega) = d\mathscr{R}(d\omega)$ by 6.1.12, we obtain

$$\sum_{\sigma\in K^{(m)}} \|\mathscr{R}(\omega)\|^p_{W_\infty(h(\sigma))} \leq C'\|\omega\|^p_{W_p(M)}. \tag{6.12}$$

Step 1.5 (estimates for K): On the left hand side of equation (6.12), we would like to replace $K^{(m)}$ with K. This can be done using the following argument, which holds for any $\eta \in W_{p,\mathrm{loc}}(M)$: For any $\tau \leq \sigma$, we certainly have

$$\|\eta\|_{W_\infty(h(\tau))} \leq \|\eta\|_{W_\infty(h(\sigma))}.$$

Consequently for any $k < m$

$$\sum_{\tau\in K^{(k)}} \|\eta\|_{W_\infty(h(\tau))} \leq \sum_{\sigma\in K^{(m)}} \|\eta\|_{W_\infty(h(\sigma))},$$

since for any $\tau \in K^{(k)}$ there exists $\sigma \in K^{(m)}$ such that $\tau \leq \sigma$. (This maybe wrong for a general simplicial complex K, but does hold in this case, since K triangulates a connected smooth manifold.) By applying this argument m-times, we obtain

$$\sum_{\sigma\in K} \|\eta\|^p_{W_\infty(h(\sigma))} = \sum_{k=0}^{m} \sum_{\sigma\in K^{(k)}} \|\eta\|^p_{W_\infty(h(\sigma))} \leq m \sum_{\sigma\in K^{(m)}} \|\eta\|^p_{W_\infty(h(\sigma))} \tag{6.13}$$

Altogether we obtain

$$\begin{aligned} \|\mathscr{R}(\omega)\|^p_{S_p(M)} &= \sum_{\sigma\in K} \|\mathscr{R}(\omega)\|^p_{W_\infty(h(\sigma))} \overset{(6.13)}{\leq} m \sum_{\sigma\in K^{(m)}} \|\mathscr{R}(\omega)\|^p_{W_\infty(h(\sigma))} \\ &\overset{(6.12)}{\leq} mC'\|\omega\|^p_{W_p(M)}. \end{aligned} \tag{6.14}$$

This completes Step 1.

Step 2: By combining Step 1 with Lemma 6.1.7(ii)(iii) we obtain the other claims.

(ii) We proceed in a similar fashion and show that there exists a constant $C' > 0$ such that $\|\mathscr{A}(\omega)\|_{L_p^{k-1}(M)} \leq C'\|\omega\|_{L_p^k(M)}$.

Step 1 (preparations): Let $\omega \in W_p^k(M)$ and $i \in \mathbb{N}$. By 6.1.10(iv), we obtain

$$A_i(\omega)|_{M\setminus Z_i} = 0. \tag{6.15}$$

Assume that $\varepsilon > 0$ is chosen so small that $\delta = \delta(\varepsilon) > 0$ from 6.1.10(iv) satisfies

$$\mathcal{O}_{N\delta}(Z_i) \subset \Sigma_i. \tag{6.16}$$

Step 2 (local estimates): Let us analyze the quantity $\|\mathscr{A}_i(\omega)\|_{L_p^{k-1}(\Sigma_i)}$ for some fixed $i \in \mathbb{N}$. By (6.15), we obtain $\operatorname{supp} A_i\omega \subset Z_i$. Now consider the operator R_j for $1 \leq j \leq i-1$. By definition 6.1.8(vi), R_j is the identity on $L_p(\Sigma_i)$ unless $\langle x_j\rangle \in \mathrm{st}_K(x_i)$. We collect all those x_j in the set

$$\{x_{j_1}, \dots, x_{j_r}\} := \mathrm{cl}(\mathrm{st}_K(x_i))^{(0)} \cap \{x_1, \dots, x_{i-1}\}$$

and obtain

$$\mathscr{A}_i(\omega)|_{\Sigma_i} = (R_1 \circ \ldots \circ R_{i-1} \circ A_i)(\omega)|_{\Sigma_i} = (R_{j_1} \circ \ldots \circ R_{j_r} \circ A_i)(\omega)|_{\Sigma_i}.$$

Since K is star-bounded, we obtain $r \leq N$. Successively applying 6.1.10(ii), we obtain

$$\begin{aligned} \|\mathscr{A}_i(\omega)\|_{L_p^{k-1}(\Sigma_i)} &= \|(R_{j_1} \circ \ldots \circ R_{j_r} \circ A_i)(\omega)\|_{L_p^{k-1}(\Sigma_i)} \\ &\leq \lambda_p^r \|A_i(\omega)\|_{L_p^{k-1}(\Sigma_i)} \leq \underbrace{\lambda_p^N \lambda_p'}_{=:C} \|\omega\|_{L_p^k(\Sigma_i)}. \end{aligned} \tag{6.17}$$

STEP 3 (global estimates): We calculate

$$\begin{aligned}
\|\mathscr{A}(\omega)\|^p_{L^{k-1}_p(M)} &= \sum_{\sigma\in K^{(m)}} \|\mathscr{A}(\omega)\|^p_{L^{k-1}_p(h(\sigma))} = \sum_{\sigma\in K^{(m)}} \Big\| \sum_{i\in\mathbb{N}} \mathscr{A}_i(\omega)\Big\|^p_{L^{k-1}_p(h(\sigma))} \\
&= \sum_{\sigma\in K^{(m)}} \Big\| \sum_{i\in I(\sigma)} \mathscr{A}_i(\omega)\Big\|^p_{L^{k-1}_p(h(\sigma))} \\
&\leq 2^{(m+1)p} \sum_{\sigma\in K^{(m)}} \sum_{i\in I(\sigma)} \|\mathscr{A}_i(\omega)\|^p_{L^{k-1}_p(\Sigma_i)} \\
&\overset{(6.17)}{\leq} 2^{(m+1)p} C^p \sum_{\sigma\in K^{(m)}} \sum_{i\in I(\sigma)} \|\omega\|^p_{L^k_p(\Sigma_i)} \\
&\overset{(6.10)}{\leq} 2^{(m+1)p} C^p N(m+1) \|\omega\|^p_{L^k_p(M)}.
\end{aligned} \tag{6.18}$$

This equation (6.18) may be applied to $d\omega$ instead of ω as well. Since $d\mathscr{A}(\omega) = \omega - \mathscr{R}(\omega) - \mathscr{A}(d\omega)$ by 6.1.12, this implies the statement (together with the fact that we already derived the analogous estimate for $\mathscr{R}$ in (i)).

(iii) Again we proceed similarly but slightly differently.

STEP 1 (preparations): Let $\omega \in S^k_p(M)$. This implies $\omega \in W^k_p(M)$, thus $\mathscr{A}(\omega) \in W^{k-1}_p(M)$ by (ii). By 6.1.7(ii) both claims follow, if we can find a constant $C > 0$ such that $\|\mathscr{A}\omega\|_{S_p(M)} \leq C\|\omega\|_{S_p(M)}$.

STEP 2 (local estimates): First of all notice that for any $\sigma \in K^{(m)}$

$$\begin{aligned}
\|\mathscr{A}(\omega)\|_{L_\infty(h(\sigma))} &= \Big\| \sum_{i\in\mathbb{N}} \mathscr{A}_i(\omega)\Big\|_{L_\infty(h(\sigma))} \leq \sum_{i\in I(\sigma)} \|\mathscr{A}_i(\omega)\|_{L_\infty(h(\sigma))} \\
&\leq \sum_{i\in I(\sigma)} \|\mathscr{A}_i(\omega)\|_{L_\infty(\Sigma_i)} \overset{(6.17)}{\leq} C' \sum_{i\in I(\sigma)} \|\omega\|_{L_\infty(\Sigma_i)}.
\end{aligned} \tag{6.19}$$

The same can be done with $d\omega$ and therefore, using $d\mathscr{A}\omega = \omega - \mathscr{R}\omega - \mathscr{A}(d\omega)$ again, we obtain a constant $C'' > 0$ such that

$$\begin{aligned}
\|\mathscr{A}(\omega)\|^p_{S^{k-1}_p(M)} &= \sum_{\sigma\in K} \|\mathscr{A}(\omega)\|^p_{W^{k-1}_\infty(h(\sigma))} \overset{(6.13)}{\leq} m \sum_{\sigma\in K^{(m)}} \|\mathscr{A}(\omega)\|^p_{W^{k-1}_\infty(h(\sigma))} \\
&\overset{(6.19)}{\leq} mC'' \sum_{\sigma\in K^{(m)}} \sum_{i\in I(\sigma)} \|\omega\|^p_{W^k_\infty(\Sigma_i)} \leq mC''(m+1) \sum_{\sigma\in K^{(m)}} \|\omega\|^p_{W^k_\infty(h(\sigma))} \\
&\leq mC''(m+1)\|\omega\|^p_{S^k_p(M)}.
\end{aligned}$$

□

We are now in a position to make the strategy outlined in 6.1.1 rigorous.

6.2.2 Corollary. If $h : |K| \to M$ is GKS, then the composition

$$S_p(K) \xrightarrow{\varphi_h^{-1}} S_p(M) \overset{\iota}{\hookrightarrow} W_p(M)$$

induces a topological isomorphism $\mathscr{H}_p(K) \to H_p(M)$.

Proof. Consider diagram (6.1) again.

Step 1 (continuity): The map φ_h^{-1} is continuous by 6.1.6, φ_h is continuous by 6.1.7(i), $\mathscr{R}$ is continuous by 6.2.1(i) and ι is continuous by 6.1.7(iii). Consequently the maps $\iota \circ \varphi_h^{-1}$ and $\varphi_h \circ \mathscr{R}$ are continuous and induce continuous maps in the cohomology.

Step 2 (bijectivity): Let $\omega \in \mathscr{Z}_p(K)$. We may combine the statements of 6.1.12 and 6.2.1 to obtain the identity

$$\mathscr{R} - \mathrm{id}_{W_p(M)} = d \circ \mathscr{A} + \mathscr{A} \circ d : (W_p(M), \|\_\|_{W_p(M)}) \to (W_p(M), \|\_\|_{W_p(M)}).$$

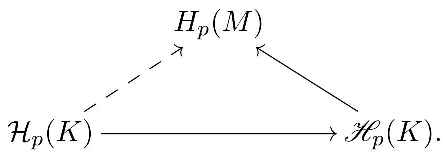
Now $S_p(M) \subset W_p(M)$ and $\mathscr{A}(S_p(M)) \subset S_p(M)$ by 6.2.1(iii). In addition φ_h is a cochain map by 2.3.13. Altogether this implies

$$\begin{aligned} [\varphi_h \circ \mathscr{R} \circ \iota \circ \varphi_h^{-1}]([\omega]) &= [\varphi_h(\mathscr{R}(\varphi_h^{-1}(\omega)))] \\ &= [\varphi_h\Big(d(\mathscr{A}(\varphi_h^{-1}(\omega))) + \mathscr{A}(d(\varphi_h^{-1}(\omega))) + \varphi_h^{-1}(\omega)\Big)] \\ &= [d(\varphi_h(\mathscr{A}(\varphi_h^{-1}(\omega)))] + [\varphi_h(\mathscr{A}(\varphi_h^{-1}(d(\omega))))] + [\varphi_h(\varphi_h^{-1}(\omega))] \\ &= [\omega]. \end{aligned}$$

Analogously let $\eta \in Z_p(M)$ and calculate

$$[\iota \circ \varphi_h^{-1} \circ \varphi_h \circ \mathscr{R}]([\eta]) = [\mathscr{R}(\eta)] = [d(\mathscr{A}(\eta)) + \mathscr{A}(d(\eta)) + \eta] = [\eta]. \qquad \square$$

6.2.3 Main Theorem. If $h : |K| \to M$ is GKS, then there exists a commutative diagram of isomorphisms



Therefore all L_p-cohomolgies of M are mutually isomorphic.

Proof. The solid arrows are given by Main Theorem 3.2.8 and Corollary 6.2.2. Define the dashed arrow to be the composition of the solid arrows. $\square$

7 Closing Remarks

In this section we will have a quick look at the history of the problem at hand, sketch some possible generalizations and give a very panoramic overview of applications and more recent developments concerning L_p-spaces and -cohomology.

7.1 Background

The isomorphism theorems presented in this thesis were published by Gol'dshtein, Kuz'minov and Shvedov in 1988. Their contribution continues the work of Dodziuk, who published his approach in 1981, [1]. Dodziuks setup differs from our approach: He restricts his attention to the case $p = 2$ and his manifold M is assumed to be complete oriented Riemannian such that the following conditions are satisfied:

(I) The manifold M has injectivity radius $d > 0$.

(C_k) The curvature tensor R of M and its covariant derivatives $\nabla^l R$, $0 \leq l \leq k$, are uniformly bounded.

He also imposes a condition, let us call it (D), on the triangulation of the manifold that is slightly different from ours, c.f. [1, (2.3)]. The most important disadvantage of Dodziuks approach is the fact that he only works with reduced L_2-cohomology. As we have pointed out in 2.1.22, this considerably changes the notion of an exact form. Nevertheless he established the following result, [1, Theorem 2.7]:

7.1.1 Theorem (Dodziuk). Let M be a complete oriented Riemannian manifold satisfying the conditions (I) and (C_k) for an integer $k > \frac{m}{2} - 1$. Let $\tau : K \to M$ be a triangulation (satisfying condition (D),[1, (2.3)]). Then integration of forms over simplices of K induces an isomorphism

$$\bar{H}_2(K) \to \bar{H}_2(M).$$

7.2 Possible generalizations

Our global assumption was that (M, g) is a smooth oriented Riemannian manifold without boundary. In case M is not orientable and has a boundary, this theory, in particular the Main Theorems 3.2.8 and 6.2.1 as well as their corollaries, are still valid with minor modifications. In general one has to replace the Riemannian volume form $d_g V \in \Omega^m(M)$ by a Riemannian volume density μ_g. In the Definition 2.1.9 of the weak differential one has to require all test forms η to be compactly supported in $\operatorname{Int} M$, the interior of M. Theorem 6.1.10 also requires a slight modification in case $x_i \in \partial M$.
One may even drop the condition that M is smooth and use the notion of a Lipschitz manifold instead (c.f. [27], [28]):

7.2.1 Definition (Lipschitz manifold). Let M be a topological m-manifold. An atlas $A = (\varphi_i : U_i \to V_i)_{i \in I}$ for M is a *Lipschitz atlas*, if for any $i, j \in I$ the transition function

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is Lipschitz continuous (with respect to any norm on $\mathbb{R}^m$).

Notions like L_p-forms, exterior derivatives and currents can be defined on Lipschitz manifolds as well. A key ingredient for this generalization is Rademachers Theorem (c.f. [31, 11A]), which states that any Lipschitz continuous map is differentiable almost everywhere. Finally, the exterior algebra is of course not the only vector bundle on which one could define the notion of L_p-sections.

7.2.2 Definition. Let (M, g) be a Lipschitz Riemannian manifold, $\pi : E \to M$ be a Lipschitz vector bundle with fibre metric h and $1 \leq p < \infty$. Then

$$L_p(E) := \{s : M \to E \mid s \text{ is a measurable section and } \|s\|^p_{L_p(E)} := \int_M \|s\|^p_h \mu_g < \infty\}$$

is the *space of L_p-sections* (similar for $p = \infty$).

However, on a general bundle there is no exterior differential and therfore this does not define a cohomology theory.

In 2.1.11 we already remarked that instead of L_p-forms one can also define $L_{p,q}$-forms for $p \neq q$. In its most general form, the definition of the corresponding cohomology has to be changed as follows:

7.2.3 Definition (weighted $L_{p,q}$-cohomology). For any $0 \leq k \leq m$ let $\sigma_k : M \to \mathbb{R}$ be some positive function. Then for any $1 \leq p, q \leq \infty$

$$W^k_{p,q}(M, \sigma_k, \sigma_{k+1}) := \{\omega \in L^k(M) \mid \|\omega\|_{W^k_{p,q}(M,\sigma_k,\sigma_{k+1})} := \|\sigma_k \omega\|_{L^k_p(M)} + \|\sigma_{k+1} d\omega\|_{L^{k+1}_q(M)}\}$$

is the space of *weighted $L_{p,q}$-forms*. In this case the *closed forms* are defined by

$$Z^k_q(M, \sigma_k) := \{\omega \in W^k_{q,q}(M, \sigma_k, \sigma_k) \mid d\omega = 0)\},$$

the *exact forms* are defined by

$$B^k_{p,q}(M, \sigma_{k-1}, \sigma_k) := \{\omega \in W^k_{q,q}(M, \sigma_k, \sigma_k) \mid \exists \eta \in W^{k-1}_{p,q}(M, \sigma_{k-1}, \sigma_k) : d\eta = \omega)\}.$$

and the *weighted $L_{p,q}$-cohomology of M* is defined by

$$H^k_{p,q}(M, \sigma_{k-1}, \sigma_k) := \frac{Z^k_q(M, \sigma_k)}{B^k_{p,q}(M, \sigma_{k-1}, \sigma_k)}.$$

Furthermore denote by

$$\bar{B}^k_{p,q}(M, \sigma_{k-1}, \sigma_k)$$

the closure of $B^k_{p,q}(M, \sigma_{k-1}, \sigma_k)$ in $L^k_q(M, \sigma_k)$. Then

$$\bar{H}^k_{p,q}(M, \sigma_{k+1}, \sigma_k) := \frac{Z^k_q(M, \sigma_k)}{\bar{B}^k_{p,q}(M, \sigma_{k-1}, \sigma_k)}$$

is the *reduced weighted $L_{p,q}$-cohomology of M* and

$$T^k_{p,q}(M, \sigma_{k-1}, \sigma_k) := \frac{\bar{B}^k_{p,q}(M, \sigma_{k-1}, \sigma_k)}{B^k_{p,q}(M, \sigma_{k-1}, \sigma_k)}$$

is the *weighted $L_{p,q}$-torsion of M*.

7.3 Applications and further theorems

As pointed out by Pansu in [18], the theory of L_p-cohomology is much less developed than L_2-cohomology. Nevertheless we want to sketch some of what is known yet. There are two types of theorems concerning L_p-cohomology one might expect: On the one hand, theorems that conclude something about L_p-cohomology (e.g. theorems that actually calculate the L_p-cohomology) and on the other hand theorems that conclude something from the L_p-cohomology.

Let us discuss the former ones first. In general it is very difficult to calculate the L_p-cohomology. However it is actually possible to give some qualitative statements at least, if the manifold is particularly simple or has additional structure. In 2.1.24, we already discussed the real half-line. There are more results on one-dimensional manifolds.

7.3.1 Intervals and warped cylinders

7.3.1 Theorem (L_p-cohomology of intervals, Kopylov, 2009, [12, Theorem 2.3]). Let $a < b$ and $\sigma_0, \sigma_1 : [a, b[\to \mathbb{R}$ be some continuous positive functions and $1 < p, q < \infty$. Define the number

$$\chi_{p,q}(a, b, \sigma_0, \sigma_1) := \begin{cases} \sup_{\tau \in [a,b]} \left\{ \left| \int_\tau^b |\sigma_0(t)|^p dt \right|^{\frac{1}{p}} \left| \int_a^\tau |\sigma_1(t)|^{-q'} dt \right|^{\frac{1}{q'}} \right\}, & p \geq q, \\ \left| \int_a^b \left(\left| \int_a^\tau |\sigma_1(t)|^{-q'} dt \right|^{p-1} \left| \int_\tau^b |\sigma_0(t)|^p dt \right| \right)^{\frac{q}{q-p}} |\sigma_1(\tau)|^{-q'} d\tau \right|^{\frac{q-p}{pq}}, & p < q. \end{cases}$$

Here p' and q' are Hölder conjugate to p respectively q. Then

(i) $H^1_{p,q}([a, b[, \{a\}, \sigma_0, \sigma_1) = 0 \iff \chi_{p,q}(a, b, \sigma_0, \sigma_1) < \infty$.

(ii) $H^1_{p,q}([a, b[, \sigma_0, \sigma_1) = 0 \iff \chi_{p,q}(a, b, \sigma_0, \sigma_1) < \infty$ or $\chi_{p,q}(b, a, \sigma_0, \sigma_1) < \infty$.

7.3.2 Theorem (Kopylov, 2009, [12, Theorem 2.5]). Let $a < b$ and $\sigma_0, \sigma_1 : [a, b[\to \mathbb{R}$ be some continuous positive functions and $1 < p, q < \infty$. Then

(i) $\bar{H}^1_{p,q}([a, b[, \sigma_0, \sigma_1) = 0$,

(ii) $\bar{H}^1_{p,q}([a, b[, \{a\}, \sigma_0, \sigma_1) = 0$ if and only if $\int_a^b \sigma_1(t)^{-q'} dt = \infty$ or $\int_a^b \sigma_0^p(t)^p dt < \infty$.

(iii) If $\bar{H}^1_{p,q}([a, b[, \{a\}, \sigma_0, \sigma_1) = 0$, then

$$\bar{\partial} : \mathbb{R} = H^0(\{a\}) \to \bar{H}^1_{p,q}([a, b[, \{a\}, \sigma_0, \sigma_1)$$

is an isomorphism of Banach spaces.

7.3.3 Definition. Let (X, g_X) and (Y, g_Y) be two Riemannian manifolds and $f : X \to \mathbb{R}_+$ be continuous. The Riemannian manifold $(X \times_f Y, g) := (X \times Y, g_X + f^2 g_Y)$ is the *warped product of* X *and* Y. In case $X = [a, b[$, we say $C^f_{a,b} Y := X \times_f Y$ is a *warped cylinder*. We set $Y_a := \{a\} \times Y$.

7.3.4 Theorem (warped cylinders, Kopylov, 2009, [12, Theorem 2.5]). Let Y be an orientable Riemannian n-manifold, $-\infty < a < b \leq \infty$, $f : [a, b[\to \mathbb{R}_+$ be continuous, $1 < p, q < \infty$. Assume there exists $\varphi \in Z^{j-1}_p(Y) \cap Z^{j-1}_q(Y)$ such that $\int_Y \varphi \wedge \gamma \neq 0$ for some $\gamma \in \Omega^{n-j+1}_c(Y)$, $d\gamma = 0$. The following hold

(i) if $\chi_{p,q}(a, b, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) = \infty$, then

$$H^j_{p,q}(C^f_{a,b}, Y_a) \neq 0;$$

(ii) if $\chi_{p,q}(a,b,f^{\frac{n}{p}-j+1},f^{\frac{n}{q}-j+1})=\infty$ and $\chi_{p,q}(b,a,f^{\frac{n}{p}-j+1},f^{\frac{n}{q}-j+1})=\infty$, then $T^j_{p,q}(C^f_{a,b})\neq 0$ and hence

$$\dim H^j_{p,q}(C^f_{a,b}Y)=\infty.$$

7.3.2 Surfaces of revolution

In case M is a surface of revolution, the following is known.

7.3.5 Theorem (normal solvability, Kopylov, 2007, [12, Theorem 1]). Let $M\subset\mathbb{R}^{n+1}$, $n\geq 1$, be a surface of revolution defined by

$$M:=M_f:=\Big\{x\in\mathbb{R}^{n+1}\mid f(x_1)^2=\sum_{j=2}^{n+1}x_j^2, 0\leq x_1<b\Big\}\subset\mathbb{R}^{n+2},$$

where $b>0$ and f is some positive smooth function. Let Γ be a closed subspace satisfying $\Omega^k_c(M)\subset\Gamma\subset W^k_{p,q}(M)$. Let f be unbounded and assume $\frac{1}{q}-\frac{1}{p}<\frac{1}{m+1}$, $0\leq k\leq m$. Then the operator $d_\Gamma: L^k_p(M)\to L^{k+1}_q(M)$ is not normally solvable[5].

The following is a very nice example of how geometric properties of a manifold can be characterized using L_p-cohomology.

7.3.6 Theorem (torsion and volume of a surface of revolution, Kopylov, 2009, [12, Theorem 3.3, 3.4]). Suppose that $1\leq p,q<\infty$, $\frac{1}{q}-\frac{1}{p}<\frac{1}{n+1}$ and let $M:=M_f$ be a surface of revolution as above.

(i) If f is unbounded, then $T^j_{p,q}(M)\neq 0$ for any $1\leq j\leq n+1$.

(ii) If $T^j_{p,q}(M)=0$ for any $1\leq j\leq n+1$, then

$$\lim_{x\to\infty}f(x)=0 \qquad \text{and} \qquad \mathrm{vol}(M)<\infty.$$

In particular f is bounded.

7.3.3 Lie groups

There are also some results for certain Lie groups.

7.3.7 Theorem (Gol'dshtein, Troyanov, 1997, [8, Theorem 1]). Let SOL be the Lie group of matrices of the form

$$\begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix}\in\mathbb{R}^{3\times 3}$$

endowed with the bi-invariant Riemannian metric $ds^2=e^{-2z}dx^2+e^{2z}dy^2+dz^2$. For every $1<p,q<\infty$

$$\dim H^2_{p,q}(SOL)=\infty.$$

[5] Recall that an unbounded linear operator $T:X\to Y$ between Banach spaces that is defined on a dense subset $A\subset X$ is *normally solvable*, if $\overline{T(A)}=T(A)$.

7.3.8 Theorem (Kopylov 2007, [12, Theorem 1]). Let $\mathbb{H}_n$ be the Heisenberg group, i.e. the Lie group of matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & E_n & z \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{(n+2)\times(n+2)},$$

where $E_n \in \mathbb{R}^{n\times n}$ is the unit matrix, $x \in \mathbb{R}^{1\times n}$, $z \in \mathbb{R}^{n\times 1}$, $y \in \mathbb{R}$. Then

$$p < q \Longrightarrow \dim H^1_{p,q}(\mathbb{H}_n) = \infty.$$

Another survey of geometric results concerning L_p-cohomology can be found in [19].

7.3.4 Hodge decomposition

There are well-established analytic results about the Laplacian on a Riemannian manifold involving L_2-spaces. Let us fix the following notation.

7.3.9 Definition (Hodge Laplacian). Let (M, g) be a complete Riemannian manifold with exterior differential d. Denote by d^* the formal L^2-adjoint of d. The operator

$$\Delta := d \circ d^* + d^* \circ d$$

is called *Hodge Laplacian*. In case L_2-norms are taken with respect to some weight function $\sigma = e^{-\phi}$, we denote the corresponding operator by Δ_ϕ. Denote by

$$H_{k,p}(M, \sigma) := \ker \Delta \cap L^k_p(M, \sigma).$$

In case $\sigma = 1$, we drop σ in our notation.

In case $p = 2$ the *Hodge decomposition* is a well-known theorem.

7.3.10 Theorem (Hodge decomposition, Kodaira, 1949, [10]). The L_2-space over M admits the following orthogonal direct sum decomposition

$$L^k_2(M) = H_{k,2}(M) \oplus \overline{d\Omega^k_c(M)} \oplus \overline{d^*\Omega^k_c(M)}.$$

For compact orientable manifolds, the Hodge decomposition can also be found in classical textbooks on global analysis, e.g. [29, Theorem 6.8]. It seems quite natural to ask if such a decomposition is still possible for $p \neq 2$. For compact manifolds, the following answer known as *strong L_p-Hodge direct sum decomposition* has been given.

7.3.11 Theorem (Hodge-decomposistion, compact case, Scott, 1995, [24, Proposition 6.5]). Let M be a compact, orientable smooth Riemannian manifold without boundary and $1 < p < \infty$. For any $1 \leq k \leq \dim M$

$$L^k_p(M) = H_{k,p}(M) \oplus d\mathcal{W}^{k-1}_p(M) \oplus d^*\mathcal{W}^{k+1}_p(M).$$

Here the Sobolev spaces are defined slightly different than in 2.1.10, namely

$$\mathcal{W}^k_p(M) := \{\omega \in L^k_p(M) \mid \|\omega\|_{\mathcal{W}^k_p(M)} := \|\omega\|_{L_p(M)} + \|d\omega\|_{L^{k+1}_p(M)} + \|d^*\omega\|_{L^{k-1}_p(M)} < \infty\}. \tag{7.1}$$

In case M is non-compact, matters are much more complicated. By a rather recent result of Li, the Hodge decomposition holds under certain restriction on the so called *Riesz potentials*. Their definition is rather involved and requires the theory of singular integral operators, c.f. [14, 3].

7.3.12 Theorem (Hodge decomposition, non-compact case, Li, 2009, [14, Theorem 2.1]). Let (M,g) be a complete Riemannian manifold, $\phi \in \mathcal{C}^2(M)$, $\sigma := e^{-\phi}$, $p > 1$, $q := \frac{p}{p-1}$. Suppose that the *Riesz transforms* $d(\Delta_\phi^k)^{-\frac{1}{2}}$, $d^*(\Delta_\phi^k)^{-\frac{1}{2}}$ are bounded in L_p and L_q and the *Riesz potential* $(\Delta_\phi^k)^{-\frac{1}{2}}$ is bounded in L_p. Then the Strong L^p-Hodge direct sum decomposition holds:

$$L_p^k(M,\sigma) = H_{k,p}(M,\sigma) \oplus d\mathcal{W}_p^{k-1}(M,\sigma) \oplus d_\phi^* \mathcal{W}_p^{k+1}(M,\sigma)$$

The definition of $\mathcal{W}_p^k(M,\sigma)$ is analogous to $\mathcal{W}_p^k(M)$ from (7.1).

7.3.5 Poincaré duality

Another natural problem closely related to Hodge decomposition is the Poincaré duality.

7.3.13 Theorem (analytic Poincaré duality). Let M be a smooth compact oriented manifold of dimension m. The bilinear pairing $\beta : H_{\mathrm{dR}}^k(M) \times H_{\mathrm{dR}}^{m-k}(M) :\to \mathbb{R}$,

$$([\omega],[\eta]) \mapsto \int_M \omega \wedge \eta$$

is well-defined and regular. The map $\Psi : H_{\mathrm{dR}}^k(M) \to (H_{\mathrm{dR}}^{m-k})^*$, $[\omega] \mapsto ([\eta] \mapsto \beta([\omega],[\eta])$, is an isomorphism.

This instance of the theorem can be found in [29, 6.13]. It is proven using the Hodge decomposition for L_2. This version of course has the disadvantage that M is required to be compact. A slightly less popular version is the following (taken from [16, Exc. 16-6])

7.3.14 Theorem. Let M be a smooth oriented m-manifold. Then the map $PD : \Omega^k(M) \to \Omega_c^{m-k}(M)^*$, $\omega \mapsto (\eta \mapsto \int_M \omega \wedge \eta)$, induces an isomorphism $H_{\mathrm{dR}}^k(M) \to H_c^{m-k}(M)^*$.

This version is proven using a similiar bootsrap argument as in the "elementary" proof of the de Rham theorem given in [16, 16.12] and relies on a Mayer-Vietoris sequence.

Again one might ask what happens if M is non-compact, but a complete Riemannian manifold, and p is arbitrary. The following answer (translated from the French article) was given by Pansu.

7.3.15 Theorem (Poincaré duality, Pansu, 2008, [18, Lemma 13]). Let M be a complete oriented Riemannian manifold of dimension m. Let $p > 1$ and let q be Hölder conjugate to p. Let $\omega \in L_p^k(M)$. Then the following holds:

(i) $0 \neq [\omega] \in \bar{H}_p^k(M)$ if and only if there exists $\eta \in L_q^{m-k}(M)$ such that

$$\int_M \omega \wedge \eta \neq 0.$$

(ii) $0 \neq [\omega] \in H_p^k(M)$ if and only if there exists a sequence $\eta_j \in L_q^{m-k}(M)$ such that

$$\int_M \omega \wedge \eta_j \geq 1 \qquad \text{and} \qquad \|d\eta_j\|_{L_q(M)} \to 0.$$

(iii) As a consequence, we obtain

$$\bar{H}_p^k(M) = 0 \Longleftrightarrow \bar{H}_q^{m-k}(M) = 0, \qquad T_p^k(M) = 0 \Longleftrightarrow T_q^{m-k}(M) = 0.$$

7.4 L_p-Duality

In functional analysis there is yet another famous duality theorem, see for instance [22, 6.16].

7.4.1 Theorem. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, $1 \leq p < \infty$ and let q be Hölder conjugate to p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. The *Hölder pairing* pairing $\beta : L_q(\mu) \times L_p(\mu) \to \mathbb{R}$, $(f, g) \mapsto \int_X fg d\mu$, is regular and the map $\Psi : L_q(\mu) \to L_p(\mu)^*$, $f \mapsto (g \mapsto \beta(f, g))$ is an isometric isomorphism.

We would like to adapt this theorem to $L_p(M)$. To that end we require some preparation.

7.4.2 Definition (Hölder pairing). Assume $1 \leq p, q \leq \infty$ are Hölder conjugate. The pairing $\beta : L_q^{m-k}(M) \times L_q^k(M) \to \mathbb{R}$,

$$(\omega, \eta) \mapsto \int_M \omega \wedge \eta$$

is called the *Hölder pairing of M*.

7.4.3 Lemma. Any $\omega \in L^m(M)$ satisfies the the standard estimate

$$\left| \int_M \omega \right| \leq \int_M |\omega|.$$

Therefore β is well-defined.

Proof. The Riemannian volume form $d_g V$ may be expressed locally by

$$d_g V = E^1 \wedge \ldots E^m,$$

where $E_1, \ldots, E_m$ is a local orthonormal frame (c.f. [16, (13.6)]). This implies

$$|d_g V| = |E^1 \wedge \ldots \wedge E^m| \overset{1.2.4}{=} 1. \tag{7.2}$$

Now the integration of functions on M may be expressed measure theoretically: Define the measure space $(X, \mathcal{A}, \mu_g)$ by $X := M$, $\mathcal{A} := \sigma(\tau_M)$, i.e. the σ-algebra generated by the topology τ_M of M and for any $A \in \mathcal{A}$ define $\mu_g(A) := \int_M \chi_A d_g V$, where χ_A is the characteristic function of A. Now the inequality

$$\left| \int_X f d\mu \right| \leq \int_X |f| d\mu \tag{7.3}$$

holds for any measure space $(X, \mathcal{A}, \mu)$. Any $\omega \in L^m(M)$ may be written as $\omega = f d_g V$ for some function f and therefore, we obtain

$$\left|\int_M \omega\right| = \left|\int_M f d_g V\right| = \left|\int_X f d\mu_g\right| \overset{(7.3)}{\leq} \int_X |f| d\mu_g = \int_M |f| d_g V$$
$$\overset{(7.2)}{=} \int_M |f||d_g V| d_g V = \int_M |f d_g V| d_g V = \int_M |\omega|.$$

This implies any $\omega \in L_q^{m-k}(M)$, $\eta \in L_p^k(M)$

$$|\beta(\omega, \eta)| \leq \int_M |\omega \wedge \eta| \overset{2.1.18}{<} \infty.$$

□

7.4.4 Definition (finitely framed). A Riemannian manifold (M, g) is *finitely framed*, if there exists a subset $M' \subset M$ of measure zero and a finite family of measurable subsets $\{U_\nu\}_{1 \leq \nu \leq N}$ of $M \setminus M'$ such that

(i) $\bigcup_{\nu=1}^N U_\nu = M \setminus M'$.

(ii) For any $\mu \neq \nu$ the set $U_\nu \cap U_\mu$ has measure zero.

(iii) For any $1 \leq \nu \leq N$ there exists an orthonormal frame on U_ν.

7.4.5 Lemma (examples of finitely framed manifolds). If (M, g) is either

(i) complete or

(ii) GKS or

(iii) an open subset of $\mathbb{R}^m$,

(iv) compact,

then M is finitely framed.

Proof.

(i) If M is complete, introduce the following notation: Let $c : \mathbb{R} \to M$ be a geodesic, $p := c(0)$, $v := \dot{c}(0)$

$$\begin{aligned} t_0 := t_0(c) := t_0(v) &:= \sup\{t > 0 \mid d(c(t), p) = t\} \in]0, \infty] \\ C_T(p) &:= \{t_0(v)v \mid v \in T_pM, \|v\| = 1, t_0(v) < \infty\} \subset T_pM \\ D_T(p) &:= \{tv \mid v \in T_pM, \|v\| = 1, 0 < t < t_0(v)\} \subset T_pM \\ C(p) &:= \exp_p(C_T(p)) \subset M. \end{aligned}$$

The set $C(p)$ is usually called the *cut locus*. By a standard theorem from differential geometry, $M' := C(p)$ is a set of measure zero and $\exp_p : D_T(p) \to M \setminus M'$ is a diffeomorphism. Therefore normal coordinates centered at p yield an orthonormal frame on $M \setminus M'$.

(ii) Since any GKS manifold is complete by Lemma 6.1.4, this follows from (i).

(iii) This is clear.

(iv) Follows from (i). □

For the following theorem we are not aware of any reference in the literature. Its proof shall conclude this thesis.

7.4.6 Theorem (Hölder duality for $L_p(M)$). For any Hölder conjugate $1 \leq p, q < \infty$ and $0 \leq k \leq m$, the Hölder pairing $\beta : L_q^{m-k}(M) \times L_p^k(M) \to \mathbb{R}$ is regular and the map $\Psi : L_q^{m-k}(M) \to L_p^k(M)^*$, $\omega \mapsto (\eta \mapsto \beta(\omega, \eta))$, is an isomorphism.

Proof. We alread established in Lemma 7.4.3 that this pairing is well-defined.

Step 1 (pairing is regular): Since $\Omega_c^{m-k}(M) \subset L_p^k(M)$ the Fundamental Lemma 2.1.13 implies that for any $\omega \in L_q^{m-k}(M)$

$$\forall \eta \in L_p^k(M) : \beta(\omega, \eta) = 0 \Longrightarrow \omega = 0.$$

This also implies that Ψ is injective. It remains to show that Ψ is surjective.

Step 2 (case $k = 0$): Again consider the measure space $(X, A, \mu_g) := (M, \tau_M, \int_M d_g V)$. Let $l \in L_p^0(M)^*$ be arbitrary. We are looking for an $\omega = f d_g V \in L_q^m(M)$ such that for any $\eta \in L_p^0(M)$

$$l(\eta) = \beta(\omega, \eta) = \int_M \omega \wedge \eta = \int_M \eta f d_g V = \int_X f \eta d\mu_g. \tag{7.4}$$

Such an f and hence $\omega := f d_g V$ is provided by Theorem 7.4.1.

Step 3 (general k): Let $l \in L_p^k(M)^*$.

Step 3.1 (local version): In a first step replace M by an measurable subset $U \subset M$, which is sufficiently small such that there exists a local orthonormal frame $E_1, \ldots, E_m \in \mathcal{T}(U)$. For any $I \in \mathcal{I}_k$ define a functional $l_I : L_p^0(U) \to \mathbb{R}$, $g \mapsto l(gE^I)$. By the previous step, there exist $\omega^I = f_I d_g V \in L_q^m(U)$ such that

$$\forall \eta \in L_p^0(U) : l_I(\eta) = \beta(\omega^I, \eta). \tag{7.5}$$

Define

$$\omega := *\Big(\sum_{I \in \mathcal{I}_k} f_I E^I \Big) = \sum_{I \in \mathcal{I}_k} f_I \delta^I_{I^c} E^{I^c}. \tag{7.6}$$

Here δ is used as in 4.1.17. Since the $\{E^{I^c}\}$ are an ONB of $\Omega^{m-k}(U)$, this implies

$$\|\delta^I_{I^c} f_I E^{I^c}\|^q_{L_q^k(U)} = \int_U |f_I E^{I^c}|^q d_g V = \int_U |f_I|^q d_g V = \int_U |f_I|^q |d_g V| d_g V = \|\omega^I\|^q_{L_q^m(U)}$$

and therefore in particular $\omega \in L_q^{m-k}(U)$. Now let $\eta = \sum_{J \in \mathcal{I}_k} \eta_J E^J \in L_p^k(U)$ be arbitrary. We obtain

$$\begin{aligned}
\beta(\omega, \eta) &\overset{(7.6)}{=} \sum_{J \in \mathcal{I}_k} \sum_{I \in \mathcal{I}_k} \beta(\delta^I_{I^c} f_I E^{I^c}, \eta_J E^J) = \sum_{J \in \mathcal{I}_k} \sum_{I \in \mathcal{I}_k} \int_U \delta^I_{I^c} f_I \eta_J E^{I^c} \wedge E^J \\
&= \sum_{J \in \mathcal{I}_k} \int_U \delta^J_{J^c} f_J \eta_J E^{J^c} \wedge E^J = \sum_{J \in \mathcal{I}_k} \int_U f_J \eta_J d_g V \\
&= \sum_{J \in \mathcal{I}_k} \beta(\omega^J, \eta_J) \overset{(7.5)}{=} \sum_{J \in \mathcal{I}_k} l_J(\eta_J) = \sum_{J \in \mathcal{I}_k} l(\eta_J E^J) = l(\eta).
\end{aligned}$$

Step 3.2 (global version): Now let $l \in L_p^k(M)^*$. Let $\{U_\nu\}_{1 \leq \nu \leq N}$ be a disjoint open cover of $M \setminus N$, where N is a set of measure zero and therefore negligible. For any $1 \leq \nu \leq N$ define the functional $l_\nu : L_p^k(U_\nu) \to \mathbb{R}$, $\xi \mapsto l(\chi_\nu \xi)$, where $\chi_\nu := \chi_{U_\nu}$ is the characteristic function of U_ν (we think of $\chi_\nu \xi \in L_p^k(M)$ as extended by zero outside U_ν). By the local version, there exist $\omega_\nu \in L_q^{m-k}(U_\nu)$ such that

$$\forall 1 \leq \nu \leq N : \forall \xi \in L_p^k(U_\nu) : l_\nu(\xi) = \beta(\omega_\nu, \xi). \tag{7.7}$$

Extend ω_ν by zero outside U_i and define

$$\omega := \sum_{i=1}^{N} \omega_\nu.$$

Since

$$\|\omega\|^q_{L_q^{m-k}(M)} = \sum_{i=1}^{N} \|\omega_\nu\|_{L_q^{m-k}(U_\nu)},$$

we obtain $\omega \in L_q^{m-k}(M)$. This part of the proof requires the cover to be finite. We notice that

$$\forall \mu \neq \nu : \beta(\omega_\mu, \chi_\nu \eta_\nu) = 0 \tag{7.8}$$

and conclude

$$\begin{aligned} l(\eta) &= \sum_{\nu=1}^{N} l(\chi_\nu \eta_\nu) = \sum_{\nu=1}^{N} l_\nu(\chi_\nu \eta_\nu) \overset{(7.7)}{=} \sum_{\nu=1}^{N} \beta(\omega_\nu, \chi_\nu \eta_\nu) \\ &\overset{(7.8)}{=} \sum_{\nu,\mu=1}^{N} \beta(\omega_\mu, \chi_\nu \eta_\nu) = \beta(\omega, \eta). \end{aligned}$$

□

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Index

List of Symbols

$\langle \omega \rangle$	homogenous current generated by ω, page 66
$\langle x_0, \dots, x_k \rangle$	simplex spanned by $x_0, \dots, x_k \in \mathbb{R}^n$, page 33
$[x_{\pi(0)}, \dots, x_{\pi(k)}]$	oriented simplex, page 34
A_ε	regularization homotopy, page 83
A'_ε	homotopy on currents, page 86
Alt	the alternator, page 13
$\mathrm{Ban}_\mathbb{R}$	category of real banach spaces and bounded linear operators
B	the Euclidean unit ball in $\mathbb{R}^m$, page 71
$\mathcal{B}_p(K)$	exact simplicial L_p-cochains, page 43
$\mathscr{B}_p(K)$	exact S-forms of type L_p, page 49
$B_p(M)$	exact L_p-forms, page 27
B_I^r	short form for a term in the Whitney transformation, page 53
$\mathrm{Ch}(\mathrm{Ban}_\mathbb{R})$	category of chain complexes over $\mathrm{Ban}_\mathbb{R}$
$C^k(K)$	simplicial cochain groups, page 40
$C_p^k(K)$	simplicial L_p-cochains, page 42
$C_k(K)$	simplicial chain groups, page 40
$C_k(K)$	simplicial homology, page 40
(C_1, C_2)-bounded	bounded diffeomorphism, page 20
(C_1, C_2)-equivalent	equivalence of norms, page 19
$\mathrm{Ch}(\mathrm{Vect}_\mathbb{R})$	category of chain complexes of $\mathbb{R}$ vector spaces
cl	(simplicial) closure, page 35
δ_J^I	Kronecker delta, page 68
Δ_k	standard k-simplex, page 35
$\mathscr{D}$	smooth compactly supported test forms, page 63
d	exterior (weak) differential, page 26
$\mathscr{D}'$	currents, page 65
$\partial^i \sigma$	i-th boundary face of the simplex σ, page 34
$\partial \varphi_i$	canonical vector field generated by φ
$\tilde{\Delta}_\infty$	"infinite dimensional standard simplex", page 45
$\tilde{\Delta}_k$	standard k simplex in $\mathbb{R}^{k+1}$, page 44

$d\varphi_i$	canonical covector field generated by φ
dV_g	volume form
$\mathscr{E}$	smooth test forms, page 63
$\mathscr{E}'$	compactly supported currents, page 65
$\mathrm{End}(M)$	fields of endomorphisms on M
$\flat$	flat operator, page 10
f^*	pullback of f, page 15
F_δ	a δ-neighborhood of a star, page 100
$\Gamma(E)$	(usually smooth) sections in the bundle E
h	homotopy operator, page 79
$H^k(K)$	simplicial cohomology, page 40
$\mathcal{H}_p(K)$	simplicial L_p-cohomology, page 43
$\mathscr{H}_p(K)$	L_p-cohomology of S-forms, page 49
$H_p(M)$	L_p-cohomology of M, page 27
$\bar{H}_p(M)$	reduced L_p-cohomology of M, page 27
$[\sigma : \tau]$	incidence coefficient, page 34
I	integration map, page 51
$\mathcal{I}_k$	set of all increasing multi-indices of length k, page 83
ι_X	interior multiplication, page 72
$j_{\tau,\sigma}$	inclusion map, page 47
$\mathrm{Jac}\, F$	the Jacobian of F, page 18
K	a simplicial complex, page 35
k	degree in a chain/cochain-complex
$K^{(k)}$	the k-simplices of K, page 35
K^k	the k-skeleton of K, page 35
$L^k_{p,\mathrm{loc}}$	locally p-integrable forms of degree k, page 23
L^k_p	p-integrable forms of degree k, page 23
lk	link, page 35
M	usually a smooth Riemannian manifold without boundary
$\mathcal{O}_\varepsilon$	ε - neighbourhood, page 71
p	power of integrablility, $1 \leq p \leq \infty$

φ	usually a chart for M
φ_ε	standard mollifier, page 71
R_ε	regularization operator, page 83
R'_ε	regularization operator on currents, page 86
RM	category of Riemannian Manifolds
$\sharp$	sharp operator, page 10
s	localized translation group, page 74
$S^k(K)$	S-forms on K of degree k, page 47
$S^k_p(K)$	simplicial L_p-chains of degree k, page 48
$\mathfrak{S}_k$	symmetric group in the k elements $\{0,\dots,k-1\}$
$S(K)$	the S-forms on K, page 47
$S_p(M)$	image of φ_h^{-1}, page 99
st	star, page 35
supp	support
S_y	localized translation homotopy, page 83
$T^k_l M$	bundle of (k,l)-tensors on M, page 9
$T^k_l E$	induced basis on the tensor algebra, page 10
$T \wedge \alpha$	wedge product between a current and a form, page 68
τ_y	translation with y, page 74
v_k	Euclidean volume of Δ_k, page 45
vol	volume, page 16
V	usually an m-dimensional inner product space, page 9
$\Omega(M)$	exterior algebra of M, page 9
$\wedge$	wedge product, page 13
w	Whitney transformation, page 52
$W^k_{p,\text{loc}}$	Sobolev space of locally p-integrable weakly differentiable forms of degree k, page 24
W^k_p	Sobolev space of p-integrable weakly differentiable forms of degree k, page 25
$\Omega^*_c(M)$	smooth differential forms having compact support
$[x_I]$	oriented simplex $[x_{i_0},\dots,x_{i_k}]$, page 52
$\mathcal{Z}_p(K)$	closed simplicial L_p-cochains, page 43

$\mathscr{Z}_p(K)$	closed S-forms of type L_p, page 49
$Z_p(M)$	closed L_p-forms, page 27