The Hodge Decomposition

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This article discusses the Hodge decomposition for a real oriented compact Riemannian manifold M roughly following the argumentation in [2, 6]. The Hodge decomposition states, that the differential forms on M can be decomposed orthogonally into image and kernel of the Laplace operator:

$$\Omega^k(M) = \operatorname{im}\Delta_k \oplus \ker\Delta_k$$

The rather extensive preparations necessary to be able to write down this statement are treated in chapters 1 and 2 in full detail. Unfortunately the proof is even more extensive and requires a statement from regularity theory, explained in chapter 3, and a compactness result discussed in chatper 4. The proofs are unfortunately beyond this article. Finally chapter 5 puts all the pieces together, states and proves the Hodge decomposition and illustrates some celebrated corollaries and applications.

Contents

We employ the following global notation conventions:

- V and W are real vector spaces of dimensions n and m.
- $\mathcal{B}$ is the set of all ordered bases of V.
- If $B = (b_1, \dots, b_n) \in \mathcal{B}$ we denote by $(b^1, \dots, b^n)$ the corresponding dual basis of V^*.
- If $I = (i_1, \dots, i_k)$ is a multi-index and $B = (b_1, \dots, b_n) \in \mathcal{B}$, we write

$$b^I := b^{i_1} \wedge \ldots \wedge b^{i_k}$$

- We set
$$\Lambda^k(V) := \Big\{ \omega : V^k \to \mathbb{R} \mid \omega \text{ is } k\text{-fold alternating multilinear } \Big\}$$
- We obtain a basis of $\Lambda^k(V)$ via
$$\Lambda^k B := \{b^{i_1} \wedge \ldots \wedge b^{i_k} | 1 \le i_1 < \ldots, < i_k \le n\}$$
- Scalar products are denoted by $\langle \_, \_ \rangle$.
- (M, g) is an ortiented compact smooth Riemannian manifold of dimension n.
- $\Omega^k M$ is the smooth bundle given pointwise by $(\Omega^k M)_p = \Lambda^k(T_p M)$.
- $\Omega^k(M)$ is the space of smooth sections in $\Omega^k M$ and $\Omega^*(M) := \bigoplus_{k=0}^n \Omega^k(M)$.
- $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is the exterior differential.
- $H^k_{dR}(M)$ is the k-th de Rham cohomology group.
- $\langle \_, \_ \rangle_{\Omega M}$ is the fibre metric in $\Omega^*(M)$ obtained by extending g pointwise via 1.2.2.
- $\mathcal{H}^k := \mathcal{H}^k(M)$ will be the harmonic k-forms on M (see 2.1.7).

1 Basics from Linear Algebra

This section introduces basic concepts from linear algebra necessary to establish certain important constructions on manifolds.

1.0.1 Theorem (Canonical Isomorphism)**.** Let V, W be real vector spaces with $\dim V = \dim W < \infty$ and let $\beta : V \times W \to \mathbb{R}$ be a regular bilinear form. Then the induced mapping $\Psi : V \to W^*$
$$v \mapsto \beta(v, \_)$$
where $\beta(v, \_) : W \to \mathbb{R}$, $w \mapsto \beta(v, w)$, is an isomorphism, the *canonical isomorphism* between V and W^*.

Proof. Because V and W have equal and finite dimension
$$V \cong V^* \cong W^* \cong W$$
So it suffices to show, that Ψ is linear and injective. The linearity follows from the bilinearity of $\langle \_, \_ \rangle$ and the injectivity follows from
$$0 = \Psi(v) \Rightarrow \beta(, \_) = 0 \Rightarrow \forall w \in W : \beta(v, w) = 0 \Rightarrow v = 0$$
since β is regular by hypothesis. So $\ker \Psi = \{0\}$ and thus Ψ is an isomorphism. $\square$

1.1 Orientations

In this section we will introduce the concept of an oriented vector space V, which is given by a very simple equivalence relation on $\mathcal{B}$, the set of all bases of V.

1.1.1 Definition (Consistently Oriented)**.** Two bases $B, C \in \mathcal{B}$ are *consistently oriented*, if the transition automorphism $T := T(B, C)$ mapping B to C satisfies
$$\det T > 0$$
We denote this by $B \sim C$.

Of course one may check this condition by calculating the matrix of T and calculate its determinant.

1.1.2 Lemma. The property of being consistently oriented defines an equivalence relation on the set $\mathcal{B}$ of all ordered bases of V. There are precisely two equivalence classes.

Proof. Let $B, C, D \in \mathcal{B}$.

(i) Reflexivity: $B \sim B$, because the transition automorphism is the identity I which clearly has determinant $\det I = 1 > 0$.

(ii) Symmetry: Let $B \sim C$ so there is a transition automorphism $T(B,C)$ sufficing $\det T(B,C) > 0$. From linear algebra we know, that

$$\det T(C,B) = \det((T(B,C))^{-1}) = (\det T(B,C))^{-1} > 0$$

and thus $C \sim B$.

(iii) Transitivity: Let $B \sim C$ and $C \sim D$, so there are transition automorphisms $T(B,C)$ and $T(C,D)$. Linear algebra and determinant multiplication theorem yield:

$$\det(T(B,D)) = \det(T(B,C)T(C,D)) = \det(T(B,C)) \cdot \det(T(C,D)) > 0$$

Thus $B \sim D$.

Since a transition automorphism T mapping one base onto another is always an isomorphism $\det(T) \neq 0$. So any such determinant is either strictly positive or strictly negative and thus $\mathcal{B}/\sim$ consists of exactly these two equivalence classes. □

1.1.3 Definition (Orientation)**.** For any $B \in \mathcal{B}$, the equivalence class $[B]$ is an *orientation* on V. A basis C of V is *positively oriented* or simply *positive*, if $C \in [B]$ and *negative* otherwise. The tupel $(V, [B])$ is an *oriented vector space*.

1.1.4 Theorem (Characterization of Orientations)**.** Let $n \geq 1$ and $0 \neq \omega \in \Lambda^n(V)$. Then the set

$$\mathcal{O}_\omega := \{B = (b_1, \dots, b_n) \,|\, \omega(b_1, \dots, b_n) > 0\}$$

is an orientation on V.
Conversely to any given orientation $[B]$ there exists a $0 \neq \omega \in \Lambda^n(V)$, such that $\mathcal{O}_\omega = [B]$.

Proof.
Step 1: Choose two bases $B, C \in \mathcal{B}$ and let $T : V \to V$ be the transition matrix mapping B onto C. Then

$$\omega(c_1, \dots, c_n) = \omega(Tb_1, \dots, Tb_n) = \det(T) \cdot \omega(b_1, \dots, b_n)$$

Step 2: Using this equation we will show, that

$$B \text{ and } C \in \mathcal{O}_\omega \Leftrightarrow B \sim C$$

"$\Rightarrow$": If $B, C \in \mathcal{O}_\omega$ then

$$\det(T) = \frac{\omega(b_1, \dots, b_n)}{\omega(c_1, \dots, c_n)} > 0$$

thus $B \sim C$.
"$\Leftarrow$": If $\det(T) > 0$, then

$$\omega(c_1, \dots, c_n) = \underbrace{\det(T)}_{>0} \cdot \omega(b_1, \dots, b_n)$$

thus either C and B are both in $\mathcal{O}_\omega$ or are both not in $\mathcal{O}_\omega$.

STEP 3: Conversely if $B \in [B]$ is any positive basis, define $\omega : V^n \to \mathbb{R}$ as the alternating multilinear extension uniquely determined by

$$\omega(b_1, \ldots, b_n) := 1$$

By construction $\omega \neq 0$ and step 1 shows, that any other $C \in [B]$ also satisfies $\omega(c_1, \ldots, c_n) > 0$. Conversely if $(c_1, \ldots, c_n)$ is any other basis such that $\omega(c_1, \ldots, c_n) > 0$ and T is the transition matrix mapping B to C step 1 again implies

$$\det(T) = \frac{\omega(b_1, \ldots, b_n)}{\omega(c_1, \ldots, c_n)} = \frac{1}{\omega(c_1, \ldots, c_n)} > 0$$

thus $C \in [B]$.

□

1.1.5 Definition (Orientation form)**.** If $(V, [B])$ is an oriented vector space a form $0 \neq \omega \in \Lambda^n(V)$ satisfying $\mathcal{O}_\omega = [B]$ is an *orientation form.*

1.1.6 Lemma (Volume Element)**.** Let $(V, \langle \_, \_ \rangle, [B])$ be an oriented inner product space. There exists exactly one $dV \in \Lambda^n(V)$ satisfying

$$dV(b_1, \ldots, b_n) = 1$$

for any positive ONB $B = (b_1, \ldots, b_n)$. For any such B

$$dV = b^1 \wedge \ldots \wedge b^n$$

Proof.
STEP 1 (Existence): By the Gram-Schmidt-Theorem there exists an ONB $(b'_1, \ldots, b'_n)$ of V. If this basis is not positive, swich e.g. the first two vectors which yields a basis $B := (b_1, \ldots, b_n)$, wich is now positive and still an ONB. Define

$$dV := b^1 \wedge \ldots \wedge b^n$$

This obviously satisfies $dV(b_1, \ldots, b_n) = \det(b^i(b_j)) = 1$. So there exists at least one positive ONB on which dV satisfies the desired property. For any other positive ONB $C = (c_1, \ldots, c_n)$ there exists a transition automorphism let T such that $c_i = Tb_i$. Since both bases are orthonormal $\det T = \pm 1$ and since both are positive $\det T = 1$. Thus

$$dV(c_1, \ldots, c_n) = dV(Tv_1, \ldots, Tv_n) = \det(T)dV(v_1, \ldots, v_n) = 1$$

STEP 2 (Uniqueness): Since $\dim \Lambda^n(V) = 1$ for any other such $dV' \in \Lambda^n(V)$ there is a $\lambda \in \mathbb{R}$ such that $dV' = \lambda dV$. By hypothesis on an arbitrary positive ONB $(b_1, \ldots, b_n)$

$$1 = dV'(b_1, \ldots, b_n) = \lambda dV(b_1, \ldots, b_n) = \lambda$$

and thus $dW = dV$.

□

1.1.7 Definition (Volume Element)**.** Let $(V, \langle \_, \_ \rangle, [B])$ be an oriented inner product space. The form $dV \in \Lambda^n(V)$ satisfying

$$dV(b_1, \ldots, b_n) = 1$$

for any positive ONB is called *volume form.* It is well defined by Lemma 1.1.6 above.

1.2 Extension of Scalar products

Given an inner product space one might ask how to extend the inner product onto the exterior algebra. There is in fact a canonical way to do this, when employing the following convention:

1.2.1 Convention. Let $(V, \langle \_, \_ \rangle_V)$ be a real inner product space of dimension n. By Theorem 1.0.1 there exists a canonical Isomorphism $\Psi : V \to V^*$ uniquely determined by $\langle \_, \_ \rangle_V$. Thus V^* is canonically an inner product space as well, by defining

$$\forall \omega, \eta \in V^* : \langle \omega, \eta \rangle_{V^*} := \langle \Psi^{-1}(\omega), \Psi^{-1}(\eta) \rangle_V$$

We will always assume that V^* is endowed with this inner product and denote both inner products just by $\langle \_, \_ \rangle$.

1.2.2 Theorem (Extension of Scalar Products)**.** Let $(V, \langle \_, \_ \rangle)$ be a real inner product space. For any $0 \leq k \leq n$ there exists exactly one scalar product $\langle \_, \_ \rangle_{\Lambda^k} : \Lambda^k(V) \times \Lambda^k(V) \to \mathbb{R}$ such that for any ONB $C = (c_1, \ldots, c_n)$ with respect to $\langle \_, \_ \rangle$, the basis $\Lambda^k C$ is an ONB with respect to $\langle \_, \_ \rangle_{\Lambda^k}$. This scalar prouduct is given as the unique bi-additive extension of

$$\langle v^1 \wedge \ldots \wedge v^k, w^1 \wedge \ldots \wedge w^k \rangle_{\Lambda^k} = \det \left(\langle v^i, w^j \rangle \right)$$

where $v^1, \ldots, v^k, w^1, \ldots, w^k \in V^*$.

Proof.
STEP 1 (Existence): Certainly there exists an ONB B of V. Since $\Lambda^k B$ is a basis of $\Lambda^k(V)$ is suffices to define

$$\langle b^{i_1} \wedge \ldots \wedge b^{i_k}, b^{j_1} \wedge \ldots \wedge b^{j_k} \rangle_{\Lambda^k} := \det \left(\langle b^{i_r}, b^{j_s} \rangle \right)$$

and extend this bi-additively onto $\Lambda^k(V)$. Then this map is also homogenous since for any $\lambda \in \mathbb{R}$

$$\langle \lambda \cdot b^{i_1} \wedge \ldots \wedge b^{i_k}, b^{j_1} \wedge \ldots \wedge b^{j_k} \rangle_{\Lambda^k} = \langle (\lambda b^{i_1}) \wedge \ldots \wedge b^{i_k}, b^{j_1} \wedge \ldots \wedge b^{j_k} \rangle_{\Lambda^k} = \lambda \det \left(\langle b^{i_r}, b^{j_s} \rangle \right)$$

Thus $\langle \_, \_ \rangle_{\Lambda^k}$ is bilinear and clearly symmetric, because $\langle \_, \_ \rangle$ is. Furthermoore

$$\langle b^{i_1} \wedge \ldots \wedge b^{i_k}, b^{i_1} \wedge \ldots \wedge b^{i_k} \rangle_{\Lambda^k} = \det \left(\langle b^{i_r}, b^{i_s} \rangle \right) = \det \left(\delta_{i_r, i_s} \right) = 1 > 0$$

So $\langle \_, \_ \rangle_{\Lambda^k}$ is positive definit, thus a scalar product for which $\Lambda^k B$ is an ONB. If $C = (c_1, \ldots, c_n)$ is any other ONB in of V, then there is an orthogonal transformation $T \in O(V)$ such that $C = TB$ and it follows from

$$\det \left(\langle c^{i_r}, c^{j_s} \rangle \right) = \det \left(\langle T b^{i_r}, T b^{j_s} \rangle \right) = \det \left(\langle b^{i_r}, T^T T b^{j_s} \rangle \right) = \det \left(\langle b^{i_r}, b^{j_s} \rangle \right)$$

that $\Lambda^k C$ is an ONB as well.

STEP 2 (Uniqueness): Let g be any other scalar product on $\Lambda^k(V)$ satisfying the required properties. Since $(b_1, \ldots, b_n)$ is still an ONB of B, $\Lambda^k B$ is also a g-ONB. Thus g and $\langle \_, \_ \rangle_{\Lambda^k}$ are equal on an ONB of $\Lambda^k(V)$ and thus equal on all of $\Lambda^k(V)$.

□

1.3 The Hodge Star Operator on Vector Spaces

The Hodge star operator is a particular isomorphism $*_k : \Lambda^k(V) \to \Lambda^{n-k}(V)$. By considering the dimensions

$$\dim \Lambda^k(V) = \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k} = \dim \Lambda^{n-k}(V)$$

its clear, that such an isomorphism exists. The following theorem asserts, that among all those isomorphisms there exists a particularly nice one.

1.3.1 Theorem (Existence of the Hodge operator). Let $(V, \langle\_,\_\rangle, [B])$ be an oriented inner product space and let dV be the associated volume element (c.f. 1.1.7). Then for any $0 \leq k \leq n$, there exists precisely one isomorphism $* = *_k : \Lambda^k(V) \to \Lambda^{n-k}(V)$, $\omega \mapsto *\omega$, which satisfies

$$\forall \omega, \eta \in \Lambda^k(V) : \omega \wedge *\eta = \langle \omega, \eta \rangle_{\Lambda^k} dV$$

This is equivalent to

$$\forall \omega, \eta \in \Lambda^k(V) : \langle \omega \wedge *\eta, dV \rangle_{\Lambda^n} = \langle \omega, \eta \rangle_{\Lambda^k}$$

Proof.
STEP 1 (Preparations): Fix any $0 \leq k \leq n$. The map $\kappa_k : \Lambda^k(V) \to (\Lambda^k(V))^*$

$$\omega \mapsto \langle \omega, \_ \rangle_{\Lambda^k}$$

is the canonical isomorphism from 1.0.1 and the extended scalar product from the right hand side is taken from 1.2.2. Similar, we define another map $\Psi_k : \Lambda^{n-k}(V) \to (\Lambda^k(V))^*$

$$\eta \mapsto \langle \_ \wedge \eta, dV \rangle_{\Lambda^n}$$

which is induced from the bilinear pairing $\beta_k : \Lambda^{n-k}(V) \times \Lambda^k(V) \to \mathbb{R}$

$$(\eta, \omega) \mapsto \langle \omega \wedge \eta, dV \rangle_{\Lambda^n}$$

We would like to apply 1.0.1 again to show, that this is an isomorphism as well. Therefore we have to show, that β_k is a regular bilinear form. Bilinearity is an immediate consequence of the bilinearity of the wedge product and $\langle\_,\_\rangle_{\Lambda^n}$. To see its regularity consider any $\eta \in \Lambda^{n-k}(V)$ and assume

$$\forall \omega \in \Lambda^k(V) : \beta_k(\eta, \omega) = \langle \omega \wedge \eta, dV \rangle_{\Lambda^n} = 0$$

Since $\langle\_,\_\rangle_{\Lambda^n}$ is regular and $dV \neq 0$, this implies

$$\forall \omega \in \Lambda^k(V) : \omega \wedge \eta = 0$$

To see that this finally this implies $\eta = 0$, we may check this on a basis. So let's assume, that $\eta = b^{i_1} \wedge \ldots \wedge b^{i_{n-k}}$, where $i_1 < \ldots < i_{n-k}$. Choose a complementary tupel of indices $j_1 < \ldots < j_k$ such that $\{i_1, \ldots, i_{n-k}, j_1, \ldots j_k\} = \{1, 2, \ldots, n\}$ and define $\omega := b^{j_1} \wedge \ldots \wedge b^{j_k} \neq 0$. It follows that

$$\omega \wedge \eta = b^{j_1} \wedge \ldots \wedge b^{j_k} \wedge b^{i_1} \wedge \ldots \wedge b^{i_{n-k}} = \pm b^1 \wedge \ldots \wedge b^n \neq 0$$

as well. Contradiction! Thus β_k is regular and Ψ_k is an isomorphism as well.
STEP 2 (Definition): Now we are able to define $*_k : \Lambda^k(V) \to \Lambda^{n-k}(V)$ by

$$* := *_k := \Psi_k^{-1} \circ \kappa_k$$

It follows immediatly that $*_k$ is an isomorphism, since it's a composition of isomorphisms. By definition

$$\forall \omega, \eta \in \Lambda^k(V) : \langle \omega \wedge *_k\eta, dV \rangle = \Psi_k(*_k\eta)(\omega) = \Psi_k(\Psi_k^{-1}(\kappa_k(\eta)))(\omega) = \kappa_k(\eta)(\omega) = \langle \eta, \omega \rangle = \langle \omega, \eta \rangle$$

Thus the second condition is satisfied.
STEP 3 (Equivalence of the conditions): Let $\omega, \eta \in \Lambda^k(V)$. Since $\dim \Lambda^n(V) = 1$ and $dV \neq 0$ there is precisely one $\lambda \in \mathbb{R}$, such that $\omega \wedge *\eta = \lambda dV$. By 1.1.6 and 1.2.2

$$\lambda = \lambda \cdot \langle dV, dV \rangle_{\Lambda^n} = \langle \lambda dV, dV \rangle_{\Lambda^n} = \langle \omega \wedge *\eta, dV \rangle_{\Lambda^n} = \langle \omega, \eta \rangle_{\Lambda^k}$$

Thus the second condition implies the first one.
Conversely the first condition implies the second since

$$\omega \wedge *\eta = \langle \omega, \eta \rangle_{\Lambda^k} dV \Longrightarrow \langle \omega \wedge *\eta, dV \rangle_{\Lambda^n} = \langle \omega, \eta \rangle_{\Lambda^k} \langle dV, dV \rangle_{\Lambda^n} = \langle \omega, \eta \rangle_{\Lambda^k}$$

□

Besides this abstract definition, the Hodge operator satisfies various other properties and can be easily computed in coordinates as the following theorem shows.

1.3.2 Theorem (Properties of the Hodge operator). Let $(V, \langle \_, \_ \rangle, [B])$ be an oriented inner product space. The Hodge operator satisfies

(i) $\forall \omega, \eta \in \Lambda^k(V) : *\eta \wedge \omega = \langle \eta, \omega \rangle_{\Lambda^k} dV = \langle \omega, \eta \rangle_{\Lambda^k} dV = \omega \wedge *\eta$

(ii) Let $\sigma \in \mathcal{S}_n$ be a permutation satisfying $\sigma(1) < \ldots < \sigma(k)$ and $\sigma(k+1) < \ldots < \sigma(n)$. Then

$$*(b^{\sigma(1)} \wedge \ldots \wedge b^{\sigma(k)}) = \operatorname{sgn}(\sigma) b^{\sigma(k+1)} \wedge \ldots \wedge b^{\sigma(n)}$$

whenever $B = (b_1, \ldots, b_n)$ is a positive ONB of V. In particular: $*(b^1 \wedge \ldots \wedge b^k) = b^{k+1} \wedge \ldots \wedge b^n$

(iii) $*1 = dV$, $*dV = 1$

(iv) $\forall \omega \in \Lambda^k(V) : *_{n-k} *_k \omega = (-1)^{k(n-k)} \omega$

(v) $\forall \omega, \eta \in \Lambda^k(V) : \langle \omega, \eta \rangle = \langle *\omega, *\eta \rangle$

(vi) $\forall \omega, \eta \in \Lambda^k(V) : \langle \omega, \eta \rangle = *(\omega \wedge *\eta) = *(\eta \wedge *\omega)$

Proof.

(i) This follows directly from 1.3.1.

(ii) Define $\eta := b^{\sigma(1)} \wedge \ldots \wedge b^{\sigma(k)} \in \Lambda^k(V)$ and $\tilde{\eta} := \operatorname{sgn}(\sigma) b^{\sigma(k+1)} \wedge \ldots \wedge b^{\sigma(n)} \in \Lambda^{n-k}(V)$. In order to show, that $\tilde{\eta} = *\eta$, we have to check

$$\forall \omega \in \Lambda^k(V) : \omega \wedge \tilde{\eta} = \langle \omega, \tilde{\eta} \rangle dV$$

by 1.3.1. Since both sides are linear in ω it suffices, to check this on a basis. So let $\omega := b^{i_1} \wedge \ldots \wedge b^{i_k}$ such that $i_1 < \ldots < i_k$. So we need to show that

$$\operatorname{sgn}(\sigma) b^{i_1} \wedge \ldots \wedge b^{i_k} \wedge b^{\sigma(k+1)} \wedge \ldots \wedge b^{\sigma(n)} = \langle b^{i_1} \wedge \ldots \wedge b^{i_k}, b^{\sigma(1)} \wedge \ldots \wedge b^{\sigma(k)} \rangle b^1 \wedge \ldots \wedge b^n$$

and distinguish the following two cases.
If $\{i_1, \ldots, i_k, \sigma(k+1), \ldots, \sigma(n)\} \neq \{1, \ldots, n\}$, this set contains a repeated index, i.e. $\exists \nu \in \{k+1, \ldots, n\} : \sigma(\nu) = i_\nu$. The left hand side is zero due to the propertys of the wedge product. But the right hand side quals zero as well, because $(i_1, \ldots, i_k)$ can no longer be a permutation of $(\sigma(1), \ldots, \sigma(k))$ and thus $b^{i_1} \wedge \ldots \wedge b^{i_k} \perp b^{\sigma(1)} \wedge \ldots \wedge b^{\sigma(k)}$.
If $\{i_1, \ldots, i_k, \sigma(k+1), \ldots, \sigma(n)\} = \{1, \ldots, n\}$ it follows that $\{\sigma(1), \ldots, \sigma(k)\} = \{i_1, \ldots, i_k\}$, so there exists $\tau \in S_k$, such that $(\sigma(1), \ldots, \sigma(k)) = (\tau(i_1), \ldots, \tau(i_k))$. This implies:

$$\begin{aligned}
&\operatorname{sgn}(\sigma) b^{i_1} \wedge \ldots \wedge b^{i_k} \wedge b^{\sigma(k+1)} \wedge \ldots \wedge b^{\sigma(n)} \\
&= \operatorname{sgn}(\sigma) b^{\tau^{-1}(\sigma(1))} \wedge \ldots \wedge b^{\tau^{-1}(\sigma(k))} \wedge b^{\sigma(k+1)} \wedge \ldots \wedge b^{\sigma(n)} \\
&= \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) b^{\sigma(1)} \wedge \ldots \wedge b^{\sigma(k)} \wedge b^{\sigma(k+1)} \wedge \ldots \wedge b^{\sigma(n)} \\
&= \operatorname{sgn}(\tau) b^1 \wedge \ldots \wedge b^k \wedge b^{k+1} \wedge \ldots \wedge b^n \\
&= \operatorname{sgn}(\tau) \langle b^{\sigma(1)} \wedge \ldots \wedge b^{\sigma(k)}, b^{\sigma(1)} \wedge \ldots \wedge b^{\sigma(k)} \rangle_{\Lambda^k} b^1 \wedge \ldots \wedge b^n \\
&= \langle b^{i_1} \wedge \ldots \wedge b^{i_k}, b^{\sigma(1)} \wedge \ldots \wedge b^{\sigma(k)} \rangle_{\Lambda^k} b^1 \wedge \ldots \wedge b^n
\end{aligned}$$

(iii) Follows now immediatly from what we have just proved via $\sigma := \mathrm{id}$ and $k := 0$ resp. $k := n$.

(iv) Again it suffives to check this on a basis. So let $\sigma \in S_n$ such that

$$\sigma(1) < \ldots < \sigma(k) \text{ and } \sigma(k+1) < \ldots < \sigma(n)$$

and define $\eta := b^{\sigma(1)} \wedge \ldots \wedge b^{\sigma(k)}$. Using (ii), we obtain

$$\begin{aligned} * * \eta &= *(\operatorname{sgn}(\sigma)(b^{\sigma(k+1)} \wedge \ldots \wedge b^{\sigma(n)})) = \operatorname{sgn}(\sigma) * (b^{\sigma(k+1)} \wedge \ldots \wedge b^{\sigma(n)}) \\ &= \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) b^{\tau(n-k+1)} \wedge \ldots \wedge b^{\tau(n)} \end{aligned}$$

where $\tau \in \mathcal{S}_n$ is a permutation such that

$$(\tau(1), \ldots, \tau(n-k)) = (\sigma(k+1), \ldots, \sigma(n)) \text{ and } \tau(n-k+1) < \ldots < \tau(n)$$

This immediately implies, that

$$(\tau(n-k+1), \ldots, \tau(n)) = (\sigma(1), \ldots, \sigma(k))$$

thus

$$* * \eta = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\eta$$

and it suffices to calculate the signs. In order to do so, we let $\pi \in \mathcal{S}_n$ be the permutation

$$(1, \ldots, k, k+1, \ldots, n) \mapsto (k+1, \ldots, n, 1, \ldots, k)$$

and observe, that $\tau = \sigma \circ \pi$ such that

$$\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma)\operatorname{sgn}(\pi) = \operatorname{sgn}(\pi) = (-1)^{k(n-k)}$$

(v) We calculate

$$\langle *\omega, *\eta\rangle dV \overset{\text{(i)}}{=} *\omega \wedge (* * \eta) \overset{\text{(iv)}}{=} (-1)^{k(n-k)} * \omega \wedge \eta = \eta \wedge *\omega \overset{\text{(i)}}{=} \langle \eta, \omega\rangle dV = \langle \omega, \eta\rangle dV$$

(vi) We calculate

$$\langle \omega, \eta\rangle = \langle \omega, \eta\rangle 1 \overset{\text{(iii)}}{=} \langle \omega, \eta\rangle * dV = *(\langle \omega, \eta\rangle dV) = *(\omega \wedge *\eta) \overset{\text{(i)}}{=} *(\eta \wedge *\omega)$$

□

2 The Laplacian on Manifolds

In this section we will apply the concepts from linear algebra on the tangent spaces of a manifold.

2.0.3 Definition (Hodge operator)**.** For any $0 \le k \le n$ define $*_k : \Omega^k(M) \to \Omega^{n-k}(M)$ pointwise, i.e. for any $p \in M$ let $*_p : \Lambda^k(T_pM) \to \Lambda^{n-k}(T_pM)$ be the Hoge operator on a single vector speace defined by 1.3.1.

2.0.4 Lemma. For any $0 \le k \le n$, the map $* : \Omega^k(M) \to \Omega^{n-k}(M)$ is a smooth map between smooth vector bundles.

Proof. Theorem 1.3.2,(ii) show that the component functions with respect to any smooth positive orthonormal frame $\{E_i\}$ are locally given by the constant functions $+1, -1, 0$. □

2.0.5 Definition (Beltrami)**.** For $k \ge 1$ define $\delta_k : \Omega^k(M) \to \Omega^{k-1}(M)$

$$\delta_k := (-1)^{n(k+1)+1} *_{n-k+1} \circ d_{n-k} \circ *_k$$

and $\delta_0 := 0$. The extension $\delta : \Omega^*(M) \to \Omega^*(M)$ is called *Beltrami-Operator*.

2.0.6 Lemma. The Beltrami-Operator satisfies:

$$\delta \circ \delta = 0$$

Proof. By Definition using 1.3.2(iv)

$$\delta \circ \delta = \pm * d * *d* = \pm * d \,\mathrm{id}\, d* = \pm * d^2 * = 0$$

□

2.0.7 Definition (Laplace-Beltrami)**.** Define $\Delta_k : \Omega^k(M) \to \Omega^k(M)$

$$\Delta_k := \delta_{k+1} \circ d_k + d_{k-1} \circ \delta_k$$

where we set $d_{-1} := 0$. This extends to a map $\Delta : \Omega^*(M) \to \Omega^*(M)$, which is called *Laplace-Beltrami-Operator*.

The next lemma should convice you, that this makes sense.

2.0.8 Lemma. For $M = \mathbb{R}^n$ and $f \in \Omega^0(\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{R}^n)$

$$\Delta f = -\sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$$

Proof. Unraveling the definitions and applying 1.3.2 we obtain

$$\Delta_0(f) = \delta_1(d_0(f)) + d_{-1}(\delta_0(f)) = \delta_1(d(f)) = \delta_1\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i\right)$$
$$= (-1)^{n(1+1)+1} \sum_{i=1}^n * \left(d\left(\frac{\partial f}{\partial x_i} * (dx_i)\right)\right) = (-1)^{2n+1} \sum_{i=1}^n * \left(d\left(\frac{\partial f}{\partial x_i}(-1)^i dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n\right)\right)$$
$$= -\sum_{i=1}^n * \left(\sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(-1)^i dx_j \wedge dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n\right)$$
$$= -\sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}(-1)^i * \left(dx_i \wedge dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n\right) = -\sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2} * dV = -\sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$$

□

2.0.9 Lemma. The Laplacian commutes with the Hodge, i.e.

$$* \circ \Delta = \Delta \circ *$$

Proof. Chose any k and calculate:

$$\begin{aligned}
*_k \circ \Delta_k &= *_k \circ \delta_{k+1} \circ d_k + *_k \circ d_{k-1} \circ \delta_k \\
&= (-1)^{n(k+1)+1} *_k \circ *_{n-k} \circ d_{n-k-1} \circ *_{k+1} \circ d_k + (-1)^{n(k+1)+1} *_k \circ d_{k-1} \circ *_{n-k+1} \circ d_{n-k} \circ *_k \\
&= (-1)^{n(k+1)+1}(-1)^{(n-k)k} d_{n-k-1} \circ *_{k+1} \circ d_k + (-1)^{n(k+1)+1} *_k \circ d_{k-1} \circ *_{n-k+1} \circ d_{n-k} \circ *_k \\
&= (-1)^{n(k+1)+1} *_k \circ d_{k-1} \circ *_{n-k+1} \circ d_{n-k} \circ *_k + (-1)^{n(k+1)+1} d_{n-k-1} \circ *_{k+1} \circ d_k (-1)^{k(n-k)} \\
&= (-1)^{n(k+1)+1} *_k \circ d_{k-1} \circ *_{n-k+1} \circ d_{n-k} \circ *_k + (-1)^{n(k+1)+1} d_{n-k-1} \circ *_{k+1} \circ d_k \circ *_{n-k} \circ *_k \\
&= \delta_{n-k+1} \circ d_{n-k} \circ *_k + d_{n-k-1} \circ \delta_{n-k} \circ *_k \\
&= \Delta_{n-k} \circ *_k
\end{aligned}$$

□

2.1 L^2 scalar product

2.1.1 Definition (L^2 scalar product)**.** For any $k \in \mathbb{N}$, the bilinear form $\langle \_, \_ \rangle_{L^2} : \Omega^k(M) \times \Omega^k(M) \to \mathbb{R}$ defined by

$$\langle \alpha, \beta \rangle_{L^2(M)} := \int_M \alpha \wedge *\beta$$

is the L^2 *scalar product on* $\Omega^k(M)$. The bilinear extension $\langle \_, \_ \rangle_{L^2}(M) : \Omega^*(M) \times \Omega^*(M) \to \mathbb{R}$ obtained by declaring the various $\Omega^k(M)$ to be mutually orthogonal, is the L^2*(M) scalar product.* This induces a norm $\|\_\|_{L^2} := \sqrt{\langle \_, \_ \rangle_{L^2}}$ on $\Omega^*(M)$. We will always shorten $\langle \_, \_ \rangle := \langle \_, \_ \rangle_{L^2(M)}$ if not explicitely mentioned otherwise.
We denote by

$$L^2(M)$$

the completion of $\Omega^*(M)$ with respect to the L^2-scalar product.

2.1.2 Lemma. The L^2 scalar product is indeed a scalar product.

Proof. We have to check:

(i) Symmetry: $a \wedge *\beta = *\beta \wedge \alpha$ by 1.3.2

(ii) Bilinearity follows from linearity of the of the integral and the multilinearity of the wedge product.

(iii) Positive Definity:

$$\langle \alpha, \alpha \rangle_{L^2(M)} = \int_M \alpha \wedge *\alpha = \int_M \langle \alpha, \alpha \rangle_{\Omega M} dV = \langle \alpha, \alpha \rangle_{\Omega M} \int_M dV > 0$$

□

2.1.3 Lemma. The Beltrami-Operator δ ist the adjoint of the exterior derivative whith respect to the L^2 scalar product, i.e.

$$\forall \alpha, \beta \in \Omega^*(M) : \langle d\alpha, \beta \rangle_{L^2} = \langle \alpha, \delta\beta \rangle_{L^2}$$

Proof. By bilinearity of both sides and the orthogonality of the various $\Omega^k(M)$ it suffices to check this for elements $\alpha \in \Omega^{k-1}(M)$, $\beta \in \Omega^k(M)$. In order to do so remember, that by 1.3.2

$$*_{n-k} \circ *_k = (-1)^{k(n-k)} \operatorname{id}_{\Omega^k(M)}$$

By definition $\delta_k = (-1)^{n(k+1)+1} *_{n-k+1} \circ d_{n-k} \circ *_k : \Omega^k(M) \to \Omega^{k-1}(M)$ thus

$$\begin{aligned} *_{k-1} \circ \delta_k &= (-1)^{n(k+1)+1} *_{k-1} \circ *_{n-k+1} \circ d_{n-k} \circ *_k = (-1)^{n(k+1)+1+(n-k+1)(n-(n-k+1))} d_{n-k} \circ *_k \\ &= (-1)^{nk+n+1+(n-k+1)(k-1)} d_{n-k} \circ *_k = (-1)^{nk+n+1+nk-n-k^2+k+k-1} d_{n-k} \circ *_k \\ &= (-1)^{-k^2} d_{n-k} \circ *_k = (-1)^k d_{n-k} \circ *_k \end{aligned}$$

where the last step holds, since if k is even / odd, so is k^2. Using this and the product rule, we obtain

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^{k-1} \alpha \wedge (d * \beta) = d\alpha \wedge *\beta - \alpha \wedge (*\delta\beta)$$

Integrating both sides and applying Stokes Theorem we obtain:

$$0 = \int_{\partial M} \alpha \wedge *\beta = \int_M d(\alpha \wedge *\beta) = \int_M (d\alpha \wedge *\beta - \alpha \wedge *\delta\beta) = \langle d\alpha, \beta \rangle - \langle \alpha, \delta\beta \rangle$$

□

2.1.4 Corollary. The Laplace-Beltrami-Operator is selfadjoint, i.e.

$$\langle \Delta\alpha, \beta \rangle = \langle \alpha, \Delta\beta \rangle$$

Proof.

$$\langle \Delta\alpha, \beta \rangle = \langle (\delta d + d\delta)\alpha, \beta \rangle = \langle \delta d\alpha, \beta \rangle + \langle d\delta\alpha, \beta \rangle = \langle \alpha, \delta d\beta \rangle + \langle \alpha, d\delta\beta \rangle = \langle \alpha, \delta d\beta + d\delta\beta \rangle = \langle \alpha, \Delta\beta \rangle$$

□

2.1.5 Convention. We will denote the adjoint of Δ by Δ^*. Although $\Delta = \Delta^*$ this notation is sometimes useful to stress the spirit of a calculation.

2.1.6 Lemma (important)**.** For any $\alpha \in \Omega^k(M)$

$$\Delta\alpha = 0 \Longleftrightarrow d\alpha = 0 \text{ and } \delta\alpha = 0$$

Proof.
"$\Leftarrow$": By definition

$$\Delta\alpha = d\delta\alpha + \delta d\alpha = 0$$

"$\Rightarrow$": Suppose $\Delta\alpha = 0$. Then

$$0 = \langle \Delta\alpha, \alpha \rangle = \langle (d\delta + \delta d)\alpha, \alpha \rangle = \langle d\delta\alpha, \alpha \rangle + \langle \delta d\alpha, \alpha \rangle = \langle \delta\alpha, \delta\alpha \rangle + \langle d\alpha, d\alpha \rangle$$

Since the exterior scalar product is positive definit, this implies $\delta\alpha = 0$ and $d\alpha = 0$. □

2.1.7 Definition (harmonic forms)**.** The elements of

$$\mathcal{H}^k(M) := \left\{ \omega \in \Omega^k(M) \mid \Delta\omega = 0 \right\} = \ker \Delta_k$$

are called *harmonic* k-forms. We define

$$\mathcal{H}^*(M) := \bigoplus_{k=0}^{n} \mathcal{H}^k(M)$$

2.1.8 Corollary. Every harmonic function $f \in \mathcal{C}^\infty(M) = \Omega^0(M)$ on a compact connected manifold M is constant.

Proof. By 2.1.6 we have

$$\Delta f = 0 \Rightarrow df = 0 \Rightarrow f = \text{const}$$

□

3 Weak Solutions and Regularity

3.0.9 Definition (Weak Solution)**.** Let $\alpha \in \Omega^k(M)$ be arbitrary. A form $\omega \in \Omega^k(M)$ such that

$$\Delta\omega = \alpha$$

is a *(strong) solution* of "$\Delta\omega = \alpha$".
A continuous linear functional $l : (\Omega^k(M), \|\_\|_{L^2}) \to (\mathbb{R}, |\_|)$ is a *weak solution* of $\Delta\omega = \alpha$, if

$$\forall \eta \in \Omega^k(M) : l(\Delta^*\eta) = \langle \alpha, \eta \rangle$$

Since we are only dealing with the Laplace equation, the word "solution" will always mean a solution of the Laplace equation.

3.0.10 Lemma (Strong to Weak)**.** Let $\alpha, \omega \in \Omega^k(M)$ such that

$$\Delta\omega = \alpha$$

i.e. ω is a strong solution. Then $l : \Omega^k(M) \to \mathbb{R}$, defined by

$$l(\eta) := \langle \omega, \eta \rangle$$

is a weak solution.

Proof. Let $\eta \in \Omega^k(M)$ be arbitrary. By the Cauchy-Schwarz inequality

$$|l(\eta)| = |\langle \omega, \eta \rangle| \leq \|\omega\| \|\eta\|$$

thus l is continuous and since

$$l(\Delta^* \eta) = \langle \omega, \Delta^* \eta \rangle = \langle \Delta\omega, \eta \rangle = \langle \alpha, \eta \rangle$$

is it a weak solution. □

3.0.11 Theorem (Weak to Strong)**.** Let $\alpha \in \Omega^k(M)$, and let l be a weak solution of $\Delta\omega = \alpha$. Then there exists $\omega \in \Omega^k(M)$ such that

$$\forall \beta \in \Omega^k(M) : l(\beta) = \langle \omega, \beta \rangle$$

This implies

$$\forall \eta \in \Omega^k(M) : \langle \Delta\omega, \eta \rangle = \langle \omega, \Delta^* \eta \rangle = l(\Delta^* \eta) = \langle \alpha, \eta \rangle$$

and consequently, ω, is a strong solution, i.e.

$$\Delta\omega = \alpha$$

For the proof one has to delve deeply into the theory of Sobolev spaces, elliptic operators and regularity theory. Unfortunately this is far beyond the scope of this article and so we just cite this theorem from [2, 6.5].

4 Compact Embedding

Will will furthermoore need the following theorem, which is also very time-consuming to prove. It is also discussed in [2, 6.6].

4.0.12 Theorem (Compactness)**.** Let $\{\alpha_n\}$ be a sequence in $\Omega^k(M)$, such that there exists $C > 0$, such that

$$\|\alpha_n\| \leq C \qquad \|\Delta\alpha_n\| \leq C$$

Then there exists a subsequence of α_n, which is Cauchy.

We will need this theorem primarily in the form of the following

4.0.13 Lemma. For any $0 \leq k \leq n$:

$$\exists C > 0 : \forall \beta \in \mathcal{H}^k(M)^\perp : \|\beta\| \leq C \|\Delta\beta\|$$

Proof. Suppose the contrary. Then for any $j \in \mathbb{N}$ there exists $\tilde{\beta}_j \in \Omega^k(M)$ such that

$$\|\tilde{\beta}_j\| > j\|\Delta\tilde{\beta}_j\|$$

Define

$$\beta_j := \frac{\tilde{\beta}_j}{\|\tilde{\beta}_j\|}$$

This sequence satisfies

$$\forall j \in \mathbb{N} : \|\beta_j\| = 1 \qquad \lim_{j\to\infty} \|\Delta\beta_j\| \leq \lim_{j\to\infty} \frac{1}{j} = 0$$

By Theorem 4.0.12 there exists a subsequence of $\{\beta_j\}$ which is Cauchy. For simplicity we may assume, that this subsequence is $\{\beta_j\}$ itself. For any $\psi \in \Omega^k(M)$

$$|\langle\beta_j, \psi\rangle - \langle\beta_k, \psi\rangle| = |\langle\beta_j - \beta_k, \psi\rangle| \leq \|\beta_j - \beta_k\|\|\psi\|$$

Thus $\{\langle\beta_j, \psi\rangle\}$ is a Cauchy sequence in $\mathbb{R}$. Consequently, the functional $l : \Omega^k(M) \to \mathbb{R}$

$$\psi \mapsto \lim_{j\to\infty} \langle\beta_j, \psi\rangle$$

is well defined, linear and bounded. Since

$$\forall\varphi \in \Omega^k(M) : 0 \leq \lim_{j\to\infty} |\langle\Delta\beta_j, \varphi\rangle| \leq \lim_{j\to\infty} \|\Delta\beta_j\|\|\varphi\| = 0$$

by construction, we obtain

$$\forall\varphi \in \Omega^k(M) : l(\Delta^*\varphi) = \lim_{j\to\infty} \langle\beta_j, \Delta^*\varphi\rangle = \lim_{j\to\infty} \langle\Delta\beta_j, \varphi\rangle = 0 = \langle 0, \varphi\rangle$$

Thus l is a weak solution of $\Delta\beta = 0$. By the regularity theorem 3.0.11

$$\exists\beta \in \Omega^k(M) : \forall\psi \in \Omega^k(M) : l(\psi) = \langle\beta, \psi\rangle \text{ and } \Delta\beta = 0$$

Since $\{\beta_j\}$ is Cauchy, it converges to some limit $\gamma \in L^2(M)$, but by continuity

$$\forall\psi \in \Omega^k(M) : \langle\gamma - \beta, \psi\rangle = \langle \lim_{j\to\infty} \beta_j, \psi\rangle - \lim_{j\to\infty} \langle\beta_j, \psi\rangle = 0$$

Thus $\beta_j \to \beta$, which implies (also by continuity)

$$\|\beta\| = 1 \qquad \beta \in \mathcal{H}^k(M)^\perp$$

On the other hand

$$\Delta\beta = 0 \Longrightarrow \beta \in \mathcal{H}^k(M)^\perp \cap \mathcal{H}^k(M) = \{0\}$$

Contradiction! □

5 The Hodge Decomposition

We are finally in the position to put all the pieces together. Remember, that M denotes an oriented compact Riemannian manifold (without boundary) of dimension n.

5.1 Statement and Proof

5.1.1 Theorem (Hodge Decomposition Theorem)**.** For any $0 \leq k \leq n$ the space $\mathcal{H}^k := \mathcal{H}^k(M)$ is of finite dimension and there are direct L^2-orthogonal decompositions:

$$\begin{aligned}\Omega^k(M) &= \operatorname{im}\Delta_k \oplus \ker \Delta_k = \Delta(\Omega^k(M)) \oplus \mathcal{H}^k \\ &= d\delta(\Omega^k(M)) \oplus \delta d(\Omega^k(M)) \oplus \mathcal{H}^k \\ &= d(\Omega^{k-1}(M)) \oplus \delta(\Omega^{k+1}(M)) \oplus \mathcal{H}^k\end{aligned}$$

For any $\alpha \in \Omega^k(M)$

$$\exists \omega \in \Omega^k(M) : \Delta\omega = \alpha \iff \alpha \perp \mathcal{H}^k.$$

Proof.
Step 1 (Finite Dimensionality): Suppose $\mathcal{H}^k$ were not finite dimensional. Then $\mathcal{H}^k$ would contain an infinite orthonormal sequence. By Theorem 4.0.12 this sequence had a Cauchy subsequence, which is a contradiction. Thus $\mathcal{H}^k$ is finite dimensional.
Step 2 (Decomposition): Certarinly there exists an orthogonal decomposition

$$\Omega^k(M) = (\mathcal{H}^k)^\perp \oplus \mathcal{H}^k$$

Consequently the statement will be proven, if we can show, that

$$(\mathcal{H}^k)^\perp = \operatorname{im}\Delta_k$$

Step 2.1 ($\operatorname{im}\Delta_k \subset (\mathcal{H}^k)^\perp$): We just calculate

$$\forall \varphi \in \Omega^k(M) : \forall \omega \in \mathcal{H}^k : \langle \Delta\varphi, \omega\rangle = \langle \varphi, \Delta\omega\rangle = 0 \Longrightarrow \Delta\varphi \in (\mathcal{H}^k)^\perp$$

Step 2.2 ($(\mathcal{H}^k)^\perp \subset \operatorname{im}\Delta_k$): Let $\alpha \in (\mathcal{H}^k)^\perp$. Define $l : \operatorname{im}\Delta_k \to \mathbb{R}$ by

$$l(\Delta\varphi) = \langle \alpha, \varphi\rangle$$

This is well defined, since $\Delta\varphi_1 = \Delta\varphi_2$ implies $\varphi_1 - \varphi_2 \in \mathcal{H}^k$ and thus $\langle \varphi_1 - \varphi_2, \alpha\rangle = 0$. It is clear, that l is linear. We claim, that l is also continuous. Denote by $h : \Omega^k(M) \to \mathcal{H}^k$ the canonical projection operator. Let $\varphi \in \Omega^k(M)$ and $\psi := \varphi - h(\varphi)$. This implies

$$|l(\Delta\varphi)| = |l(\Delta\psi)| = |\langle \alpha, \psi\rangle| \leq \|\alpha\|\|\psi\| \overset{4.0.13}{=} c\|\alpha\|\|\Delta\psi\| = c\|\alpha\|\|\Delta\varphi\|$$

thus l is continuous. By the Hahn-Banach theorem there exists a continuous extention of l on $\Omega^k(M)$. Consequently l is a weak solution of $\Delta\omega = \alpha$. By Theorem 3.0.11 there exists a strong solution $\omega \in \Omega^k(M)$, i.e. $\alpha = \Delta\omega \in \operatorname{im}\Delta_k$.
Step 3 (Equivalent formulations): By what we have proven so far and by definition

$$\begin{aligned}\Omega^k(M) = \Delta(\Omega^k(M)) \oplus \mathcal{H}^k = (d\delta + \delta d)(\Omega^k(M)) \oplus \mathcal{H}^k \subset (d\delta((\Omega^k(M))) + \delta d(\Omega^k(M))) \oplus \mathcal{H}^k \\ \subset (d(\Omega^{k-1}(M))) + \delta(\Omega^{k+1}(M))) \oplus \mathcal{H}^k \subset \Omega^k(M)\end{aligned}$$

which implies, that all subsets are actually equalities. Since

$$\forall \omega, \eta \in \Omega^k(M) : \langle d\delta\omega, \delta d\eta\rangle \overset{2.1.3}{=} \langle dd\delta\omega, d\eta\rangle = 0$$

we obtain

$$d\delta((\Omega^k(M))) \perp \delta d(\Omega^k(M)).$$

Similarly since

$$\forall \omega \in \Omega^{k-1}(M) : \forall \eta \in \Omega^{k+1}(M) : \langle d\omega, \delta\eta \rangle \overset{2.1.3}{=} \langle dd\omega, \eta \rangle = 0$$

we obtain

$$d(\Omega^{k-1}(M))) \perp \delta(\Omega^{k+1}(M))$$

Since

$$\forall \omega, \eta \in \Omega^k(M) : \forall \alpha \in \mathcal{H}^k : \langle d\delta\omega + \delta d\eta, \alpha \rangle = \langle \delta\omega, \delta\alpha \rangle + \langle d\eta, d\alpha \rangle \overset{2.1.6}{=} 0$$

we get

$$(d\delta((\Omega^k(M))) \oplus \delta d(\Omega^k(M))) \perp \mathcal{H}^k$$

Finally

$$\forall \omega \in \Omega^{k-1}(M) : \forall \eta \in \Omega^{k+1}(M) : \forall \alpha \in \mathcal{H}^k : \langle d\omega + \delta\eta, \alpha \rangle = \langle \omega, \delta\alpha \rangle + \langle \eta, d\alpha \rangle \overset{2.1.6}{=} 0$$

implies

$$(d(\Omega^{k-1}(M))) \oplus \delta(\Omega^{k+1}(M))) \perp \mathcal{H}^k$$

□

5.2 Applications and Corollaries

5.2.1 Definition (Green's Operator)**.** Let $\alpha \in \Omega^k(M)$ be arbitrary. Then there exists a unique representation

$$\alpha = \varphi + \psi \in \operatorname{im} \Delta_k \oplus \ker \Delta_k$$

Define the orthogonal projection $\pi_k : \Omega^k(M) \to (\mathcal{H}^k)^\perp$ by $\pi_k(\alpha) := \varphi$ and the *harmonizator* $h_k : \Omega^k(M) \to \mathcal{H}^k$ by $h_k(\alpha) := \psi$. Since the Laplacian restricts so an automorphism $\Delta_k : (\mathcal{H}^k)^\perp \to (\mathcal{H}^k)^\perp$, we may define

$$G_k : \Omega^k(M) \to (\mathcal{H}^k)^\perp, G_k := \Delta|_{(\mathcal{H}^k)^\perp}^{-1} \circ \pi_k$$

This map is *Green's Operator*. In other words $G(\alpha)$ is the unique form satisfying

$$\Delta G(\alpha) = \alpha - h(\alpha) = \pi(\alpha)$$

5.2.2 Lemma.

(i) The map h is a linear operator with norm...

(ii) The map π is a linear operator with norm...

(iii) The map G is a linear operator with norm...

(iv) G is self-adjoint and takes bounded sequences to sequences with Cauchy subsequences.

5.2.3 Theorem. Green's operator G commutes with any linear operator which commutes with the Laplacian. In particular G commutes with d, δ and Δ.

Proof. Let $T : \Omega^k(M) \to \Omega^l(M)$ be a linear operator such that $T\Delta = \Delta T$. This implies immediately

$$\forall \alpha \in \mathcal{H}^k : \Delta(T(\alpha)) = T(\Delta(\alpha)) = 0$$

and thus $T(\mathcal{H}^k) \subset \mathcal{H}^l$ as well as

$$T((\mathcal{H}^k)^\perp) = T(\Delta(\Omega^k(M)) = \Delta(T(\Omega^k(M))) \subset (\mathcal{H}^l)^\perp$$

This means

$$\forall \alpha = \varphi + \psi \in (\mathcal{H}^k)^\perp \oplus \mathcal{H}^k : T_k(\pi_k(\alpha)) = T_k(\varphi) = \pi_l(T_k(\varphi)) = \pi_l(T_k(\varphi) + T_k(\psi)) = \pi_l(T_k(\alpha))$$

so $T \circ \pi = \pi \circ T$, which implies by hypothesis

$$T \circ G = T \circ \Delta|_{(\mathcal{H}^k)^\perp}^{-1} \circ \pi = \Delta|_{(\mathcal{H}^k)^\perp}^{-1} \circ T \circ \pi = \Delta|_{(\mathcal{H}^k)^\perp}^{-1} \circ \pi \circ T = G \circ T$$

For the second statement we notice that Δ clearly commutes with itself and by definition

$$d\Delta = d(d\delta + \delta d) = d\delta d = (d\delta + \delta d)d = \Delta d$$
$$\delta\Delta = \delta(d\delta + \delta d) = \delta d\delta = (d\delta + \delta d)\delta = \Delta\delta$$

□

5.2.4 Theorem (Harmonic de Rahm cohomology)**.** Every de Rahm cohomology class has a unique harmonic representative. If $[\alpha] \in H^k_{dR}(M)$, then $[\alpha] = [h(\alpha)]$.

Proof.
STEP 1 (Existence): Let $[\alpha] \in \mathcal{H}^k_{dR}(M)$ be arbitrary. We calculate

$$\alpha = \Delta G\alpha + h\alpha = d\delta G\alpha + \delta dG\alpha + h\alpha = d\delta G\alpha + \delta Gd\alpha + h\alpha = d\delta G\alpha + h\alpha$$

Thus $[\alpha] = [h\alpha]$ and every cohomology class contains at least one harmonic representative.
STEP 2 (Uniqueness): Assume that $\alpha_1, \alpha_2 \in \mathcal{H}^k(M)$ differ by an exact form, i.e.

$$\exists \beta \in \Omega^{k-1}(M) : d\beta = \alpha_2 - \alpha_1$$

This implies

$$\langle d\beta, \alpha_2 - \alpha_1 \rangle = \langle \beta, \delta\alpha_2 \rangle - \langle \beta, \delta\alpha_1 \rangle \overset{2.1.6}{=} 0$$

and the decomposition

$$(\alpha_2 - \alpha_1) - d\beta = 0$$

is orthogonal. Consequently $d\beta = 0$ and $\alpha_2 = \alpha_1$. So every cohomology class contains at most one harmonic representative.

□

5.2.5 Corollary. The map $h : \Omega^k(M) \to \mathcal{H}^k(M)$ induces a vector space isomorphism

$$[h] : \mathcal{H}^k_{dR}(M) \xrightarrow{\sim} \mathcal{H}^k(M), [\alpha] \mapsto h(\alpha)$$

Proof. By 5.1.1

$$\forall \beta \in \Omega^{k-1}(M) : d\beta + 0 + 0 \in d(\Omega^{k-1}(M)) \oplus \delta(\Omega^{k-1}(M)) \oplus \mathcal{H}^k(M)$$

thus $h(d\beta) = 0$ and $[h]$ is well-defined. Any harmonic form is also closed by 2.1.6 and thus determines a cohomology class. By Theorem 5.2.4 above

$$\forall \alpha \in \mathcal{H}^k(M) : [h]([\alpha]) = h(\alpha) = \alpha$$

so $[h]$ is surjective. Analogously $[h]$ is also injective since

$$\forall [\alpha] \in H^k_{dR} : 0 = [h]([\alpha]) = h(\alpha) \Longrightarrow [\alpha] = [h(\alpha)] = [0]$$

□

5.2.6 Corollary (Finite Dimensionality)**.** The de Rahm cohomology groups of compact orientable smooth manifolds are all finite dimensional.

Proof. Every smooth manifold admits a Riemannian metric, so the statement follows from 5.2.4 and 5.1.1. □

5.2.7 Theorem (Analytic Poincaré duality)**.** Let M be a smooth compact oriented manifold of dimension n. The bilinear pairing $(\_, \_) : H^k_{dR} \times H^{n-k}_{dR} \to \mathbb{R}$ defined by

$$([\omega], [\eta]) \to \int_M \omega \wedge \eta$$

is well-defined and regular. Thus it determines isomorphisms

$$H^k_{dR} \xrightarrow{\sim} (H^{n-k}_{dR})^*$$

Proof. The bilinearity is clear. To show that the pairing is well-defined consider closed forms $\omega \in \Omega^k(M)$, $\eta \in \Omega^{n-k}(M)$ and assume e.g. that $\omega' = \omega + d\alpha \in \Omega^k M$ for some $\alpha \in \Omega^{k-1}(M)$. This implies

$$\int_M \omega' \wedge \eta = \int_M \omega \wedge \eta + \int_M d\alpha \wedge \eta = \int_M \omega \wedge \eta + \int_M d(\alpha \wedge \eta) = \int_M \omega \wedge \eta$$

using Stokes' theorem. Analogously the pairing is well-defined in the second argument.
To see the regularity, consider any $[\omega] \in H^k_{dR}(M)$. We have to show

$$\forall [\eta] \in H^{n-k}_{dR}(M) : (\omega, \eta) = 0 \implies [\omega] = 0$$

We assume $[\omega] \neq 0$ and choose any Riemannian metric on M. By 5.2.4 we can assume that ω is harmonic and thus $\omega \neq 0$. Since $*\Delta = \Delta *$ by 2.0.9, this implies, that $\eta := *\omega$ is harmonic as well and by 2.1.6 $*\omega$ is also closed. This implies

$$([\omega], [\eta]) = \int_M \omega \wedge *\omega = \|\omega\|^2_{L^2(M)} \neq 0$$

The statement now follows from 1.0.1. □

References

[1] Lee, Introduction to Smooth Manifolds

[2] Warner, Foundations of Differentitable Manifolds and Lie-Groups