Geodesics, Energy and Variations

Nikolai Nowaczyk

June 2009

This is a script for talk in Prof. Ballmanns seminar "Differential Topology and Morse Theory" held in the summer term 2009 at the University of Bonn. It discusses the three different notions of straight lines, length minimizers and energy minimizers which are all related to geodesics on the level of Riemannian manifolds. We develop and apply the calculus of variations to proof the first variation formula and the central theorem, that geodesics are critical points of the energy function. Please send any comments or mistakes to nikno@uni-bonn.de.

Contents

0.1 Convention. For this entire article M denotes a smooth manifold of dimension m. For all statements involving the geometry of M we fix an arbitrary Riemannian metric g. For any $p \in M$, $V \in T_pM$ we denote by

$$||V|| := \sqrt{g_p(V,V)}$$

the norm induced by g on the tangent space.
We denote by $\nabla : TM \times TM \to TM$ the Riemannian connection on M and by D the covariant differentiation operator on vector fields along curves. $R(\_,\_)(\_)$ is the curvature endomorphism.

1 Remainder: Tangent Space of Manifolds

Before we start let us briefly collect some basic facts about the tangent space of a smooth manifold.

1.1 Definition (Tangent Space via Derivations)**.** Let $p \in M$ be a point and denote by $\mathcal{C}_p^\infty(M)$ the space of germs $[f]_p$ of smooth functions $f : M \to \mathbb{R}$. Then

$$T_pM := \mathrm{Der}_p(\mathcal{C}_p^\infty(M))$$

which is just the set of all linear functions $v : \mathcal{C}_p^\infty(M) \to$ satisfying the product rule

$$v(fg) = v(f)g(p) + f(p)v(g)$$

These spaces assemble to

$$TM := \coprod_{p \in M} T_pM$$

which can be given the structure of a smooth manifold of as well.

1.2 Definition (Pushforward)**.** Let M, N be smoth manifolds and $F : M \to N$ be a smooth map. The *pushforward* of F is a map $F_* : TM \to TN$ defined for $p \in M$ as the map $F_*|_p : T_pM \to T_{F(p)}N$

$$v \mapsto (g \mapsto v(g \circ F))$$

One can show that this is a smooth map between TM and TN.

1.3 Theorem (Curves as Derivations)**.** Let $\gamma : I \to M$ be a smooth curve and $t_0 \in I$. Then γ induces a derivation $\dot\gamma(t_0) \in T_{\gamma(t_0)}M$ via

$$f \mapsto \partial_t(f \circ \gamma)|_{t=t_0}$$

Conversely every derivation $v_p \in T_pM$ can be written as $\dot\gamma(0)$ for some curve $\gamma :]-\varepsilon, \varepsilon[\to M$, $\gamma(0) = p$, $\dot\gamma(0) = v_p$.

1.4 Lemma (Curves in Coordinates)**.** Let $\gamma : I \to M$ be a smooth curve and $\varphi : U \to V$ be a chart such that $\gamma(I) \subset U$. Denote by $\gamma_\varphi : I \to V$, $\gamma_\varphi := \varphi \circ \gamma$, the path γ in coordinates φ. On the one hand, we have a coordinate representation

$$\dot\gamma(t) = \sum_{i=1}^m \dot\gamma^i(t) \partial\varphi|_{\gamma(t)}$$

of the derivation $\dot\gamma(t)$. On the other hand, we have the classical derivative

$$\dot\gamma_\varphi(t) = \sum_{i=1}^m \dot\gamma_\varphi^i(t) e_i$$

For any $1 \leq i \leq m$

$$\dot\gamma_\varphi^i(t) = \dot\gamma^i(t)$$

Proof. By definition

$$\dot\gamma_\varphi^i(t) = (\varphi_i \circ \gamma)'(t) = \dot\gamma(t)(\varphi_i) = \sum_{j=1}^m \dot\gamma^j(t) \partial\varphi_j|_{\gamma(t)}(\varphi_i) = \dot\gamma^i(t)$$

□

1.5 Theorem (Pushforward for Curves)**.** Let $F : M \to \mathbb{R}$ be a smooth function, $p \in M$ and $X \in T_pM$, which is represented by $\gamma :]-\varepsilon, \varepsilon[\to$, $\gamma(0) = p$, $\dot\gamma(0) = X$. Then

$$F_*|_p X = \partial_t (F \circ \gamma)|_{t=0} \partial t|_{F(p)}$$

where ∂_t is the coordinate field induces by $\mathrm{id} : \mathbb{R} \to \mathbb{R}$.

Proof. Chose any $g \in \mathcal{C}^\infty_{F(p)}(\mathbb{R})$ and calculate

$$(F_*|_p X)(g) = X(g \circ F) = \dot\gamma(0)(g \circ F) = \partial_t (g \circ F \circ \gamma)|_{t=0} = \partial_t g|_{F(\gamma(0))} \partial_t (F \circ \gamma)|_{t=0} = \partial_t (F \circ \gamma)|_{t=0} \partial_t|_{F(p)}(g)$$

□

1.6 Definition (Critical Point)**.** A point $p \in M$ of $F \in \mathcal{C}^\infty(M)$ is a *critical point* for F, if $F_*|_p : T_pM \to T_{F(p)}\mathbb{R}$ is the zero map.

1.7 Remark. If $F : M \to \mathbb{R}$ is a function and $p \in M$, then p is a critical point for F if and only if for all $V \in T_pM$, we have $F_*|_p(V) = 0$. If V is represented by a curve $\gamma :]-\varepsilon, \varepsilon[\to M$, $\gamma(0) = p$, $\dot\gamma(0) = V$, the theorem above shows, that $F_*|_p V = 0 \Leftrightarrow \partial_t(F \circ \gamma)|_{t=0} = 0$. Thus p is a critical point for F if and only if for every curve $\gamma :]-\varepsilon, \varepsilon[\to M$, $\gamma(0) = p$, we have $\partial_t(F \circ \gamma)|_{t=0} = 0$.
There is a canonical alternative definition for a critical point: If F is a function on M one defines the *differential* of F at $p \in M$ to be $dF|_p : T_pM \to \mathbb{R}$ by sending $V \mapsto dF|_p(V) := V(F)$. It is natural to say, that F has a critical point at p, if $dF|_p = 0$. Notice that these definitions are equivalent: Since if V is again represented by γ, this implies $dF|_p(V) = V(F) = \dot\gamma(0)(V) = \partial_t(F \circ \gamma)|_{t=0}$. Thus $dF|_p = 0 \Leftrightarrow F_*|_p = 0$.
If one introduces the gradient $\operatorname{grad} f$ as $df^\sharp$, where $\sharp : T^*M \to TM$ denotes the canonical tangent / cotangent isomorphism on a Riemannian manifold, we can even summarize

$$dF|_p = 0 \Leftrightarrow F_*|_p = 0 \Leftrightarrow \operatorname{grad} f|_p = 0 \Leftrightarrow \forall 1 \le i \le n : \partial\varphi_i|_p(f) = 0$$

2 Paths, Energy and Geodesics

2.1 Paths in Manifolds

2.1 Definition (Path Space)**.** Let M be a smooth manifold and $p, q \in M$. A *piecewise smooth path from p to q* is a continuous map $\omega : [0,1] \to M$ such that

(a) There exists a finite subdivision $0 = t_0 < t_1 < \ldots < t_k = 1$ of $[0,1]$ such that $\omega|_{[t_{i-1}, t_i]} \in \mathcal{C}^\infty([t_{i-1}, t_i], M)$.
(b) $\omega(0) = p$ and $\omega(1) = q$

The collection of all those paths is denoted by

$$\Omega(M; p, q)$$

When there is no chance of confusion, we also write $\Omega := \Omega(M; p, q)$.

2.2 Definition (Vector fields along curves)**.** Let $\omega \in \Omega$ be a smooth path. A *smooth vector field along ω* is a smooth map $W : [0,1] \to TM$ such that

$$\forall t \in [0,1] : W(t) \in T_{\omega(t)}M$$

If ω is only piecewise smooth, we define a *piecewise smooth vector field along ω* to be a continuous map $W : [0,1] \to TM$ such that there exists a finite partition $0 = t_0 < \ldots < t_k = 1$ such that every $W|_{[t_{i-1}, t_i]}$ is a smooth vector field along $\omega|_{[t_{i-1}, t_i]}$.
We denote by

$$\Gamma(\omega) := \{W : [0,1] \to TM | \text{W is a smooth vector field along } \omega\}$$

From now on we assume M to be endowed with a Riemannian metric. This enables us to define geodesics: The Riemannian metric g induces a unique symmetric compatible linear connection $\nabla : TM \times TM \to TM$, which induces a covariant differentiation operator $D_t : \Gamma(\omega) \to \Gamma(\omega)$ on the vector fields along any smooth path $\omega \in \Omega$. So we may just define

2.3 Definition (Geodesic). A smooth path $\omega \in \Omega$ is a geodesic, if

$$D_t \dot{\omega} \equiv 0$$

We would like to characterize geodesics in terms of length and energy, making rigorous our intuition that geodesics are paths of minimal energy.

2.2 Length and Energy

2.4 Definition (Length Function). For any $0 \leq a \leq b \leq 1$ we can define the *length from a to b* to be

$$L_a^b(\omega) := \int_a^b ||\dot{\omega}(t)||dt$$

The *length* of ω is just

$$L(\omega) := L_0^1(\omega)$$

We can view L as a Function $L : \Omega(M; p, q) \to \mathbb{R}$, called the *length Function.*

2.5 Definition (Energy Function). Analogously, for any $0 \leq a \leq b \leq 1$ we can define the *energy from a to b* to be

$$E_a^b(\omega) := \int_a^b ||\dot{\omega}(t)||^2 dt$$

The *energy* of ω is just

$$E(\omega) := E_0^1(\omega)$$

We can view E as a Function $E : \Omega(M; p, q) \to \mathbb{R}$, called the *energy Function.*

2.6 Lemma (Length/Energy-Relation). Let $\omega \in \Omega$ and $0 \leq a \leq b \leq 1$ be arbitrary. Then

$$L_a^b(\omega)^2 \leq (b - a)E_a^b(\omega)$$

where equality holds if and only if ω has constant speed.

Proof. Consider $f, g \in L^2([0, 1])$ defined by $f(t) := 1$, $g(t) := ||\dot{\omega}(t)||$. By the Cauchy/Schwarz - Inequality with respect to the L^2 scalar product, we obtain

$$L_a^b(\omega)^2 = \left(\int_a^b 1 \cdot ||\dot{\omega}(t)||dt\right)^2 = \langle f, g\rangle_{L^2}^2 \leq ||f||_{L^2}^2 ||g||_{L^2}^2 = \int_a^b 1^2 dt \cdot \int_a^b ||\dot{\omega}(t)||^2 dt = (b - a)E_a^b(\omega)$$

Equality holds if and only if f and g are linear dependent which is the case if and only if $||\dot{\omega}(t)||$ is constant. □

2.7 Theorem. Let (M, g) be a complete Riemannian manifold and let $p, q \in M$ have distance $d(p, q) =: d$. Then the energy Function $E : \Omega(M; p, q) \to \mathbb{R}$ attains its infimum d^2 precisely on the set of minimal geodesics from p to q.

Proof. From the definition of E it follows that

$$\inf_{\omega \in \Omega} E(\omega) = \inf_{\omega \in \Omega} \left(\int_0^1 ||\dot{\omega}(t)||^2 dt \right) \geq 0$$

So the at least this infimum always exists. Since M is complete, there exists a minimal geodesic γ from p to q by theorem of Hopf-Rinow, [2, 6.13, 6.15]. This says by definition

$$\inf_{\omega \in \Omega} L(\omega) = L(\gamma) = d(p, q) = d$$

i.e. this infimum is attained by some γ. Since γ is a geodesic and thus has constant speed, we obtain for any $\omega \in \Omega$:

$$E(\gamma) \overset{2.6}{=} L(\gamma)^2 \leq L(\omega)^2 \leq E(\omega)$$

Thus for every minimizing geodesic γ and any path ω, we obtain $E(\omega) \geq E(\gamma)$. Thus $\inf_{\omega \in \Omega}(\omega) \leq E(\gamma)$ as well.
The equality $L(\gamma)^2 = L(\omega)^2$ can hold if and only if ω also is a length minimizer. On the other hand, again by 2.6, the equality $L(\omega)^2 = E(\omega)$ can hold if and only if has constant speed. So unless ω is also a minimal geodesic, we have $E(\gamma) < E(\omega)$ (c.f. [1, 10.7], [2, 6.6]).
Since $E(\gamma) = L(\gamma)^2$ for any minimizing geodesic γ, this shows in particular, that E really attains the infimum d^2, thus the infimum is a minimum. □

3 The Calculus of Variations

Although Ω is not a smooth manifold, we will think of it as some sort of infinite dimensional generalization of a smooth manifold. As a first step, we will change our terminology conventions by setting

3.1 Convention. Let $\omega \in \Omega$. Then we call

$$T_\omega \Omega := \{W : [0, 1] \to TM | \text{W is a piecewise smooth vector field along } \omega \text{ and } W(0) = W(1) = 0\}$$

the *tangent space of* ω *in* Ω

The strategy is now to use an analogous construction as in 1.3. We need the notion of curves in Ω and a notion of their velocity, i.e. their derivatives, . This leads to variations and variation fields.

3.2 Definition (Variation)**.** Let $\omega \in \Omega(M; p, q) =: \Omega$ and $U \subset \mathbb{R}^n$ be a neighbourhood of 0. A map $\bar{\alpha} : U \to \Omega$ is an *n-parameter varation of* ω, if

(a) $\bar{\alpha}(0) = \omega$

(b) There exists a finite subdivision $0 = t_0 < \ldots < t_k = 1$ of $[0, 1]$ such that the map $\alpha : U \times [0, 1] \to M$ defined by

$$(u, t) \mapsto \bar{\alpha}(u)(t)$$

is continuous on $U \times [0, 1]$ and smooth on each $U \times [t_{i-1}, t_i]$ for any $1 \leq i \leq k$.

If $n = 1$ and $U =]-\varepsilon, \varepsilon[$, $\varepsilon > 0$, we just say $\bar{\alpha} :]-\varepsilon, \varepsilon[\to \Omega$ is a *variation of* ω.
Note that for each $u \in U$ we have $a(u, 0) = p$ and $a(u, 1) = q$.

We may think of $\bar{\alpha} :]-\varepsilon, \varepsilon[\to \Omega$ as a "smooth path" in Ω, i.e. as a path of paths. Every such α gives rise to two systems of curves: For any fixed $u \in]-\varepsilon, \varepsilon[$ we obtain a *main curve* $\alpha_u : [0, 1] \to M$, $t \mapsto \alpha(u, t)$, which is only piecewise smooth and for any fixed $t \in [0, 1]$ a *transversal curve* $\alpha_t :]-\varepsilon, \varepsilon[\to M$, $u \mapsto (u, t)$, which is smooth. In fact the later ones are most interesting, because they tell us, how the variation varies the original path. So it is only natural to ask, what their "velocity vector" is. In fact this is the following.

3.3 Definition (Variation field). Let $\bar{\alpha}$ be an n-parameter variation of $\omega \in \Omega$. For any $1 \leq k \leq n$ $W_k : [0,1] \to TM$

$$t \mapsto \partial_k \alpha(u,t)|_{u=0} = \partial_k \alpha(0,t)$$

is the *variation field of* α *in direction* k and $(W_1, \ldots, W_k)$ is the *variation field system of* $\bar{\alpha}$. If $n = 1$ we just say $W := W_1$ is *the variation field of* $\bar{\alpha}$.

Clearly $W \in T_\omega \Omega$: For any $t \in [0,1]$, $x := \alpha(0,t)$ and $f \in \mathcal{C}_x^\infty(M)$ we have

$$W(t)(f) = \partial_u (f(\alpha(u,t))|_{u=0}$$

So any variation field W of a variation $\bar{\alpha}$ of ω is a vector field along ω. The converse is also true. In fact

3.4 Lemma. Let $\omega \in \Omega$ be a path and $(W_1, \ldots, W_n) \in (T_\omega \Omega)^n$ be a system of n vector fields along ω. Then there exists $U \subset \mathbb{R}^n$ open and an n-parameter variation $\bar{\alpha} : U \to \Omega$ of ω such that

$$\partial_k \alpha(u,t)|_{u=0} = W_k(t)$$

Proof. Define $\alpha :]-\varepsilon, \varepsilon[^n \times [0,1] \to M$ to be

$$(u,t) \mapsto \exp_{\omega(t)} \left(\sum_{i=1}^n u_i W_i(t) \right)$$

where $\varepsilon > 0$ is sufficiently small, such that exp is well defined. Fix and index $1 \leq k \leq n$, points $u_1, \ldots, u_{k-1}, u_{k+1}, \ldots, u_n \in]-\varepsilon, \varepsilon[$, a time $t \in I$ and set $p := \omega(t)$. Chose normal coordinates $\varphi : U \to V$ near p. For any $X \in T_pM$ denote by γ_X the unique geodesic through p with inital vector X. In coordinates we may write (c.f. [2, 5.11])

$$\gamma_X(s) = \varphi^{-1} \circ (sX^1, \ldots, sX^m)$$

Consequently, for any $f \in \mathcal{C}_p^\infty(M)$ we can use Lemma 1.4 to compute

$$\begin{aligned}
\partial_k \alpha(u,t)|_{u_k=0}(f) &= \partial_k \left(f \circ \exp_{\omega(t)} \left(\sum_{i=1}^n u_i W_i(t)) \right) \right) |_{u_k=0} \\
&= \partial_k \left(f \circ \gamma_{\sum_{i=1}^n u_i W_i(t)}(1) \right) |_{u_k=0} \\
&= \partial_k \left(f \circ \varphi^{-1} \left(\sum_{i=1}^n s u_i W_i^1(t), \ldots, \sum_{i=1}^n s u_i W_i^m(t))(1) \right) \right) |_{u_k=0} \\
&\overset{1.4}{=} \left(\varphi^{-1} \left(W_k^1(t), \ldots, W_k^m(t) \right) \right)(f) \\
&= \sum_{i=1}^m W_k^i(t) \partial \varphi_i|_{\omega(t)}(f) \\
&= W_k(f)
\end{aligned}$$

□

In case $n = 1$ there is a more standard version of this Lemma, which we discuss here as well. The proof is a bit simpler.

3.5 Lemma. Let $W \in T_\omega \Omega$ be arbitrary. Then there exists a variation $\bar{\alpha} :]-\varepsilon, \varepsilon[\to \Omega$, such that its variation field satisfies

$$\partial_u \alpha(0,t) = W(t)$$

Proof. Define $\alpha :]-\varepsilon, \varepsilon[\times[0,1] \to M$ to be

$$(u,t) \mapsto \exp_{\omega(t)}(uW(t))$$

For any $p \in M$, $X \in T_pM$ denote by γ_V the unique geodesic through p with inital vector X. The properties of the exponential map imply:

$$\partial_u \alpha(u,t)|_{u=0} = \partial_u(\exp_{\omega(t)}(uW(t)))|_{u=0} = \partial_u(\gamma_{uW(t)}(1))|_{u=0} \overset{(*)}{=} \partial_u(\gamma_{W(t)}(u))|_{u=0} = W(t)$$

The keystep $(*)$ uses the Rescaling Lemma, which can be found in [2, 5.8]. □

3.6 Definition (Critical path)**.** A path $\omega \in \Omega$ is a *critical path* for a function $F : \Omega \to \mathbb{R}$, if for any variation $\bar{\alpha}$ of ω

$$\partial_u(F \circ \bar{\alpha})|_{u=0}$$

exists and equals zero.

3.1 First Variation Formula

3.7 Convention. For any piecewise continuous vector field $V : [0,1] \to TM$ along any $\omega \in \Omega$ and, we define

$$\forall 0 < t \leq 1 : V(t \swarrow) := \lim_{s \searrow t} V(s)$$
$$\forall 0 \leq t < 1 : V(\nearrow t) := \lim_{s \nearrow t} V(s)$$
$$\forall 0 < t < 1 : \Delta_t V := V(t \swarrow) - V(\nearrow t)$$

Since there is a partition $0 = t_0 < \ldots < t_k = 1$ of $[0,1]$ such that V is continuous on each $[t_{i-1}, t_i]$, $\Delta_t V = 0$ for all but the finitely many $t_1, \ldots, t_{k-1}$. We denote these vectors by $\Delta_i V := \Delta_{t_i} V$.

3.8 Theorem (First Variation Formula)**.** Let (M, g) be a Riemannian manifold $p, q \in M$, $\omega \in \Omega(M; p, q) = \Omega$ and let $\bar{\alpha} :]-\varepsilon, \varepsilon[\to \Omega$ be a variation of ω. Let $0 = t_0 < \ldots < t_k = 1$ be the partition of $[0,1]$ such that $\alpha \in \mathcal{C}^\infty(]-\varepsilon, \varepsilon \times [t_{i-1}, t_i], M)$. Let $W(t) := \partial_u \alpha(u,t)|_{u=0}$ be the associated variation field, $V(t) := \dot{\omega}(t)$ the velocity vector of ω, and $A(t) := D_t V(t)$ be the acceleration vector of ω. Then the *first varation formula* holds:

$$\frac{1}{2}\partial_u E(\bar{\alpha}(u))|_{u=0} = -\sum_{i=1}^{k-1} \langle W(t_i), \Delta_i V \rangle - \int_0^1 \langle W(t), A(t) \rangle dt$$

Proof. Since the Riemannian connection is by definition compatible with the Riemannian metric we have

$$\partial_u \langle \partial_t \alpha, \partial_t \alpha \rangle \overset{A.1}{=} \langle D_u \partial_t \alpha, \partial_t \alpha \rangle + \langle \partial_t \alpha, D_u \partial_t \alpha \rangle = 2\langle D_u \partial_t \alpha, \partial_t \alpha \rangle \overset{A.2}{=} 2\langle D_t \partial_u \alpha, \partial_t \alpha \rangle$$

The identity

$$\partial_t \langle \partial_u \alpha, \partial_t \alpha \rangle = \langle D_t \partial_u \alpha, \partial_t \alpha \rangle + \langle \partial_u \alpha, D_t \partial_t \alpha \rangle \tag{1}$$

again obtained by the covariant product rule (Lemma A.1), allows us to to the following "integration by parts":

$$\begin{aligned} \int_{t_{i-1}}^{t_i} \langle D_t \partial_u \alpha, \partial_t \alpha \rangle dt &= \int_{t_{i-1}}^{t_i} \partial_t \left(\langle \partial_u \alpha, \partial_t \alpha \rangle \right) - \langle \partial_u \alpha, D_t \partial_t \alpha \rangle dt \\ &= \langle \partial_u \alpha, \partial_t \alpha \rangle |_{t_{i-1} \swarrow}^{\nearrow t_i} - \int_{t_{i-1}}^{t_i} \langle \partial_u \alpha, D_t \partial_t \alpha \rangle dt \end{aligned} \tag{2}$$

Putting all this together, we obtain

$$\begin{aligned}
\frac{1}{2}\partial_u E(\bar{\alpha}(u))|_{u=0} &= \frac{1}{2}\partial_u \int_0^1 ||\partial_t \alpha(u,t)||^2 dt|_{u=0} \overset{(*)}{=} \frac{1}{2}\int_0^1 \partial_u \langle \partial_t\alpha(u,t), \partial_t\alpha(u,t)\rangle|_{u=0} dt \\
&\overset{(1)}{=} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \langle D_t \partial_u \alpha(u,t), \partial_t \alpha(u,t)\rangle|_{u=0} dt \\
&\overset{(2)}{=} \sum_{i=1}^{k} \left(\langle \partial_u\alpha(u,t), \partial_t\alpha(u,t)\rangle|_{t_{i-1}\nwarrow}^{\nearrow t_i}|_{u=0} - \int_{t_{i-1}}^{t_i} \langle \partial_u\alpha(u,t), D_t\partial_t\alpha(u,t)\rangle|_{u=0} dt \right) \\
&= \sum_{i=1}^{k} \langle W(t_i), V(\nearrow t_i)\rangle - \langle W(t_{i-1}), V(t_{i-1} \swarrow)\rangle - \int_0^1 \langle W(t), A(t)\rangle dt \\
&\overset{(**)}{=} -\sum_{i=1}^{k-1} (-\langle W(t_i), V(\nearrow t_i)\rangle + \langle W(t_i), V(t_i \swarrow)\rangle) - \int_0^1 \langle W(t), A(t)\rangle dt \\
&= -\sum_{i=1}^{k-1} \langle W(t_i), \Delta_i V\rangle - \int_0^1 \langle W(t), A(t)\rangle dt
\end{aligned}$$

In $(*)$ we have used Lebesgue dominated convergence theorem or more directly the theorem concerning differentiation of parameter dependent integrals. This is possible here since $(u,t) \mapsto ||\alpha(u,t)||$ is piecewise $\mathcal{C}^1$ and $[0,1]$ is compact. This is the point of the proof where we actually show that the derivative $\partial_u(E(\bar{\alpha}(u)))$ exists.
In $(**)$ we have reindexed the sum using the fact that $W(0) = W(1) = 0$. □

3.9 Theorem. A path $\omega \in \Omega$ is a critical point for E if and only if ω is a geodesic.

Proof. "$\Leftarrow$": If ω is a geodesic, then it is smooth, so we may choose $t_0 = 0 < t_1 = 1$ as a partition of $[0,1]$. By definition the accerleration vector $A(t) = D_t\dot{\omega}(t) = 0$ and thus

$$\frac{1}{2}\partial_u E(\bar{\alpha}(u))|_{u=0} = -\sum_{i=1}^{k-1} \langle W(t_i), \Delta_i V\rangle - \int_0^1 \langle W(t), A(t)\rangle dt = 0$$

"$\Rightarrow$": Suppose

$$0 = \frac{1}{2}\partial_u E(\bar{\alpha}(u))|_{u=0} = -\sum_{i=1}^{k-1} \langle W(t_i), \Delta_i V\rangle - \int_0^1 \langle W(t), A(t)\rangle dt$$

Construct a variation field $W \in T_\omega\Omega$ such that

$$W(t) = f(t)A(t)$$

where $f : I \to \mathbb{R}$ is piecewise smooth and positive everywhere except that it vanishes at the t_i, and $A(t) = D_t\dot{\omega}$ is the acceleration of ω. (Remember, that Lemma 3.5 allows us to write any vector field along ω as the variation field of a variation.) Then

$$0 = -\int_0^1 f(t)\langle A(t), A(t)\rangle dt = -\int_0^1 f(t)||A(t)||^2 dt$$

This implies $\forall t \in [0,1] : f(t)||A(t)|| = 0$. By hypothesis for any $t \notin \{t_1, \ldots, t_{k-1}\}$

$$f(t) > 0 \Rightarrow A(t) = 0$$

Piecewise continuity implies $A \equiv 0$. Thus every $\omega|_{[t_{i-1},t_i]}$ is a geodesic.
To show that ω is smooth on the whole of $[0,1]$, pick a variation field W such that $W(t_i) = \Delta_i V$, $1 \leq i \leq k-1$. This implies

$$0 = -\sum_{i=1}^{k-1} \langle \Delta_i V, \Delta_i V \rangle$$

Thus $\Delta_i V = 0$. So it follows that ω is at least of class $\mathcal{C}^1$.
Chose any $1 \leq i \leq k-1$. Since ω is of class $\mathcal{C}^1$, there exists a well defined velocity vector $V := \dot\omega(t_i)$. By the existence and uniqueness theorem for geodesics (see A.4) there exists $\varepsilon > 0$ and a geodesic $\gamma_V :]t_i - \varepsilon, t_i + \varepsilon[\to M$ such that $\gamma_V(t_i) = \omega(t_i)$ and $\dot\gamma(t_i) = V$. In particular γ_V is smooth and from the uniqueness statement (after shrinking ε if necessary), it follows that:

$$\forall t \in]t_i - \varepsilon, t_i + \varepsilon[: \gamma_V(t) = \omega(t)$$

So in particular ω is smooth in a neighbourhood of t_i as well, hence of class $\mathcal{C}^\infty$.

□

3.2 The Energy Hessian and the Second Variation Formula

3.10 Definition (Hessian of a function)**.** Let $f \in \mathcal{C}^\infty(M)$ with critical point $p \in M$. The map $f_{**} : T_pM \times T_pM \to \mathbb{R}$ which is defined as follows is the *Hessian of f at p*: For $X_1, X_2 \in T_pM$ choose an open neighbourhood $V \subset \mathbb{R}^2$ and a smooth map $\alpha : V \to M$ such that

$$\alpha(0,0) = p \qquad \partial_1\alpha(0,0) = X_1 \qquad \partial_2\alpha(0,0) = X_2$$

Then define

$$f_{**}(X_1, X_2) := \partial_1\partial_2(f \circ \alpha)(0,0)$$

3.11 Remark. Like in 1.7 other definitions are possible. The most general one would be to define the Hessian of f to be the 2-tensor field

$$\nabla^2 f(X,Y) := X(Y(f)) - (\nabla_Y X)(f)$$

(see [2, (4.8)]. Note that if f has a critical point at $p \in M$, we may choose a chart φ at p and calculate

$$\begin{aligned}(\nabla_Y X)|_p(f) &= Y^i(p)(\nabla_{\partial\varphi_i}(X^j \partial\varphi_j))|_p(f) \\ &= Y^i(p)(\partial\varphi_i|_p(X_j)\partial\varphi_j|_p(f) + X^j(p)(\nabla_{\partial\varphi_i}\varphi_j)|_p(f)) \\ &= Y^i(p)(\partial\varphi_i|_p(X_j)\partial\varphi_j|_p(f) + X^j(p)\Gamma^k_{i,j}\partial\varphi_k|_p(f)) \\ &= 0\end{aligned}$$

Thus at a critical point p

$$\nabla^2 f(X,Y)|_p = X|_p(Y(f))$$

which at least agrees with Milnors own definition in [1, I.§1]. Writing X_1 and X_2 as in 3.10 we see that all these notions agree:

$$\nabla^2 f(X_1, X_2)|_p = X_1(X_2(f)) = X_1((\partial_2\alpha)(f)) = X_1(\partial_2(f \circ \alpha)) = \partial_1\partial_2(f \circ \alpha)(0,0) = f_{**}(X_1, X_2)$$

Motivated by 3.10 we may now extend this definition to our energy function.

3.12 Definition (Energy Hessian)**.** Let (M,g) be a Riemannian manifold and $E : \Omega(M;p,q) \to \mathbb{R}$ be the Energy Function. Then the *Hessian of the Energy Function* at a critical path $\omega \in \Omega$ is a map

$E_{**} : T_\omega\Omega \times T_\omega\Omega \to \mathbb{R}$ defined as follows: Given vector fields $W_1, W_2 \in T_\omega\Omega$, choose a 2-parameter variation $\alpha : U \times [0,1] \to M$, where $U \subset \mathbb{R}^2$ is an open neighbourhood of $(0,0)$, such that

$$\alpha(0,0,t) = \omega(t) \qquad \partial_1\alpha(0,0,t) = W_1(t) \qquad \partial_2\alpha(0,0,t) = W_2(t)$$

Then define

$$E_{**}(W_1, W_2) := \partial_1\partial_2(E \circ \bar{\alpha})(0,0)$$

where $\bar{\alpha}(u_1, u_2) \in \Omega$ is the path $\bar{\alpha}(u_1,u_2)(t) = \alpha(u_1,u_2,t)$.

Of course we have to show, that this is well defined, i.e. it does not depend on the chosen variation.

3.13 Theorem (Second Variation Formula)**.** Let (M,g) be a Riemannian manifold and γ be a geodesic. Let $\bar{\alpha} : U \to \Omega$ be a 2-parameter variation of γ with variation fields

$$W_k := \partial_k\bar{\alpha}(0,0) \in T_\gamma\Omega \qquad k = 1,2$$

Denote by $V := \dot{\gamma}$ the velocity vector field and chose a partition $0 = t_0 < \ldots < t_k = 1$ such that D_tW_1 is smooth on each subinterval $[t_{i-1}, t_i]$. Then the *second variation formula* holds:

$$\frac{1}{2}\partial_1\partial_2(E(\bar{\alpha}(0,0)) = -\sum_{i=1}^{k-1}\langle W_2(t_i), \Delta_i D_t W_1\rangle - \int_0^1 \langle W_2(t), D_t^2 W_1(t) + R(V, W_1)(V)\rangle dt$$

Proof. For each fixed u_1, $u_2 \mapsto \bar{\alpha}(u_1,u_2)$ is a one parameter variation of $\bar{\alpha}(u_1, 0)$. So according to the first variation formula (Theorem 3.8), we have

$$\frac{1}{2}\partial_2 E(\bar{\alpha}(u_1,0)) = -\sum_{i=1}^{k-1}\langle \partial_2\alpha(u_1,0,t_i), \Delta_i\partial_t\bar{\alpha}(u_1,0)\rangle - \int_0^1 \langle \partial_2\alpha(u_1,0,t), D_t\partial_t\alpha(u_1,0,t)\rangle dt$$

We will now differentiate this again and insert the various definitions. This is a bit complicated, but straightfoward. We will give the complete calculation first and mark some points, which will be explained in more detail below.

$$\begin{aligned}
\frac{1}{2}\partial_1\partial_2 E(0,0) &= -\sum_{i=1}^{k-1}\langle D_1\partial_2\alpha(0,0,t_i), \underbrace{\Delta_i\partial_t\bar{\alpha}(0,0)}_{(1)}\rangle - \sum_{i=1}^{k-1}\langle \partial_2\alpha(0,0,t_i), D_1\Delta_i\partial_t\alpha(0,0,t_i)\rangle \\
&\quad - \int_0^1 \langle D_1\partial_2\alpha(0,0,t), \underbrace{D_t\partial_t\alpha(0,0,t)}_{(2)}\rangle dt - \int_0^1 \langle \partial_2\alpha(0,0,t), D_1D_t\partial_t\alpha(0,0,t)\rangle dt \\
&= -\sum_{i=1}^{k-1}\langle W_2(t_i), \underbrace{D_1\Delta_i\partial_t\alpha(0,0,t_i)}_{(3)}\rangle - \int_0^1 \langle W_2(t), \underbrace{D_1D_tV(t)}_{(4)}\rangle \\
&= -\sum_{i=1}^{k-1}\langle W_2(t_i), \Delta_i D_t W_1(t)\rangle - \int_0^1 \langle W_2(t), R(V(t), W_1(t))(V(t)) + \underbrace{D_tD_1V(t)}_{(5)}\rangle dt \\
&= -\sum_{i=1}^{k-1}\langle W_2(t_i), \Delta_i D_t W_1(t)\rangle - \int_0^1 \langle W_2(t), R(V(t), W_1(t))(V(t)) + D_t^2 W_1(t)\rangle dt
\end{aligned}$$

Auxiliary Calculations:

(1) Since $\gamma = \bar{\alpha}(0,0)$ is a geodesic and hence smooth, we have

$$\Delta_i \partial_t \bar{\alpha}(0,0) = 0$$

Thus the entire sum vanishes:

$$-\sum_{i=1}^{k-1} \langle D_1 \partial_2 \alpha(0,0,t_i), \Delta_i \partial_t \bar{\alpha}(0,0) \rangle = 0$$

(2) Again since γ is a geodesic

$$D_t \partial_t \bar{\alpha}(0,0) = 0$$

thus the entire integral vanishes:

$$-\int_0^1 \langle D_1 \partial_2 \alpha(0,0,t), D_t \partial_t \alpha(0,0,t) \rangle dt = 0$$

(3) By the symmetry Lemma A.2

$$D_1 \partial_t \alpha(0,0,t) = D_t \partial_1 \alpha(0,0,t) = D_t W_1(t)$$

The operator Δ_i commutes with D_1 by Lemma A.5.

(4) This is the curvature formula A.3 from the appendix

$$D_1 D_t V - D_t D_1 V = R(V, W_1) V$$

(5) This is the symmetry relation from A.2 from the appendix

$$D_1 V = D_1 \partial_t \alpha = D_t \partial_1 \alpha = D_t W_1$$

□

3.14 Corollary. The map $E_{**} : T_\gamma M \times T_\gamma M \to \mathbb{R}$ is well defined, symmetric and bilinear.

Proof. Theorem 3.13 shows, that $\partial^2 E(0,0)$ always exists and depends only on W_1, W_2, i.e. it does not on the chosen variation.
The bilinearity also follows from the second variation formula since this formula is bilinear in (W_1, W_2). The symmetry is not at all obvious from this formula, but it follows directly from the theorem of Schwarz and some other basic calculus outsourced into lemma 3.15: Given vector fields $W_1, W_2 \in T_\omega \Omega$, choose a 2-parameter variation $\alpha : U \times [0,1] \to M$, where $U = I^2 \subset \mathbb{R}^2$ and $I \subset \mathbb{R}$ is an open invterval containing $0 \in \mathbb{R}$, such that

$$\alpha(0,0,t) = \omega(t), \qquad \partial_1 \alpha(0,0,t) = W_1(t), \qquad \partial_2 \alpha(0,0,t) = W_2(t)$$

Define $\bar{\beta} : U \to \Omega$ by $\bar{\beta}(u_1, u_2) := \bar{\alpha}(u_2, u_1)$. Then

$$\beta(0,0,t) = \omega(t), \qquad \partial_1 \beta(0,0,t) = W_2(t), \qquad \partial_2 \beta(0,0,t) = W_1(t)$$

Since $\partial^2 E(0,0)$ does not depend on the chosen variation, we may use $\bar{\beta}$ to compute (introducing the notation of 3.15)

$$\begin{aligned} E_{**}(W_2, W_1) &= \partial_1 \partial_2 (E \circ \bar{\beta})(0,0) = \partial_1 \partial_2 (E \circ \bar{\alpha} \circ s)(0,0) \overset{3.15}{=} \partial_2 \partial_1 (E \circ \bar{\alpha})(s(0,0)) \\ &= \partial_2 \partial_1 (E \circ \bar{\alpha})(0,0) \overset{\text{Schwarz}}{=} \partial_1 \partial_2 (E \circ \bar{\alpha})(0,0) = E_{**}(W_1, W_2) \end{aligned}$$

□

3.15 Lemma. Let $s : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (y, x)$ be the swap of arguments. Clearly $s \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2)$. Consider any open Intervall $I \subset \mathbb{R}$, define $U := I^2$ and consider any function $f \in \mathcal{C}^2(U, \mathbb{R})$. Then

$$\partial_1\partial_2(f \circ s) = (\partial_2\partial_1 f) \circ s \qquad \partial_2\partial_1(f \circ s) = (\partial_1\partial_2 f) \circ s$$

Proof. Define $g := f \circ \tau$. Since $s(U) = U$ for any $(x, y) \in U$, the following expressions are all well defined and the chain rule implies

$$\begin{aligned}(\partial_1 g(x, y), \partial_2 g(x, y)) &= \nabla g(x, y) = \nabla f(s(x, y))\nabla s(x, y) \\ &= (\partial_1 f(y, x), \partial_2 f(y, x)) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (\partial_2 f(y, x), \partial_1 f(y, x))\end{aligned}$$

Thus

$$\partial_1 g = (\partial_2 f) \circ s \qquad \partial_2 g = (\partial_1 f) \circ s$$

Repeating this procedure for $\partial_1 g$ instead of g we obtain

$$\nabla\partial_1 g(x, y) = \nabla(\partial_2 f)(s(x, y))\nabla s(x, y) = (\partial_2\partial_2 f(y, x), \partial_1\partial_2 f(y, x))$$

and for $\partial_2 g$

$$\nabla\partial_2 g(x, y) = \nabla(\partial_1 f)(s(x, y))\nabla s(x, y) = (\partial_2\partial_1 f(y, x), \partial_1\partial_1 f(y, x))$$

Alltogether we obtain

$$\partial_2\partial_1 g(x, y) = \partial_1\partial_2 f(y, x) \qquad \partial_1\partial_2 g(x, y) = \partial_2\partial_1 f(y, x)$$

□

Remember that the *index* of a bilinear form is the dimension of a maximal negative definite subspace.

3.16 Theorem. Let $\gamma \in \Omega$ be a minimal geodesic and $\bar{\alpha} :]-\varepsilon, \varepsilon[\to \Omega$ be a variation of γ with variation field W. The diagonal terms of the Energy Hessian can be described as

$$E_{**}(W, W) = \partial_u^2(E \circ \bar{\alpha})(0)$$

The pairing $E_{**} : T_\gamma\Omega \times T_\gamma\Omega \to \mathbb{R}$ is positive semi-definite and hence has index zero.

Proof. For the first statement just define a two parameter variation $\bar{\beta} :]-\varepsilon, \varepsilon[^2 \to \Omega$

$$\bar{\beta}(u_1, u_2) := \bar{\alpha}(u_1 + u_2)$$

denote $h : \mathbb{R}^2 \to \mathbb{R}$, $(u_1, u_2) \mapsto u_1 + u_2$, and use the chain rule:

$$\nabla(E \circ \bar{\beta})(u_1, u_2) = \nabla(E \circ \bar{\alpha} \circ h)(u_1, u_2) = \partial_u(E \circ \bar{\alpha})(u_1 + u_2) \begin{pmatrix} 1 & 1 \end{pmatrix} = \big(\partial_u(E \circ \bar{\alpha}), \partial_u(E \circ \bar{\alpha})\big)(u_1 + u_2)$$

So for $i \in \{1, 2\}$

$$\partial_i(E \circ \bar{\beta})(u_1, u_2) = (\partial_u(E \circ \bar{\alpha}) \circ h)(u_1, u_2)$$

Thus by the same argumentation

$$\partial_1\partial_2(E \circ \bar{\beta})(0, 0) = \partial_u^2(E \circ \bar{\alpha})(0)$$

The second statement now follows from the first one: Since $E(\bar{\alpha}(u)) \geq E(\gamma) = E(\bar{\alpha}(0))$ the function $u \mapsto E(\bar{\alpha}(u))$ has a local minimum at zero. Since γ is a geodesic the first derivative

$$\partial_u E(\bar{\alpha}(u))|_{u=0}$$

is zero by Theorem 3.9. Thus by elemental calculus

$$\partial_u^2(E \circ \bar{\alpha})(u)|_{u=0} \geq 0$$

because $\partial_u^2(E \circ \bar{\alpha})(u)|_{u=0} < 0$ would imply that $E(\bar{\alpha}(u))$ had a local maximum at zero. □

A Notation and Prerequisites from Riemannian Geometry

A.1 Lemma (Covariant Product Rule)**.** Let $\omega \in \Omega$ and $V, W \in T_\omega\Omega$. Then

$$\partial_t\langle V(t), W(t)\rangle = \langle D_tV(t), W(t)\rangle + \langle V(t), D_tW(t)\rangle$$

Proof. This is the compatibility condition used to define D. □

A.2 Lemma (Symmetry Lemma)**.** Let $\omega \in \Omega$ and $\bar{\alpha} :]\varepsilon, \varepsilon[\to \Omega$ be a variation of ω. Then the function $\alpha :] - \varepsilon, \varepsilon[\times [0, 1] \to M$, $(u, t) \mapsto \bar{\alpha}(u)(t)$ satisfies

$$D_u\partial_t\alpha(u, t) = D_t\partial_u\alpha(u, t)$$

Proof. Can be found in [1, 8.7] or [2, 6.3]. □

A.3 Lemma (Curvature Formula)**.** Let $\omega \in \Omega$, $V \in T_\omega\Omega$ and $\bar{\alpha} :]\varepsilon, \varepsilon[\to \Omega$ be a variation of ω. Then the function $\alpha :] - \varepsilon, \varepsilon[\times [0, 1] \to M$, $(u, t) \mapsto \bar{\alpha}(u)(t)$ satisfies

$$D_uD_tV - D_tD_uV = R(\partial_u\bar{\alpha}, \partial_t\bar{\alpha})(V)$$

Proof. Can be found in [1, 9.2] or [2, 10.1] □

A.4 Theorem (Existence and Uniqueness of Geodesics)**.** Let M be a Riemannian manifold. For any $p \in M$, any $V \in T_pM$ and any $t_0 \in \mathbb{R}$ there exists an open Intervall $I \subset \mathbb{R}$, $t_0 \in I$, and a geodesic $\gamma_V : I \to M$ satisfying $\gamma_V(t_0) = p$ and $\dot{\gamma_V}(t_0) = V$. Any two such geodesics agree on their common domain.

Proof. Can be found in [2, 4.10]. □

A.5 Lemma. Let $\alpha :] - \varepsilon, \varepsilon[\times [0, 1]$, $(u, t) \mapsto \alpha(u, t)$, be a two parameter variation of $\omega \in \Omega$. Let $0 = t_0 < \ldots < t_k = 1$ be a partition of $[0, 1]$ such that $\omega|_{[t_{i-1},t_i]}$ is smooth. For any $1 \leq i \leq k - 1$

$$D_u(\lim_{t\nearrow t_i} \partial_t\alpha(u, 0))(0) = \lim_{t\nearrow t_i} D_u(\partial_t\alpha(u, 0))(0)$$

Proof. Denote by $V(u, t) := \partial_t\alpha(u, 0)$ the velocity fields of α. Choose local coordinates near $\alpha(0, t_i)$, denote by E_k the local coordinate frame and by $\Gamma^k_{i,j}$ the Christoffel symbols of the Riemannian connection in this coordinate frame. By the explicit formula for covariant differentiation, which can e.g. be found in [2, (4.10)], we have

$$\begin{aligned}
\lim_{t\nearrow t_i} D_u(\partial_t\alpha(u, 0))(0) &= \lim_{t\nearrow t_i} \left(\left(\partial_uV^k(0, t) + V^j(0, t)\partial_u\alpha(0, t)\Gamma^k_{i,j}(\alpha(0, t)) \right) E_k \right) \\
&= \left(\lim_{t\nearrow t_i} (\partial_uV^k(0, t)) + \lim_{t\nearrow t_i} V^j(0, t)\partial_u\alpha(0, t)\Gamma^k_{i,j}(\alpha(0, t)) \right) E_k \\
&= \left(\partial_u \left(\lim_{t\nearrow t_i} V^k(0, t) \right) + \lim_{t\nearrow t_i} V^j(0, t)\partial_u\alpha(0, t)\Gamma^k_{i,j}(\alpha(0, t)) \right) E_k \\
&= D_u(\lim_{t\nearrow t_i} \partial_t\alpha(u, 0))(0)
\end{aligned}$$

Where we have used the fact, that $V(u, t)$ is smooth on the strip $] - \varepsilon, \varepsilon[\times [t_{i-1}, t_i]$. □

References

[1] Milnor, John: ”Morse Theory”

[2] Lee, John M.: ”Riemannian Manifolds - Introduction to Curvature”