# The de Rham Isomorphism and the $L_p$-Cohomology of non-compact Riemannian Manifolds

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## 1 Introduction

- Vorstellung: Name, Thema, Uni Bonn, Fachgebiet Differentialgeometrie, Betreuer Prof. Ballmann, Dr. Jan Swoboda.
- Arbeit basiert auf Paper
- Gol'dshtein, Kuz'minov, Shvedov: *De Rham Isomorphism of the $L_p$-Cohomology of non-compact Riemannian Manifolds*, Siberian Math. J. 29 (1988), no. 2, 190–197
- Hat nur 8 Seiten. Arbeit hat 130 S., daher nur Überblick über das Thema.
- Ziel ist Formulierung des Hauptresultats.
- Arbeit online verfügbar.

## 2 Overview and Main Result

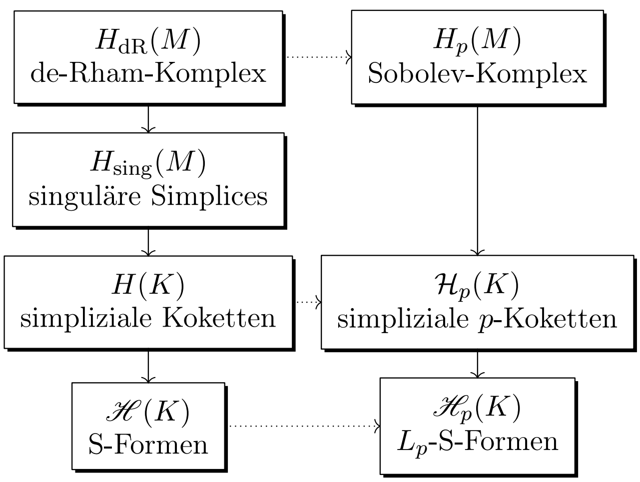


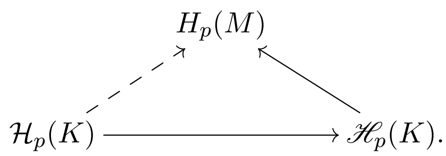

Figure 2.1: Kohomologie

**Catalogue of Prerequesites:**

- $M$ be a smooth oriented Riemannian $m$- manifold without boundary.
- $K$ be a simplicial complex in some $\mathbb{R}^n$ with star-bound $N$ (c.f. 3.9).
- Let $h : |K| \to M$ be a smooth triangulation of $M$
- $1 \leq p < \infty$

Warning: $L_p$ cohomology theories do not satisfy the Eilenberg/Steenrod Axioms.

**2.1 Main Theorem.** If $h : |K| \to M$ is GKS (c.f. 4.1), then there exists a commutative diagram of isomorphisms




Therefore all $L_p$-cohomolgies of $M$ are mutually isomorphic.

# 3 Definition of $L_p$-spaces

## 3.1 Differential Forms

**3.1 Definition** **($L_p$-spaces).** Let

$$L^k(M) := \{\omega : M \to \Lambda^k TM \mid \omega \text{ is a measurable section}\}/\sim,$$

where $\omega \sim \omega'$ if and only if $\omega$ equals $\omega'$ outside a set of measure zero. For any $1 \leq p < \infty$, define

$$L^k_p(M) := \{\omega \in L^k(M) \mid \|\omega\|^p_{L^k_p(M)} := \int_M |\omega|^p d_g V < \infty\}$$

the *p-integrable forms.* In case $p = \infty$ the norm is replaced by

$$\|\omega\|_{L^k_\infty(M)} := \text{ess sup}_{x\in M} |\omega(x)|.$$

Sometimes it is nice to have the following local version of $L_p$-spaces:

$$L^k_{p,\text{loc}}(M) := \{\omega \in L^k(M) \mid \forall K \subset M \text{ compact } : \omega \in L^k_p(K)\}.$$

The *modulus* $|\omega|$ of a differential form is defined below.

**3.2 Definition** **(modulus).** For any Riemannian metric $g$ on $M$ there exists exactly one fibre metric $\tilde{g}$ on $\Lambda^k TM$ such that for any $g$-ONB $B = (b_1, \ldots, b_k)$ of any $T_pM$ the set

$$\Lambda^k B := \{b_{i_1} \wedge \ldots \wedge b_{i_k} \mid 0 \leq i_1 < \ldots < i_k \leq m\}$$

is a $\tilde{g}$-ONB for $\Lambda^k T_pM$. Denote by $|\_|$ the norm generated by $\tilde{g}$. Then $|\omega| : M \to \mathbb{R}$ is the *modulus* of $\omega$.

**3.3 Theorem** **(completeness of $L_p$-spaces).** For every $1 \leq p \leq \infty$ and every $0 \leq k \leq m$ the space $L^k_p(M)$ is a Banach space.

**3.4 Definition** **(weak differential).** Let $\omega \in L^k_{p,\text{loc}}(M)$ and $\omega' \in L^{k+1}_{p,\text{loc}}(M)$. Then $\omega'$ is a *weak differential* of $\omega$ if

$$\forall \eta \in \Omega^{m-k-1}_c(M) : \int_M \omega' \wedge \eta = (-1)^{k+1} \int_M \omega \wedge d\eta.$$

In that case we denote $d\omega := \omega'$. The space of all those forms is denoted by $W^k_{p,\text{loc}}(M)$.

**3.5 Lemma.** The weak differential of a form $\omega \in L^k_{p,\text{loc}}(M)$ is uniquely determined (if it exists). If $\omega$ is smooth, the weak differential equals the exterior differential.

**Proof**. The first claim follows from the Fundamental Lemma of the calculus of variations. For the second, notice that by Stokes' theorem and Leibniz rule

$$\forall \eta \in \Omega^{m-k-1}_c(M) : 0 = \int_{\partial M} \omega \wedge \eta = \int_M d(\omega \wedge \eta) = \int_M d\omega \wedge \eta + (-1)^k \int_M \omega \wedge d\eta. \qquad \square$$

**3.6 Definition** **(exterior Sobolev spaces).** Employing the notation

$$\|\omega\|^p_{W^k_p(M)} := \|\omega\|^p_{L^k_p(M)} + \|d\omega\|^p_{L^{k+1}_p(M)}, \qquad \|\omega\|_{W^k_\infty(M)} := \max\{\|\omega\|_{L^k_\infty(M)}, \|\omega\|_{L^{k+1}_\infty(M)}\},$$

we define the *(exterior) Sobolev spaces*

$$W^k_p(M) := \{\omega \in W^k_{p,\text{loc}}(M) \mid \|\omega\|_{W^k_p(M)} < \infty\}.$$

**3.7 Lemma (completeness of exterior Sobolev spaces).** For every $1 \le p \le \infty$ and every $0 \le k \le m$ the map $d : W_p^k(M) \to W_p^{k+1}(M)$ is a bounded linear operator between Banach spaces.

**3.8 Definition ($L_p$-cohomology).** The cohomology of the exterior Sobolev spaces called the *$L_p$-complex of $M$*. Its cohomology groups

$$H_p^k(M) := \frac{Z_p^k(M)}{B_p^k(M)} := \frac{\ker\left(d^k : W_p^k(M) \to W_p^{k+1}(M)\right)}{\operatorname{im}\left(d^k : W_p^{k-1}(M) \to W_p^k(M)\right)} = H^k((W_p(M), d))$$

are called *$L_p$-cohomology of $M$*. This is a $\mathbb{Z}$-indexed system of vector spaces endowed with the ordinary quotient semi-norm $\|\_\|_{H_p^k}$ induced by $\|\_\|_{W_p^k}$. The spaces

$$\bar{H}_p^k(M) := \frac{Z_p^k(M)}{\overline{B_p^k(M)}} \cong \frac{H_p^k(M)}{\{x \in H_p^k(M) \,|\, \|x\|_{H_p^k} = 0\}}$$

are called *reduced $L_p$-cohomology of $M$*. The space

$$T_p^k(M) := \frac{\overline{B_p^k(M)}}{B_p^k(M)}$$

is the *Torsion of $M$*.

## 3.2 Simplicial $L_p$-cohomology

We assume the audience to be familiar with the basic notions about simplices and simplicial complexes. Therefore we will briefly recall some selected definitions of particular importance.

**3.9 Definition (star-bounded).** Let $K$ be a simplicial complex and $S \subset K$ be an arbitrary subset. The *star* of $S$ in $K$ is the set

$$\operatorname{st}(S) := \operatorname{st}_K(S) := \{\sigma \in K \,|\, \exists \tau \in S : \tau \le \sigma\}.$$

A simplicial complex $K$ is *star-bounded* $N$, if the stars of all the simplices in $K$ contain no more than $N$ simplices, i.e.

$$\exists N \in \mathbb{N} : \forall \sigma \in K : \sharp \operatorname{st}_K(\sigma) \le N.$$

**3.10 Definition (simpliziale $L_p$-Kohomologie).** Es sei $(C_\bullet(K), d_\bullet)$ der simpliziale Kettenkomplex von $K$ und $(C^\bullet(K), d^\bullet) = \operatorname{Hom}_{\mathbb{R}}((C_\bullet(K), d_\bullet))$ der simpliziale Kokettenkomplex von $K$. Wir nennen

$$C_p^k(K) := \left\{ c \in C^k(K) \;\middle|\; \|c\|_{C_p^k(K)}^p := \sum_{\sigma \in K^{(k)}} |c(\sigma)|^p < \infty \right\}$$

die *$k$-te simpliziale $L_p$-Kokettengruppe von $K$*, wobei wir mit $K^{(k)}$ die $k$-Simplices von $K$ bezeichnen. Es ist dann

$$\mathcal{H}_p^k(K) := H^k(C_p^\bullet(K), d^\bullet)$$

die *simpliziale $L_p$-Kohomologie von $K$*.

### 3.3 S-Forms

**3.11 Definition ($S$-form).** Let $K$ be a star-bounded simplicial complex. For any two simplices $\tau, \sigma \in K$, $\tau \leq \sigma$ consider the inclusion map $j_{\tau,\sigma} : \tau \hookrightarrow \sigma$. A collection of forms

$$\theta := \{\theta(\sigma) \in W^k_\infty(\sigma) \mid \sigma \in K\}$$

such that

$$\forall \tau \leq \sigma \in K : j^*_{\tau,\sigma}(\theta(\sigma)) = \theta(\tau),$$

is a *simplicial differential form of degree* $k$ or just an "*S-form*". The space of all these $S$-forms of degree $k$ on $K$ is denoted by $S^k(K)$. For any $S$-form $\theta := \{\theta(\sigma)\}_{\sigma \in K}$ of degree $k$, the collection $d\theta := \{d\theta(\sigma)\}_{\sigma\in K}$ is an $S$-form of degree $k+1$. Thus the $S$-forms assemble to a cochain complex $(S^*(K), d^*)$, the *cochain complex of $S$-forms on $K$*.

**3.12 Definition ($L_p$-cohomology of $S$-forms).** Define

$$\forall 1 \leq p < \infty : S^k_p(K) := \{\theta \in S^k_p(K) \mid \|\theta\|^p_{S^k_p(K)} := \sum_{\sigma\in K} \|\theta(\sigma)\|^p_{W^k_\infty(\sigma)} < \infty\}$$
$$S^k_\infty(K) := \{\theta \in S^k_p(K) \mid \|\theta\|_{S^k_\infty(K)} := \sup_{\sigma\in K} \|\theta(\sigma)\|_{W^k_\infty(\sigma)} < \infty\}.$$

The spaces

$$\mathscr{H}^k_p(K) := H^k(S_p(K), d)$$

are called *$L_p$-cohomology of S-forms on $K$*. The corresponding closed and exact forms are denoted by $\mathscr{Z}_p(K)$ and $\mathscr{B}_p(K)$.

**3.13 Lemma.** For every $1 \leq p \leq \infty$ the $S^*_p(K)$ assemble to a cochain complex of Banach spaces.

# 4 Isomorphism Theorem (Main Result)

**4.1 Definition (GKS-Bedingung).** Eine glatte Triangulierung $h : |K| \to M$ *erfüllt die Gol'dshtein-Kuz'minov-Shvedov-Bedingung* ("*GKS-Bedingung*"), falls gilt: Es existieren globale Konstanten $C_1, C_2 > 0$, sodass für jeden Simplex $\sigma \in K$

$$\sup_{x\in\sigma} \|h_*|_x\| \leq C_1, \qquad \text{und} \qquad \sup_{x\in\sigma} \|h^{-1}_*|_{h(x)}\| \leq C_2.$$

Hierbei ist $\|\_\|$ die Operatornorm des Pushforwards $h_*$, die punktweise bezüglich der Riemannschen Metrik $g$ auf $M$ und der *S-Metrik* (siehe Lemma 4.2 unten) auf $K$ gebildet wird. Diese Bedingung bedeutet anschaulich, dass die Geometrien $(K, g_S)$ und $(M, g)$ unter der Triangulierung $h$ nicht beliebig stark verzerrt werden dürfen.

**4.2 Lemma (S-Metrik).** Es existiert eine Menge Riemannscher Metriken $g_S = \{g(\sigma)\}_{\sigma\in K}$, sodass zum einen jeder $k$-Simplex $(\sigma, g(\sigma))$ isometrisch ist zum euklidischen Standardsimplex $(\Delta_k, \bar{g})$ und zum anderen folgende Kompatibilitätsbedingung erfüllt ist: Seien $\tau, \sigma \in K$, $\tau$ eine Seite von $\sigma$, und es sei $j : \tau \hookrightarrow \sigma$ die Inklusion. Dann gilt $j^*_{\tau,\sigma}(g(\sigma)) = g(\tau)$.

## 5 Applications and Recent Developments

- $L_p$-cohomology is invariant under quasi isometries
- Hodge decomposition
- Poincaré Duality
- Hölder duality

We give a short overview of more recent results concerning $L_p$-spaces and $L_p$-cohomology on manifolds.

**5.1 Theorem (Kopylov, 2009, [3, Theorem 3.3, 3.4]).** Suppose that $1 \leq p, q < \infty$, $\frac{1}{q} - \frac{1}{p} < \frac{1}{n+1}$ and let $M := M_f$ be a surface of revolution as above.

(i) If $f$ is unbounded, then $T^j_{p,q}(M) \neq 0$ for any $1 \leq j \leq n+1$.

(ii) If $T^j_{p,q}(M) = 0$ for any $1 \leq j \leq n+1$, then

$$\lim_{x\to\infty} f(x) = 0 \qquad \text{and} \qquad \operatorname{vol}(M) < \infty.$$

In particular $f$ is bounded.

**5.2 Definition (Hodge Laplacian).** Let $(M, g)$ be a complete Riemannian manifold with exterior differential $d$. Denote by $d^*$ the formal $L^2$-adjoint of $d$. The operator

$$\Delta := d \circ d^* + d^* \circ d$$

is called *Hodge Laplacian*. In case $L_2$-norms are taken with respect to some weight function $\sigma = e^{-\phi}$, we denote the corresponding operator by $\Delta_\phi$. Denote by

$$H_{k,p}(M, \sigma) := \ker \Delta \cap L^k_p(M, \sigma),$$

where $L^k_p(M, \sigma)$ is the $L_p$*-space with weight function* $\sigma : M \to \mathbb{R}$.

**5.3 Theorem (Hodge decomposition, Kodaira, 1949, [2]).** The $L_2$-space over $M$ admits the following orthogonal direct sum decomposition

$$L^k_2(M) = H_{k,2}(M) \oplus \overline{d\Omega^k_c(M)} \oplus \overline{d^*\Omega^k_c(M)}.$$

**5.4 Theorem (Hodge decomposition, non-compact case, Li, 2009, [4, Theorem 2.1]).** Let $(M, g)$ be a complete Riemannian manifold, $\phi \in \mathcal{C}^2(M)$, $\sigma := e^{-\phi}$, $p > 1$, $q := \frac{p}{p-1}$. Suppose that the *Riesz transforms* $d(\Delta^k_\phi)^{-\frac{1}{2}}$, $d^*(\Delta^k_\phi)^{-\frac{1}{2}}$ are bounded in $L_p$ and $L_q$ and the *Riesz potential* $(\Delta^k_\phi)^{-\frac{1}{2}}$ is bounded in $L_p$. Then the Strong $L^p$-Hodge direct sum decomposition holds:

$$L^k_p(M, \sigma) = H_{k,p}(M, \sigma) \oplus d\mathcal{W}^{k-1}_p(M, \sigma) \oplus d^*_\phi \mathcal{W}^{k+1}_p(M, \sigma)$$

(Warning: The definition of $\mathcal{W}^k_p(M, \sigma)$ is slightly different than $W^k_p(M)$.)

**5.5 Theorem (analytic Poincaré duality).** Let $M$ be a smooth compact oriented manifold of dimension $m$. The bilinear pairing $\beta : H^k_{\mathrm{dR}}(M) \times H^{m-k}_{\mathrm{dR}}(M) :\to \mathbb{R}$,

$$([\omega], [\eta]) \mapsto \int_M \omega \wedge \eta$$

is well-defined and regular. The map $\Psi : H^k_{\mathrm{dR}}(M) \to (H^{m-k}_{\mathrm{dR}})^*$, $[\omega] \mapsto ([\eta] \mapsto \beta([\omega], [\eta]))$, is an isomorphism.

**5.6 Theorem (Poincaré duality, Pansu, 2008, [5, Lemma 13]).** Let $M$ be a complete oriented Riemannian manifold of dimension $m$. Let $p > 1$ and let $q$ be Hölder conjugate to $p$. Let $\omega \in L_p^k(M)$. Then the following hold:

(i) $0 \neq [\omega] \in \bar{H}_p^k(M)$ if and only if there exists $\eta \in L_q^{m-k}(M)$ such that

$$\int_M \omega \wedge \eta \neq 0.$$

(ii) $0 \neq [\omega] \in H_p^k(M)$ if and only if there exists a sequence $\eta_j \in L_q^{m-k}(M)$ such that

$$\int_M \omega \wedge \eta_j \geq 1 \qquad \text{and} \qquad \|d\eta_j\|_{L_q(M)} \to 0.$$

(iii) As a consequence, we obtain

$$\bar{H}_p^k(M) = 0 \Longleftrightarrow \bar{H}_q^{m-k}(M) = 0, \qquad T_p^k(M) = 0 \Longleftrightarrow T_q^{m-k}(M) = 0.$$

**5.7 Definition (Hölder pairing).** Assume $1 \leq p, q \leq \infty$ are Hölder conjugate. The pairing $\beta : L_q^{m-k}(M) \times L_p^k(M) \to \mathbb{R}$,

$$(\omega, \eta) \mapsto \int_M \omega \wedge \eta$$

is called the *Hölder pairing of* $M$.

**5.8 Theorem (Hölder duality for $L_p(M)$).** For any Hölder conjugate $1 \leq p, q < \infty$ and $0 \leq k \leq m$, the Hölder pairing $\beta : L_q^{m-k}(M) \times L_p^k(M) \to \mathbb{R}$ is regular and the map $\Psi : L_q^{m-k}(M) \to L_p^k(M)^*$, $\omega \mapsto (\eta \mapsto \beta(\omega, \eta))$, is an isomorphism.

## References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