Basics of Complex Manifolds

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1 Complex Manifolds

1.1 Definition (Complex Manifold)**.** Let M be a smooth manifold of dimension $2m$. A smooth atlas $\mathscr{A}$ is *holomorphic*, if for any two charts $z : U \to U'$, $w : V \to V'$, the transition function $z \circ w^{-1} : w(U \cap V) \to z(U \cap V)$ is holomorphic, if we identify $\mathbb{R}^{2m} = \mathbb{C}^m$. In this case $\mathscr{A}$ determines a maximal holomorphic atlas, called a *complex structure* on M. The charts of an holomorphic atlas are called *holomorphic charts*. The tupel $(M, \mathscr{A})$ is a *complex manifold* of complex dimension m. If $m = 1$, M is a *Riemann Surface*.

1.2 Definition (Holomorphic maps)**.** Let M, N be complex manifolds. Then a map $f : M \to N$ is *holomorphic* , if for all holomorphic charts z of M and w of N the map $w \circ f \circ z^{-1}$ is holomorphic on its domain of definition.
If $N = \mathbb{C}$ we say f is a *holomorphic function.* For any open subset $U \subset M$ we denote by $\mathscr{O}(U)$ the set of all holomorphic functions on U.
We say f is *biholomorphic* if f is bijective and f and f^{-1} are both holomorphic. If $M = N$ we say f is an *automorphism.*

2 Almost complex Structures

2.1 Definition. Let V be an n-dimensional $\mathbb{R}$ vector space. Then a $J \in \mathrm{End}_{\mathbb{R}}(V)$ satisfying

$$J^2 = -\,\mathrm{id}_V$$

is a *complex structure on V*

2.2 Example. $\mathbb{C}^m$ as an $\mathbb{R}$ vector space with the almost complex structure J given by $v \mapsto iv$. Seen over the reals we can identify

$$(z^1, \dots, z^m) = (x^1 + iy^1, \dots, x^m + iy^m) \hat{=} (x^1, y^1, \dots, x^m, y^m)$$

Then J is given on $\mathbb{R}^{2m}$ by

$$(x^1, y^1, \dots, x^m, y^m) \mapsto (y^1, -x^1, \dots, y^m, -x^m)$$

2.3 Lemma. In this situation J determines a natural complex vector space structure on V by defining $V_J := V$ as a set and extending complex scalar multiplication by the linear extension of

$$\forall v \in V : iv := J(v)$$

We can interpred $J \in \operatorname{End}_{\mathbb{C}}(V_J)$ in this case.

Proof. Most of the axioms of scalar multiplication are more or less obvious, except maybe

$$\begin{aligned}\forall a, b, c, d \in \mathbb{R} : (a + ib) \cdot ((c + id) \cdot v) &= (a + ib) \cdot (cv + dJ(v)) = acv + adJ(v) + cbJ(v) - bcv \\ &= (ac - bc)v + (ad + cb)J(v) = ((a + ib)(c + id))v\end{aligned}$$

By definition J is $\mathbb{R}$ linear. By by definition

$$J(iv) = J(J(v)) = iJ(v)$$

□

2.4 Lemma. Let W be a $\mathbb{C}$ vector space with $\dim_{\mathbb{C}} W = m$. Denote by $W_{\mathbb{R}}$ the underlying real vector space, i.e. $W_{\mathbb{R}} = W$ as a set and scalar multiplication on W is restricted to the reals. Then

$$\dim_{\mathbb{R}} W_{\mathbb{R}} = 2m$$

Proof. Let $b_1, \dots, b_m$ be a $\mathbb{C}$-basis of W and $w \in W$. Then there exist $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ such that

$$w = \sum_{k=1}^{m} \lambda_k b_k = \sum_{k=1}^{m} \operatorname{Re}(\lambda_k) b_k + \sum_{k=1}^{m} \operatorname{Im}(\lambda_k) i b_k$$

so $(b_1, ib_1, \dots, b_m, ib_m)$ $\mathbb{R}$-generates $W_{\mathbb{R}}$. They are $\mathbb{R}$ - linear independent since

$$\forall \alpha_k, \beta_k \in \mathbb{R} : 0 = \sum_{k=1}^{m} \alpha_k b_k + \sum_{k=1}^{m} \beta_k i b_k = \sum_{k=1}^{m} (\alpha_k + i\beta_k) b_k$$

implies $\forall 1 \leq k \leq m : \alpha_k = \beta_k = 0$ since $b_1, \dots, b_m$ are $\mathbb{C}$ - linear independent. □

2.5 Corollary. If V is a finite dimensional $\mathbb{R}$-vector space which admits a complex structure J then its dimension is even. Especially if $n := \dim_{\mathbb{R}} V$ and $m := \dim_{\mathbb{C}} V_J$ we obtain:

$$n = \dim_{\mathbb{R}} V = \dim_{\mathbb{R}} V_J = 2m$$

2.6 Definition (Complexification)**.** Let V be an n-dimensional $\mathbb{R}$ vector space. The space

$$V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$$

is the *complexification of* V. By the dimension formula for tensor products $\dim_{\mathbb{R}} V^{\mathbb{C}} = 2n$. So $V_{\mathbb{C}}$ can be seen as a complex vector space with scalar multiplication defined by

$$\forall z, \alpha \in \mathbb{C} : \forall v \in V : z \cdot v \otimes \alpha := v \otimes (z\alpha)$$

There is a canonical embedding $V \to V^{\mathbb{C}}$, $v \mapsto v \otimes 1$. The image of this embedding is precisely the subset of $V^{\mathbb{C}}$, that is invariant under complex conjugation which is defined by $\overline{v \otimes \alpha} := v \otimes \overline{\alpha}$.
If W is another $\mathbb{R}$ vector space, any $f \in \operatorname{Hom}_{\mathbb{R}}(V, W)$ extends to an $f^{\mathbb{C}} \in \operatorname{Hom}_{\mathbb{C}}(V, W)$ via

$$f^{\mathbb{C}}(v \otimes \alpha) := f(v) \otimes \alpha$$

If J is a complex structure on V, it especially as a $\mathbb{C}$-linear extension $J^{\mathbb{C}} : V_C \to V_C$ sometimes also denoted by J

2.7 Lemma. Complexification is a covariant additive functor from the category of $\mathbb{R}$ vector spaces to the category of $\mathbb{C}$ vector spaces, i.e.

(i) For any $g : U \to V$, $f : V \to W$: $(f \circ g)^{\mathbb{C}} = f^{\mathbb{C}} \circ g^{\mathbb{C}}$

(ii) $(\mathrm{id}_V)^{\mathbb{C}} = \mathrm{id}_{V^{\mathbb{C}}}$

(iii) For any $f, g : V \to W$: $(f + g)^{\mathbb{C}} = f^{\mathbb{C}} + g^{\mathbb{C}}$

(iv) For any $a \in \mathbb{R}$: $(af)^{\mathbb{C}} = af^{\mathbb{C}}$

2.8 Lemma. [Properties of complex Structures] Let V be an n-dimensional $\mathbb{R}$ vector space with a complex structure J and $V^{\mathbb{C}}$ its complexification. Then $J = J^{\mathbb{C}}$ still satisfies $J^{\mathbb{C}} \circ J^{\mathbb{C}} = -\mathrm{id}_{V^{\mathbb{C}}}$.

(i) J is an isomorphism

(ii) The only eigenvalues of J are $+i, -i$.

(iii) J is diagonizable.

(iv) If V' is the eigenspace of $+i$ and V'' is the eigenspace of $-i$, we have

$$V^{\mathbb{C}} = V' \oplus V''$$

(v) The maps $\varphi : V_J \to V'$ and $\psi : V_J \to V''$

$$v \mapsto \frac{1}{2}(v \otimes 1 - iJ(v) \otimes 1) \qquad\qquad v \mapsto \frac{1}{2}(v \otimes 1 + iJ(v) \otimes 1)$$

are complex linear / complex antilinear resp. isomorphisms.

Proof.

(i) Injectivity:

$$v \in \ker J \Rightarrow -v = J(J(v)) = J(0) = 0 \Rightarrow v = 0$$

Surjectivity: Let $v \in V^{\mathbb{C}}$ be arbitrary and define $w := -J(v)$ then

$$J(w) = J(-J(v)) = -J(J(v)) = v$$

(ii) Assume $\lambda \in \mathbb{C}$ is an eigenvalue of J. Then for any $v \in V_{\mathbb{C}}$

$$-v = J(J(v)) = J(\lambda v) = \lambda J(v) = \lambda^2 v$$

and thus $\lambda^2 = 1$.

(iii) We claim that $(X + i)(X - i)$ is the minimal polynomial of J, by simply calculating

$$(J + i)(J - i) = J^2 - iJ + iJ - i^2 = -\mathrm{id} + \mathrm{id} = 0$$

(iv) This is a consequence of the preceeding statements.

(v) First of all, the image of φ really is contained in V' since

$$J(\varphi(v)) = J(\frac{1}{2}(v \otimes 1 - iJ(v) \otimes 1)) = \frac{1}{2}(J(v) \otimes 1 + iv \otimes 1) = \frac{1}{2}(iv \otimes 1 + J(v) \otimes 1) = i\varphi(v)$$

Complex linearity follows by definition via

$$\varphi(iv) = \varphi(J(v)) = \frac{1}{2}(J(v) \otimes 1 - iJ(J(v)) \otimes 1) = \frac{1}{2}(J(v) \otimes 1 + iv \otimes 1) = i\frac{1}{2}(-iJ(v) \otimes 1 + v \otimes 1) = i\varphi(v)$$

Now assume

$$v \in \ker \varphi \Rightarrow 0 = \frac{1}{2}(v \otimes 1 - iJ(v) \otimes 1) \Rightarrow v \otimes 1 = J(v) \otimes i$$

On the other hand complex conjugation yields

$$\Rightarrow v \otimes 1 = v \otimes \bar{1} = \overline{v \otimes 1} = \overline{J(v) \otimes i} = J(v) \otimes \bar{i} = -J(v) \otimes i$$

This implies $v \otimes 1 = 0 \Rightarrow v = 0$. So φ in injective and thus an isomorphism.
The discussion of ψ is entirely analogous.

□

2.9 Lemma. Let V be an $\mathbb{R}$ vector space endowed with an almost complex structure J. Then the dual space $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ has a natural complex structure as well given by $J(f)(v) := f(J(v))$.

2.10 Definition (Almost complex structure)**.** Let M be a smooth manifold. Let J be a smooth $(1,1)$ tensor field on M. For any $p \in M$ we can interpret $J(p) \in \mathrm{End}_{\mathbb{R}}(T_pM)$. If J satisfies

$$\forall p \in M : J(p)^2 = -\,\mathrm{id}_{T_pM}$$

J is an *almost complex structure* on M.

2.11 Lemma (Complex to Almost Complex)**.** Let M be a smooth manifold. If M admits a complex structure $\mathscr{A}$, then M admits an almost complex structure J. Let $\dim_{\mathbb{C}} M = m$ and (z, U) be any holomorphic chart inducing a coordinate frame $\partial x_1, \partial y_1, \dots, \partial x_m, \partial y_m$. Then J is given locally as

$$J_p(\partial x_i|p) = \partial y_i|p \qquad\qquad J_p(\partial y_i|p) = -\partial x_i|p$$

where $1 \le i \le m$ and $p \in U$.

Proof. We have no chance but to define J as required. Clearly J satisfies $J^2 = -1$. We have to show that this definition does not depend on the chosen coordinate chart. So let (z', U') be another holomorphic coordinate chart and suppose $p \in U \cap U'$. Denote by J'_p the induced almost complex structure defined by z'. Fix any $1 \le i \le m$. To simplify notation write

$$\begin{aligned} J &:= J_p & J' &:= J'_p \\ X_i &:= \partial x_i|p & Y_i &:= \partial y_i|p \\ X'_i &:= \partial x'_i|p & Y'_i &:= \partial y'_i|p \end{aligned}$$

By definition we have

$$\begin{aligned} J(X_i) &= Y_i & J(Y_i) &= -X_i \\ J'(X'_i) &= Y'_i & J'(Y'_i) &= -X'_i \end{aligned}$$

We have to show, that

$$J(X'_i) = J'(X'_i) \qquad\qquad J(Y'_i) = J'(Y'_i)$$

Let $\psi := z' \circ z^{-1}$ be the transition function and $T := d\psi(z(p))$ be its differential. Let's discuss the case $n = 1$ first by by dropping all the i in notation. Since ψ is holomorphic, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$T = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

This uses the Cauchy-Riemann differential equations. It follows that the coordinate frames are transformed via

$$\begin{aligned} X' &= \alpha X + \beta Y \\ Y' &= -\beta X + \alpha Y \end{aligned}$$

This implies

$$J(X') = J(\alpha X + \beta Y) = \alpha J(X) + \beta J(Y) = \alpha Y - \beta X = Y' = J'(X')$$

And similar for Y'.
The multidimensional case follows similar since the multidimensional Cauchy-Riemann differential equations state that T is of the same form as above, but with α, β being matrices. □

2.12 Definition (Almost Complex To Complex)**.** An almost complex structure J on M is a *complex structure*, if it is induced by some complex structure on M as above.

2.13 Remark. One can show, that on a surface any almost complex structure is a complex structure.

3 Tangent Bundle and Vector fields

3.1 Definition (Complexified Tangent Bundle)**.** Let M be a smooth manifold. Then

$$T_{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C}$$

is the *complexified tangent bundle* of M.

3.2 Lemma. If J is an almost complex structure, the complexified tangent bundle admits an eigenspace decomposition

$$T_{\mathbb{C}}M = T'M \oplus T''M$$

with respect to J. Notice that $T'M$ and $T''M$ are smooth subbundles of $T_{\mathbb{C}}M$ and the isomorphisms in lemma 2.8 extend to isomorphisms of complex vector bundles.

3.3 Definition (Complex Vector Fields)**.** A smooth section Z of $T_{\mathbb{C}}M$ is a *complex vector field.* Any such field can be written as

$$Z = X + iY$$

where $X, Y \in \Gamma(M)$ are smooth real vector fields. We denote the space of all complex vector fields by $\Gamma_{\mathbb{C}}(M)$.

3.4 Lemma (Vector fields as Derivations)**.** Any complex vector field Z acts on $\mathscr{C}^\infty(M, \mathbb{C})$ as a complex linear derivation in the following way: Write $Z = X + iY \in \Gamma_{\mathbb{C}}(M)$, $X, Y \in \Gamma(M)$ and $f = u + iv \in \mathscr{C}^\infty(M, \mathbb{C})$ with $u, v \in \mathscr{C}^\infty(M, \mathbb{R})$. Define

$$Z(f) := X(u) - Y(v) + i(X(v) + Y(u))$$

Then Z is a complex linear derivation.

Proof. Clearly Z is $\mathbb{R}$-linear, so it suffices to check

$$\begin{aligned} Z(if) &= Z(i(u+iv)) = Z(-v+iu) = -X(v) - Y(u) + iX(u) - iY(v) \\ &= iX(u) - iY(v) - X(v) - Y(u) = i(X(u) - Y(v) + i(X(v) + Y(u))) = iZ(f) \end{aligned}$$

We have to check, that Z satisfies the product rule with respect to complex multiplication as well. So take another $g = a + ib \in \mathscr{C}^\infty(M, \mathbb{C})$ and calculate:

$$\begin{aligned} Z(fg) &= Z(au - bv + i(bu + av)) = X(au - bv) - Y(bu + av) + i(X(bu + av) + Y(au - bv)) \\ &= aX(u) + uX(a) - bX(v) - vX(b) - bY(a) - uY(b) - aY(v) - vY(a) \\ &+ i(bX(u) + uX(b) + aX(v) + vX(a) + uY(a) + aY(u) - bY(v) - vY(b)) \\ &= (a + ib)(X(u) - Y(v) + i(X(v) + Y(u))) + (u + iv)(X(a) - Y(b) + i(X(b) + Y(a))) \\ &= gZ(f) + fZ(g) \end{aligned}$$

□

3.5 Definition (Lie - Bracket)**.** Remember that for any smooth vector fields $X, Y \in \Gamma(M)$ we obtain a new vector field $[X, Y] \in \Gamma(M)$ by defining

$$[X, Y](f) := X(Yf) - Y(Xf)$$

where $f \in \mathscr{C}^\infty(M)$. The operation $[\_, \_]$ is called *Lie-Bracket*.

3.6 Definition (Complex Lie - Bracket)**.** We extend the Lie Bracket to complex vector fields, by defining

$$[X + iY, U + iV] := [X, U] - [Y, V] + i([X, V] + [Y, U])$$

3.7 Definition. Let (z, U) be a holomorphic coordinate chart for M. Write $z^j = x^j + iy^j$. Define

$$\partial z^j := \frac{1}{2}(\partial x^j - i\partial y^j) \qquad \partial \overline{z}^j := \frac{1}{2}(\partial x^j + i\partial y^j)$$

Any complex vector field Z on M has a decomposition $Z = Z' + Z''$ where

$$Z' = \frac{1}{2}(Z - iJZ) \in TM' \qquad Z'' = \frac{1}{2}(Z + iJZ) \in TM''$$

3.8 Lemma. Let $dx^j, dy^j \in T^*U$ be the dual operators of $\partial x^j, \partial y^j \in TU$. Define

$$dz^j := dx^j + idy^j \qquad d\overline{z}^j := dx^j - idy^j$$

Then $dz^j, d\overline{z}^j$ are the dual basis corresponding to ∂z^j, $\partial \overline{z}^j$.

Proof. We check this for one arbitrary j, which we drop in notation for simplicity, by checking, that dz has the dual basis property:

$$dz(\partial z) = (dx + idy)(\frac{1}{2}(\partial x - i\partial y)) = \frac{1}{2}dx(\partial x) - \frac{i}{2}dx(\partial y) + \frac{i}{2}dy(\partial x) - \frac{i^2}{2}dy(\partial y) = 1$$

and

$$dz(\partial \overline{z}) = (dx + idy)(\frac{1}{2}(\partial x + i\partial y)) = \frac{1}{2}dx(\partial x) - \frac{i}{2}dx(\partial y) + \frac{i}{2}dy(\partial x) + \frac{i^2}{2}dy(\partial y) = 0$$

The calculation for $d\overline{z}$ is entirely analogous. □

4 Compatible Metrics and Hermitian Manifolds

4.1 Definition (Compatible Metric, Hermitian Manifold)**.** Let M be a complex manifold with corresponding almost complex structure J. A Riemannian metric $g = \langle \_, \_ \rangle$ on M is *compatible* with J if

$$\forall X, Y \in \Gamma(M) : \langle JX, JY \rangle = \langle X, Y \rangle$$

The triple (M, J, g) is an *Hermitian Manifold.*

4.2 Lemma. Let (M, J, g) be a Hermitian Manifold. Then there is a complex linear extension $g^{\mathbb{C}}$ of $g = \langle \_, \_ \rangle$ to $T_{\mathbb{C}}M$ given by

$$\langle X_1 \otimes z_1, X_2 \otimes z_2 \rangle := z_1 z_2 \langle X_1, X_2 \rangle$$

This extension is symmetric and satisfies the following conditions

(i) For all complex vector fields Z_1, Z_2: $\langle \overline{Z}_1, \overline{Z}_2 \rangle = \overline{\langle Z_1, Z_2 \rangle}$

(ii) For all $Z_1, Z_2 \in T'M$: $\langle Z_1, Z_2 \rangle = 0$

(iii) If $Z \neq 0$: $\langle \overline{Z}, Z \rangle > 0$

Proof. The symmetry is obvious from the definition. Write $Z_1 = X_1 \otimes z_1$, $Z_2 = X_2 \otimes z_2$, $Z = X \otimes z$, where $X_1, X_2, X \in \Gamma(M)$, $z_1, z_2, z \in \mathbb{C}$

(i) Unraveling the definitions we obtain

$$\begin{aligned}\langle \overline{Z}_1, \overline{Z}_2 \rangle &= \langle \overline{X_1 \otimes z_1}, \overline{X_2 \otimes z_2} \rangle = \langle X_1 \otimes \overline{z_1}, X_2 \otimes \overline{z_2} \rangle = \overline{z_1}\,\overline{z_2} \langle X_1, X_2 \rangle \\ &= \overline{z_1 z_2 \langle X_1, X_2 \rangle} = \overline{\langle X_1 \otimes z_1, X_2 \otimes z_2 \rangle} = \overline{\langle Z_1, Z_2 \rangle}\end{aligned}$$

(ii) On the one hand we obtain by definition

$$\langle Z_1, Z_2 \rangle = z_1 z_2 \langle X_1, X_2 \rangle$$

Now assume $Z_1, Z_2 \in T'M$, i.e. $J(Z_1) = iZ_1$, $J(Z_2) = iZ_2$. Then on the other hand

$$\langle Z_1, Z_2 \rangle = \langle J(Z_1), J(Z_2) \rangle = \langle iZ_1, iZ_2 \rangle = \langle X_1 \otimes iz_1, X_2 \otimes iz_2 \rangle = iz_1 iz_2 \langle X_1, X_2 \rangle = -z_1 z_2 \langle X_1, X_2 \rangle$$

So $\langle Z_1, Z_2 \rangle = 0$.

(iii) If $Z \neq 0$ then $X \neq 0$, $z \neq 0$ as well, so

$$\langle \overline{Z}, Z \rangle = \langle \overline{X \otimes z}, X \otimes z \rangle = \langle X \otimes \overline{z}, X \otimes z \rangle = z\overline{z} \langle X, X \rangle = |z|^2 ||X||^2 > 0$$

□

4.3 Lemma. Let M be a complex manifold with complex structure J. Let $g_{\mathbb{C}} : T_{\mathbb{C}}M \times T_{\mathbb{C}}M \to \mathbb{C}$ be a symmetric complex bilinear form on $T_{\mathbb{C}}M$ satisfying the conditions (i)-(iii) in the lemma above. Then there exists a Riemannian metric $g_{\mathbb{R}}$ on TM, such that $g_{\mathbb{R}}$ is compatible with J and the complex bilinear extension $(g_{\mathbb{R}})^{\mathbb{C}}$ of $g_{\mathbb{R}}$ is equal to $g_{\mathbb{C}}$. In this case, the conditions (i)-(iii) hold for complex vector fields as well.

Proof. Let $X = X \otimes 1$, $Y = Y \otimes 1$ be real vector fields. Then property (i) ensures

$$g_{\mathbb{C}}(X, Y) = g_{\mathbb{C}}(\overline{X}, \overline{Y}) = \overline{g_{\mathbb{C}}(X, Y)}$$

which implies, that $g_{\mathbb{C}}|_{TM} : TM \times TM \to \mathbb{R}$. So define

$$g_{\mathbb{R}} := \mathrm{Re}(g_{\mathbb{C}}) = g_{\mathbb{C}}|_{TM}$$

By construction $g_{\mathbb{R}}$ is $\mathbb{R}$-bilinear and symmetric. Property (iii) ensures

$$g_{\mathbb{R}}(X, X) = g_{\mathbb{R}}(\overline{X}, X) > 0$$

for any real vector field $X \neq 0$. So $g_{\mathbb{R}}$ is a Riemannian metric.
Consider arbitrary complex vector fields $X_1 \otimes z_1, X_2 \otimes z_2$ and the complexification of $g_{\mathbb{R}}$:

$$(g_{\mathbb{R}})^{\mathbb{C}}(X_1 \otimes z_1, X_2 \otimes z_2) = z_1 z_2 g_{\mathbb{R}}(X_1, X_2) = z_1 z_2 g_{\mathbb{C}}(X_1, X_2) = g_{\mathbb{C}}(X_1 \otimes z_1, X_2 \otimes z_2)$$

So $(g_{\mathbb{R}})^{\mathbb{C}} = g_{\mathbb{C}}$.
By property (ii) $g_{\mathbb{C}}$ vanishes on $T'M$. Let $Z_1, Z_2 \in T''M$. Property (i) immediately implies

$$\overline{g_{\mathbb{C}}(Z_1, Z_2)} = g_{\mathbb{C}}(\overline{Z_1}, \overline{Z_2}) = 0$$

since $\overline{Z_1}, \overline{Z_1} \in T'M$. So $g_{\mathbb{C}}$ vanishes on $T''M$ as well. Now let $X, Y \in TM$ be real vector fields. Interpreted as the complex vector fields $X \otimes 1, Y \otimes 1$ they admit a uinque decomposition $X = X' + X'' \in T'M \oplus T''M$, $Y = Y' + Y'' \in T'M \oplus T''M$. We obtain

$$\begin{aligned}
g_{\mathbb{R}}(J(X), J(Y)) &= g_{\mathbb{C}}(J(X), J(Y)) \\
&= g_{\mathbb{C}}(J(X') + J(X''), J(Y') + J(Y'')) \\
&= g_{\mathbb{C}}(J(X'), J(Y') + g_{\mathbb{C}}(J(X'), J(Y'')) + g_{\mathbb{C}}(J(X''), J(Y')) + g_{\mathbb{C}}(J(X''), J(Y'')) \\
&= g_{\mathbb{C}}(iX', iY') + g_{\mathbb{C}}(iX', -iY'') + g_{\mathbb{C}}(-iX'', iY') + g_{\mathbb{C}}(-iX'', -iY'') \\
&= g_{\mathbb{C}}(X', Y'') + g_{\mathbb{C}}(X'', Y') \\
&= g_{\mathbb{C}}(X', Y') + g_{\mathbb{C}}(X', Y'') + g_{\mathbb{C}}(X'', Y') + g_{\mathbb{C}}(X'', Y'') \\
&= g_{\mathbb{C}}(X' + X'', Y' + Y'') \\
&= g_{\mathbb{R}}(X, Y)
\end{aligned}$$

So $g_{\mathbb{R}}$ is compatible with J. For any complex vector fields $X_1 \otimes z_1, X_2 \otimes z_2$:

$$\begin{aligned}
&g_{\mathbb{C}}(J(X_1 \otimes z_1), J(X_2 \otimes z_2))) = z_1 z_2 \cdot g_{\mathbb{C}}(J(X_1), J(X_2)) \\
&= z_1 z_2 \cdot g_{\mathbb{R}}(J(X_1), J(X_2)) = z_1 z_2 \cdot g_{\mathbb{R}}(X_1, X_2) = g_{\mathbb{C}}(X_1 \otimes z_1, X_2 \otimes z_2)
\end{aligned}$$

□

4.4 Lemma (Existence of Compatible Metrics)**.** Let M be a complex manifold and J be the induced complex structure on M. Then there exists a compatible Riemannian metric $g = \langle \_, \_ \rangle$ on M.

Proof. First we define g locally. Let $p \in M$ and (z, U) be a holomorphic chart near p. Let $V := T_pM$ and $J := J_p$. The induced coordinate frame of z yields a basis $(X^1, Y^1, \dots, X^m, Y^m)$ for V. Since J is induced by the complex structure

$$J(X^i) = Y^i \qquad\qquad J(Y^i) = -X^i$$

Define $\langle \_, \_ \rangle$ by declaring $(X^1, Y^1, \dots, X^m, Y^m)$ to be an orthonormal basis. Then

$$\begin{aligned}
\langle J(X^i), J(Y^j) \rangle &= -\langle Y^i, X^j \rangle = 0 = \langle X^i, Y^j \rangle \\
\langle J(X^i), J(X^j) \rangle &= \langle Y^j, Y^i \rangle = \delta_{ij} = \langle X^i, X^j \rangle \\
\langle J(Y^i), J(Y^j) \rangle &= \langle -X^j, -X^i \rangle = \delta_{ij} = \langle Y^i, Y^j \rangle
\end{aligned}$$

So J is compatible. This process can be done in any T_qM where $q \in U$ and any holomorphic coordinate chart. This yields to an open cover U_α of M where each TU_α is endowed with a compatible Riemannian Metric. Patching together with a smooth partition of unity, we obtain the desired result. □

4.5 Theorem. Let M be a complex manifold with complex structure J and let $g = \langle \_, \_ \rangle$ be a Riemannian metric on M. Then g is compatible with J iff around each point $p \in M$ there are holomorphic coordinates

$$z = (z^1, \ldots, z^m) = (x^1 + iy^1, \ldots, x^m + iy^m)$$

such that the associated coordinate frame $(X_1, Y_1, \ldots, X_m, Y_m)$ at p is a g - orthonormal basis of T_pM.

4.6 Corollary. Let M be a Hermitian manifold. Then the type decomposition

$$A^r(M, \mathbb{C}) = \oplus_{p+q=r} A^{p,q}(M, \mathbb{C})$$

is orthogonal with respect to the induced Hermetian metric $(\varphi, \psi) := \langle \overline{\varphi}, \psi \rangle$.
Let $\varphi \in A^{p,q}$, then it follows that $*\varphi \in A^{m-q,m-p}$.

5 Nijenhuis Tensor

5.1 Definition (Nijenhuis / Torsion Tensor)**.** Let M be a smooth manifold and J be an almost complex structure on M. For any smooth vector fields $X, Y \in \Gamma(M)$ we defined

$$N_J(X, Y) := 2([JX, JY] - [X, Y] - J([X, Y]) - J([JX, Y]))$$

N is the *associated Nijenhuis* or *torsion Tensor*.

5.2 Definition (Connection)**.** Let $\pi : E \to M$ be a smooth vector bundle over a smooth manifold M. Denote by $\mathscr{E}(M)$ the space of smooth sections over M and by $\mathscr{T}(M)$ the space of smooth sections over TM. A *connection* is a map

$$\nabla : \mathscr{T}(M) \times \mathscr{E}(M) \to \mathscr{E}(M)$$

written $(X, Y) \mapsto \nabla_X Y$ satisfying the following properties:

(i) Linearity over $\mathscr{C}^\infty(M)$ in X:

$$\forall f, g \in \mathscr{C}^\infty(M) : \nabla_{fX_1 + gX_2} = f\nabla_{X_1} Y + g\nabla_{X_2} Y$$

(ii) Linearity over $\mathbb{R}$ in Y:

$$\forall a, b \in \mathbb{R} : \nabla_X(aY_1 + bY_2) = a\nabla_X(Y_1) + b\nabla_X(Y_2)$$

(iii) Product rule:

$$\forall f \in \mathscr{C}^\infty(M) : \nabla_X(fY) = f\nabla_X Y + (Xf)Y$$

A connection is a *linear connection*, if $\mathscr{E}(M) = \mathscr{T}(M)$.

5.3 Definition (Torsion)**.** A linear connection is *torsion free*, if

$$\forall X, Y \in TM : \nabla_X Y - \nabla_Y X = [X, Y]$$

5.4 Lemma. Let M be a smooth manifold and J be an almost complex structure on M.

(i) N_J is a tensor.

(ii) Let ∇ be a torsion free connection. Then

$$\frac{1}{2} N_J(X, Y) = \nabla J(JX, Y) - J\nabla J(X, Y) - \nabla J(JY, X) + J\nabla J(Y, X)$$

(iii) J is a complex structure, if $\nabla J = 0$.

5.5 Theorem. Let M be a smooth manifold and J be an almost complex structure. Then the associated torsion tensor N_J vanishes if and only if $T'M$ is an involutive distribution, i.e. forall $Z_1, Z_2 \in E(T'M) : [Z_1, Z_2] \in E(T'M)$.

5.6 Theorem (Newlander - Nirenberg)**.** An almost complex structure J on M is a complex structure if and only if N_J vanishes.

6 Differential forms and Dolbeault cohomology

Splitting into types

$\bigwedge^k V_{\mathbb{C}}^* = \bigwedge^k (V \oplus V'')^* = \bigoplus_{k=p+q} (\bigwedge^p V'^* \bigotimes \bigwedge^q V''^*)$. By "=" we express that there is a basis-independent isomorphism between these complex vector spaces. Let $\varphi \in \bigwedge^p V'^* \bigotimes \bigwedge^q V''^*$, that is, $\varphi = \varphi_1 \otimes \varphi_2$. Define a morphism A by

$$(A\varphi)(v_1' + v_1'', \ldots, v_k' + v_k'') := \frac{1}{p!q!} \sum_{\sigma \in S_k} \varepsilon(\sigma) \varphi_1(v'_{\sigma(1)}, \ldots, v'_{\sigma(p)}) \varphi_2(v''_{\sigma(p+1)}, \ldots, v''_{\sigma(k)}).$$

By choosing bases, one can show that A defines an isomorphism.

Therefore any form in $\bigwedge^k V_{\mathbb{C}}^*$ is a linear combination of forms of type (p, q) where $p + q = k$.

Let M a complex Manifold and $T_{\mathbb{C}}M = T^*M \otimes \mathbb{C}$ its complexified tangent bundle. The pointwise splitting into types now becomes global: Define $A^r(M, \mathbb{C}) := \bigwedge^r T_{\mathbb{C}}^*M$. Then $A^r(M, \mathbb{C}) = \bigoplus_{p+q=r} A^{p,q}(M, \mathbb{C})$. Denote with $\mathcal{A}^{p,q}(M, \mathbb{C})$ the space of *smooth* sections of $A^{p,q}(M, \mathbb{C})$. These sections are called *differential forms of type* (p, q). Let $z^j = x^j + \mathrm{i}y^j$ components of holomorphic coordinates on $U \subseteq M$. Let $\mathrm{d}z^j = \mathrm{d}x^j + \mathrm{i}\mathrm{d}y^j$ and $\mathrm{d}\overline{z}^j = \mathrm{d}x^j - \mathrm{i}\mathrm{d}y^j$. Recording $Z_j = \frac{1}{2}(X_j - \mathrm{i}Y_j)$, $\overline{Z}_j = \frac{1}{2}(X_j + \mathrm{i}Y_j)$, we can now check relations as $\mathrm{d}z^j(Z_k) = \delta_{jk}$ and $\mathrm{d}\overline{z}^j(\overline{Z}_k) = \delta_{jk}$ applying the definitions and calculating. This shows that $\left(\mathrm{d}z^j, \mathrm{d}\overline{z}^j\right)$ is a dual basis of $\left(Z_j, \overline{Z}_j\right)$. Further calculations show $J(\mathrm{d}z) = \mathrm{i}\mathrm{d}z$ and $J(\mathrm{d}\overline{z}) = -\mathrm{i}\mathrm{d}\overline{z}$. Thus $\mathrm{d}z \in \mathcal{A}^{1,0}(M, \mathbb{C})$ and $\mathrm{d}\overline{z} \in \mathcal{A}^{0,1}(M, \mathbb{C})$. Using the definitions again we obtain $\mathrm{d}z(JV) = \mathrm{i}\mathrm{d}z(V)$ and $\mathrm{d}\overline{z}(JV) = -\mathrm{i}\mathrm{d}z(V)$ for any vector field V on M. For $\omega \in \mathcal{A}^{p,q}(M, \mathbb{C})$ we examine the complex structure J on M and denote $\alpha v = (a + b\mathrm{i})v = av + bJv$ for the multiplication of $z \in \mathbb{C}$ with a vectorfield v on M. The above equalities show that $\omega(\alpha v_1, \ldots, \alpha v_r) = \alpha^p \overline{\alpha}^q \omega(v_1, \ldots, v_r)$.

Exterior differential

If $\omega \in \mathcal{A}^{p,q}$ we have a local coordinate representation

$$\omega = \sum_{IJ} a_{IJ} \mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J.$$

Apply d:

$$\mathrm{d}\omega = \sum_{iIJ} X_i(a_{IJ}) \mathrm{d}x^i \wedge \mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J + \sum_{iIJ} Y_i(a_{IJ}) \mathrm{d}y^i \wedge \mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J.$$

Using $X_i = Z_i + \overline{Z}_i$ and $Y_i = i(Z_i - \overline{Z}_i)$ this is

$$\mathrm{d}\omega = \sum_{iIJ} Z_i(a_{IJ})(\mathrm{d}x^i + \mathrm{i}\mathrm{d}y^i) \wedge \mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J + \sum_{iIJ} \overline{Z}_i(a_{IJ})(\mathrm{d}x - \mathrm{i}y^i) \wedge \mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J.$$

By calling the summands $\partial\omega$ and $\overline{\partial}\omega$, we have $\mathrm{d} = \partial + \overline{\partial}$. Everything is well-defined and we have $\partial : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}$ and $\overline{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}$. In the *smooth* category one cannot define $\overline{\partial}$ because there is no natural $(Z_j, \overline{Z}_j)$. In the *almost complex* category there are more subtle difficulities.

Dolbeault cohomology

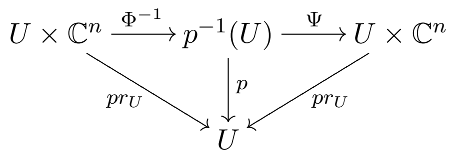
Now we use the fact $\operatorname{im} \partial \cap \operatorname{im} \overline{\partial} = 0$. From $0 = \mathrm{d}^2 = (\partial + \overline{\partial})^2$ we deduce $\partial^2 = 0 = \overline{\partial}^2$. Therefore $\overline{\partial}$ defines a cochain complex for each integer p:

$$\cdots \to \mathcal{A}^{p,q-1} \to \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1} \to \cdots$$

Its cohomology is called *Dolbeault cohomology*, the cohomology groups $H^{p,q}$ are called *Dolbeault cohomology groups* and their complex vector space dimensions are the *Hodge numbers* $h^{p,q}$ of M. Lastly, the first non-trivial cohomology group has a special meaning. We use the notation $\Omega^p(M) := \ker(\overline{\partial} : \mathcal{A}^{p,0} \to \mathcal{A}^{p,1})$, the *holomorphic forms of degree p*.

Holomorphic vector bundles

For any complex vector bundle of rank n we have the following typical diagram:



We have $(\Psi \circ \Phi^{-1})(u, v) = (u, Av) = (u, \gamma(u)v)$ where $\gamma : U \to \mathrm{Gl}(n, \mathbb{C})$ is the transition map. If one can find an atlas for a given complex vector bundle such that all γ and p are holomorphic, the bundle together with this atlas is called a *holomorphic vector bundle*. At this moment we do not know if there are complex vector bundles that admit several distinct holomorphic structures.

For coordinates z and w on $U \subseteq M$ we examine the cotangent bundle $T^*_{\mathbb{C}}U$ with induced coordinates $\mathrm{d}z^j = \sum \frac{\partial z^j}{\partial w^k}\mathrm{d}w^k$ and $\mathrm{d}\overline{z}^j = \sum \frac{\partial \overline{z}^j}{\partial \overline{w}^k}\mathrm{d}z^k$. The derivatives in the sums are the components of the transition function γ. Therefore γ is holomorphic with induced coordinates on tensor bundles $T^k_l U$ of U iff there do never appear any $\mathrm{d}\overline{z}^j$ coordinates in local coordinate representations of sections of $T^k_l U$. This shows that the bundle $\mathcal{A}^{p,0}$ has a natural holomorphic structure while the related bundles $\mathcal{A}^{p,q}$, $\mathcal{A}^k$ and $\mathcal{A}^*$ do not carry such a structure in a natural way for $q \neq 0$ (of course only if their rank is at least one).

Review of Poincaré Duality

We review the *Hodge operator* $*$ on an oriented real Manifold M of dimension n with volume form vol. For any $\varphi \in \mathcal{A}^r(M, \mathbb{R})$ the equation

$$*\varphi \wedge \psi = \langle \varphi, \psi \rangle \mathrm{vol} \quad \forall \psi \in \mathcal{A}^r$$

defines a unique form $*\varphi$. There are some well-known relations: $*1 = \mathrm{vol}$, $*\mathrm{vol} = 1$ and $** : \mathcal{A}^r \to \mathcal{A}^r$ is given by multiplication with $(-1)^{r(n-r)}$. For the exterior differential d we define the adjoint operator $\mathrm{d}^* : \mathcal{A}^r \to \mathcal{A}^{r-1}$ by

$$\int_M \langle \mathrm{d}\varphi, \psi \rangle \mathrm{vol} = \int_M \langle \varphi, \mathrm{d}^*\psi \rangle \mathrm{vol} \quad \forall \varphi \in \mathcal{A}^r_c, \psi \in \mathcal{A}^{r+1}_c$$

where $\mathcal{A}_c$ is the space of forms with compact support.

We define the *Laplace operator* $\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$ and say that a form φ is *harmonic* if $\Delta\varphi = 0$. We write $\mathcal{H}^r$ for the subspace of harmonic functions. Now we can formulate *Poincaré duality*: $* : \mathcal{H}^n \to \mathcal{H}^{n-r}$ is an isomorphism.

Serre Duality

Let M a complex Manifold of complex dimension m. Define a hermitian metric $(\cdot, \cdot)$ by $(\varphi, \psi) := \langle \overline{\varphi}, \psi \rangle$. We need two facts:

(i) The splitting into types $\mathcal{A}^r = \bigoplus_{p+q=r} \mathcal{A}^{p,q}$ is orthogonal with respect to $(\cdot, \cdot)$.

(ii) $* : \mathcal{A}^{p,q} \to \mathcal{A}^{m-q,m-p}$

The proof of (i) consists in calculating $(\mathrm{d}z^i, \mathrm{d}z^j)$, $(\mathrm{d}z^i, \mathrm{d}\overline{z}^j)$ etc. To prove (ii), one shows $(\psi, *\varphi) = 0$ for all $\psi \in \mathcal{A}^{m-q',m-p'}$. Thus, with $\bar{*}\varphi := \overline{*\varphi}$ we have $\bar{*} : \mathcal{A}^{p,q} \to \mathcal{A}^{m-p,m-q}$. Now we derive a useful expression for the adjoint operator $\overline{\partial}^*$: Let $\alpha \in \mathcal{A}^{p,q}_c$, $\beta \in \mathcal{A}^{p,q-1}_c$. Thus $\bar{*}\alpha \wedge \beta \in \mathcal{A}^{m,m-1}$ and therefore $\mathrm{d}(\bar{*}\alpha \wedge \beta) = \overline{\partial}(\bar{*}\alpha \wedge \beta) = \overline{\partial}\bar{*}\alpha \wedge \beta + (-1)^{2m-r}(\bar{*}\alpha \wedge \overline{\partial}\beta)$. Using $\bar{*}\bar{*} = (-1)^r$ for the spaces concerned we get $\mathrm{d}(\bar{*}\alpha \wedge \beta) = (-1)^{2m-r}\left(-(\bar{*}\overline{\partial}\bar{*}\alpha, \beta) + (\alpha, \overline{\partial}\beta)\right)\mathrm{vol}$, which by integration (using Stoke's theorem on the l.h.s.) yields $\int_M (\bar{*}\overline{\partial}\bar{*}\alpha, \beta)\mathrm{vol} = \int_M (\alpha, \overline{\partial}\beta)\mathrm{vol}$. Hence we have $\overline{\partial}^* = \bar{*}\overline{\partial}\bar{*}$. Setting

$\Delta_{\overline{\partial}} := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ another (easier but longer) calculation shows $\left[\Delta_{\overline{\partial}}, \overline{*}\right] = 0$. Thus $\overline{*} : \mathcal{H}^{p,q} \to \mathcal{H}^{m-p,m-q}$ is an isomorphism between the spaces $\mathcal{H}$ of $\overline{\partial}$-harmonic forms. It follows from Hodge theory that the Dolbeault cohomology groups inherit this isomorphism. This gives the final result: $H^{p,q} \cong H^{m-p,m-q}$, known as *Serre duality*.

References

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