The Collected Trivialities of Homological Algebra usually left to the Reader as an Exercise

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The following lemmata are widely used in a large variety of mathematical subjects such as Global Analysis, Algebraic Topology and Algebraic Geometry. In these areas, they are often labeled as boring and technical. As a consequence, their proofs are often unloving, short, inprecise, incomplete, left to the reader as an "exercise", handwaved or simply omitted. It is the aim of this article to fill this gap by giving complete proofs including not only some diagrams, but also classical written calculations. We will also discuss some topological applications. For more on topology see [1].

Comments or suggestions are very welcome, just email to: **n.nowaczyk@web.de**

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1 Diagram Lemmas

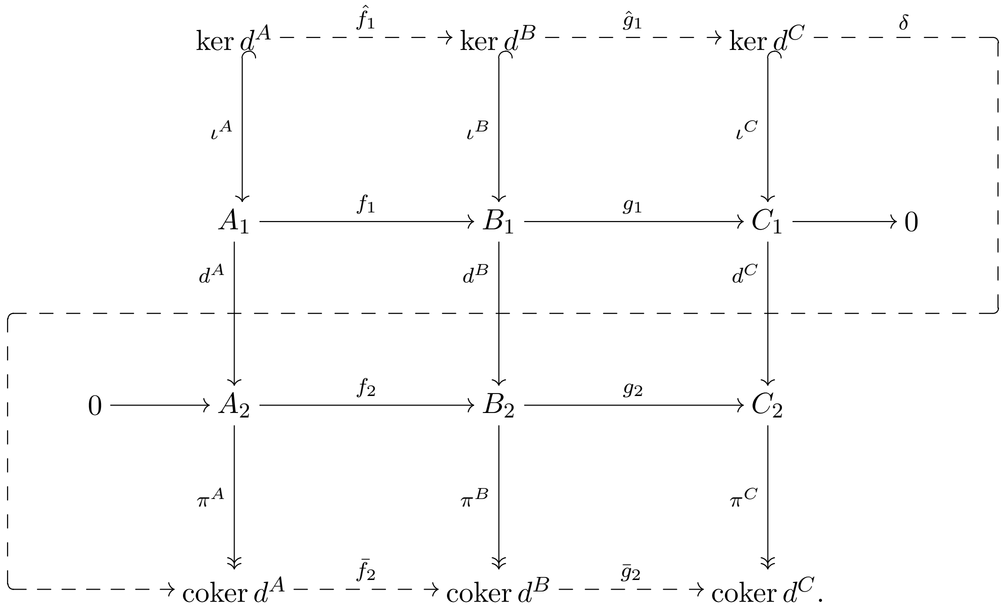
1.1 The Snake Lemma

1.1.1 Statement

1.1 Theorem (Snake Lemma)**.** Let R be a ring and suppose we are given the following commutative diagram in the category of R-modules

$$\begin{array}{ccccccc} & & A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \longrightarrow & 0 \\ & & \downarrow d^A & & \downarrow d^B & & \downarrow d^C & & \\ 0 & \longrightarrow & A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & & \end{array}$$

where the rows are short exact sequences. Then there exists the following morphisms making the following diagramm commutative:



(1.1)

In particular there exists a *connection morphism*[1] $\delta : \ker d^C \to \operatorname{coker} d^A$ such that furthermoore the sequence

$$\ker d^A \xrightarrow{\hat{f}_1} \ker d^B \xrightarrow{\hat{g}_1} \ker d^C \xrightarrow{\delta} \operatorname{coker} d^A \xrightarrow{\bar{f}_2} \operatorname{coker} d^B \xrightarrow{\bar{g}_2} \operatorname{coker} d^C$$

is exact. In addition the following statements are also true:

(i) If f_1 is injective, $\hat{f}_1$ is injective as well. If g_2 is surjective, $\bar{g}_2$ is surjective as well.

(ii) The maps ι^A, ι^B, ι^C are the canonical inclusions and the maps π^A, π^B, π^C are the canonical projections.

(iii) The maps $\hat{f}_1$,$\hat{g}_1$ are the restrictions of f_1,g_1 to $\ker d^A$ resp. $\ker d^B$. The maps $\bar{f}_2$, $\bar{g}_2$ are the maps induced by f_2,g_2 on the quotient $\operatorname{coker} d^A$ resp. $\operatorname{coker} d^B$.

(iv) By slight abuse [2] of notation, the connection morphism δ can be expressed as

$$\delta = \pi^A \circ f_2^{-1} \circ d^B \circ g_1^{-1} \circ \iota^C.$$

[1] Due to the form of the arrow in diagram (1.1) this is "the snake".

[2] The map g_1 is not an isomorphism, so g_1^{-1} has to be read as "chose an arbitrary preimage". This choice is arbitrary, but we will show that this will be compensated by π^A. The map f_2^{-1} is injective and thus can only be inverted on its image.

By defining

$$\text{chase}(c_1) := \{(b_1, b_2, a_2) \in B_1 \times B_2 \times A_2 \mid b_1 \in g_1^{-1}(c_1), d^B(b_1) = b_2, f_2(a_2) = b_2\}$$

to be a *chase tripel of* $c_1 \in \ker d^C$, the connection morphism δ can be expressed as

$$\delta(c_1) = \pi^A(a_2) = [a_2]_A,$$

whenever there are $b_1 \in B_1$, $b_2 \in B_2$ such that $(b_1, b_2, a_2) \in \text{chase}(c_1)$.

1.1.2 Proof

1.1.3 The Upper Row

- Existence of $\hat{f}_1, \hat{g}_1$: Define $\hat{f}_1 := f_1 \circ \iota^A : \ker \delta^A \to B_1$ and $\hat{g}_1 := g_1 \circ \iota^B : \ker d^B \to C_1$. We have to show

$$\operatorname{im} \hat{f}_1 \subseteq \ker d^B, \qquad \operatorname{im} \hat{g}_1 \subseteq \ker d^C.$$

Clearly $d^A \circ \iota^A = 0$. Thus by commutativity:

$$d^B \circ \hat{f}_1 = d^B \circ f_1 \circ \iota^A = f_2 \circ d^A \circ \iota^A = 0.$$

And thus $\operatorname{im} \hat{f}_1 \subseteq \ker d^B$ as claimed. The statement for $\hat{g}_1$ follows analogously. Furthermoore

$$\iota^B \circ \hat{f}_1 = \iota^B \circ f_1 \circ \iota^A = f_1 \circ \iota^A$$

So the upper left square is commutative. Analogously the upper right square is commutative as well.

- Injectivity of $\hat{f}_1$: Suppose f_1 is injective and $a_1 \in \ker \hat{f}_1$, i.e.

$$0 = \hat{f}_1(a_1) = \iota^A(f_1(a_1))$$

This implies

$$f_1(a_1) \in \ker \iota^A = \{0\} \Rightarrow a_1 \in \ker f_1 = \{0\}$$

since f_1 is injective by hypothesis.

- "$\operatorname{im} \hat{f}_1 \subseteq \ker \hat{g}_1$": By hypothesis $g_1 \circ f_1 = 0$, thus

$$\hat{g}_1 \circ \hat{f}_1 = g_1 \circ \iota^B \circ f_1 \circ \iota^A = 0$$

as well.

- "$\ker \hat{g}_1 \subseteq \operatorname{im} \hat{f}_1$": Let $b_1 \in \ker d^B$ thus

$$0 = \hat{g}_1(b_1) = (g_1 \circ \iota^B)(b_1)$$

This implies

$$\iota^B(b_1) \in \ker g_1 = \operatorname{im} f_1 \Rightarrow \exists a_1 \in A_1 : f_1(a_1) = \iota^B(b_1)$$

It remains to show, that $a_1 \in \ker d^A$. But by commutativity

$$(f_2 \circ d^A)(a_1) = (d^B \circ f_1)(a_1) = (d^B \circ \iota^B)(b_1) = 0$$

and thus $d^A(a_1) \in \ker f_2 = \{0\}$. So $a_1 \in \ker d^A$ as claimed and $\hat{f}_1(a_1) = b_1$.

1.1.4 The Lower Row

- Existence of $\bar{f}_2, \bar{g}_2$: By composition we immediately obtain the map $\pi^B \circ f_2 : A_2 \to \operatorname{coker} d^B$. By definition $\operatorname{im} d^B = \ker \pi^B$ and thus commutativity implies
$$\pi^B \circ f_2 \circ d^A = \pi^B \circ d^B \circ f_1 = 0$$
So $\operatorname{im} d^A \subset \ker(\pi^B \circ f_2)$. So by the universal property of the quotient, we obtain $\bar{f}_2 : \operatorname{coker} d^A \to \operatorname{coker} d^B$ such that
$$\hat{f}_2 \circ \pi^A = \pi^B \circ f_2$$
So the bottom left square is commutative. The statements concerning $\bar{g}_2$ follow analogously.
- "$\operatorname{im} \bar{f}_2 \subseteq \ker \bar{g}_2$": By hypothesis $g_2 \circ f_2 = 0$, so by construction
$$\bar{g}_2 \circ \bar{f}_2 \circ \pi^A = \bar{g}_2 \circ \pi^B \circ f_2 = \pi^C \circ g_2 \circ f_2 = 0$$
Since π^A is surjective, $\bar{g}_2 \circ \bar{f}_2 = 0$.
- "$\ker \bar{g}_2 \subseteq \operatorname{im} \bar{f}_2$": Let $[b_2]_B \in \ker \bar{g}_2$, i.e.
$$0 = \bar{g}_2([b_2]_B) = [g_2(b_2)]_C$$
so there exists $c_1 \in C_1 : d^C(c_1) = g_2(b_2)$. By surjectivity of g_1 there exists $b_1 \in B_1 : g_1(b_1) = c_1$. This implies
$$(g_2 \circ d^B)(b_1) = (d^C \circ g_1)(b_1) = d^C(c_1) = g_2(b_2)$$
and thus $g_2(b_2 - d^B(b_1)) = 0$. By exactness
$$b_2 - d^B(b_1) \in \ker g_2 = \operatorname{im} f_2 \Rightarrow \exists a_2 \in A_2 : f_2(a_2) = b_2 - d^B(b_1)$$
This implies
$$\bar{f}_2([a_2]_A) = [f_2(a_2)]_B = [b_2 - d^B(b_1)]_B = [b_2]_B$$
- Surjectivity of $\bar{g}_2$: Suppose g_2 is surjective and let $[c_2]_C \in \operatorname{coker} d^C$ be arbitrary. Since $c_2 \in C_2 = \operatorname{im} g_2$ there exists $b_2 \in B_2 : g_2(b_2) = c_2$. Thus
$$\bar{g}_2([b_2]_B) = [g_2(b_2)]_C = [c_2]_C$$

1.1.5 The Snake

- Existence of δ: Define $\delta : \ker d^C \to \operatorname{coker} d^A$ as follows: Let $c_1 \in \ker d^C \subset C_1$ be arbitrary. Since g_1 is surjective
$$\exists b_1 \in B_1 : g_1(b_1) = c_1$$
but this choice is arbitrary ! Define
$$b_2 := d^B(b_1)$$
Commutativity implies
$$g_2(b_2) = (g_2 \circ d^B)(b_1) = (d^C \circ g_1)(b_1) = d^C(c_1) = 0$$
Thus by exactness and since f_2 is injective
$$b_2 \in \ker g_2 = \operatorname{im} f_2 \Rightarrow \exists! a_2 \in A_2 : f_2(a_2) = b_2.$$
Define
$$\delta(c_1) := \pi^A(a_2) = (\pi^A \circ f_2^{-1} \circ d^B \circ g_1^{-1} \circ \iota^C)(c_1).$$
In this diagram chase only the $b_1 \in g_2^{-1}(c_1)$ was an arbitrary choice. We have just shown that chase(c_1) is never emtpy.

- δ is well defined: We have to show, that δ does not depend on the choice of the chase triple. So chose another
$$(b_1', b_2', a_2') \in \text{chase}(c_1)$$
This implies
$$g_1(b_1' - b_1) = c_1 - c_1 = 0 \Rightarrow b_1' - b_1 \in \ker g_1 = \operatorname{im} f_1 \Rightarrow \exists a_1 \in A_1 : f_1(a_1) = b_1' - b_1$$
Thus
$$b_2' - b_2 = d^B(b_1' - b_1) = (d^B \circ f_1)(a_1) = (f_2 \circ d^A)(a_1),$$
which implies
$$a_2' - a_2 = f_2^{-1}(b_2' - b_2) = d^A(a_1) \in \operatorname{im} d^A = \ker \pi^A.$$
Thus finally
$$\pi^A(a_2') = \pi^A(a_2)$$
This shows that δ is well defined and that for any
$$(b_1, b_2, a_2) \in \text{chase}(c_1)$$
we have $\delta(c_1) = \pi^A(a_2)$. It also justifies the expression
$$\delta = \pi^A \circ f_2^{-1} \circ d^B \circ g_1^{-1} \circ \iota^C.$$
- δ is a homomorphism: Let $\lambda \in R$ and $c_1, c_1' \in d^C$ be arbitrary and
$$(b_1, b_2, a_2) \in \text{chase}(c_1) \qquad\qquad (b_1', b_2', a_2') \in \text{chase}(c_1')$$
Sinc all the maps in the diagram are homomorphisms, we get
$$\begin{aligned} &g_1(b_1 + \lambda b_1') = g_1(b_1) + \lambda g_1(b_1') = c_1 + \lambda c_1' \Rightarrow b_1 + \lambda b_1' \in g_1^{-1}(c_1 + \lambda c_1'),\\ &d^B(b_1 + \lambda b_1') = d^B(b_1) + \lambda d^B(b_1') = b_2 + \lambda b_2',\\ &f_2(a_2 + \lambda a_2') = f_2(a_2) + \lambda f_2(a_2'). \end{aligned}$$
Thus
$$(b_1 + \lambda b_1', b_2 + \lambda b_2', a_2 + \lambda a_2') \in \text{chase}(c_1 + \lambda c_1'),$$
which means
$$\delta(c_1 + \lambda c_1') = \pi^A(a_2 + \lambda a_2') = \pi^A(a_2) + \lambda \pi^A(a_2') = \delta(c_1) + \lambda \delta(c_1').$$
- "$\operatorname{im} \hat{g}_1 \subseteq \ker \delta$": Let $b_1 \in \ker d^B$ be arbitrary. Then $\hat{g}_1(b_1) = (g_1 \circ \iota^B)(b_1)$ and thus especially $b_1 \in g_1^{-1}(b_1)$. This implies
$$(\delta \circ \hat{g}_1)(b_1) = (\pi^A \circ f_2^{-1} \circ d^B)(b_1) = 0,$$
since $b_1 \in \ker d^B$ by hypothesis.
- "$\ker \delta \subseteq \operatorname{im} \hat{g}_1$": Let $c_1 \in \ker \delta$ and chose $(b_1, b_2, a_2) \in \text{chase}(c_1)$. This implies
$$0 = \delta(c_1) = \pi^A(a_2) \Rightarrow \exists a_1 \in A_1 : d^A(a_1) = a_2 = (f_2^{-1} \circ d^B)(b_1)$$
Define $b_1' := f_1(a_1)$. This implies
$$d^B(b_1') = (d^B \circ f_1)(a_1) = (f_2 \circ d^A)(a_1) = (f_2 \circ f_2^{-1} \circ d^B)(b_1) = d^B(b_1)$$
Consequently $b_1 - b_1' \in \ker d^B$. By construction
$$\hat{g}_1(b_1 - b_1') = (g_1 \circ \iota^B)(b_1 - b_1') = g_1(b_1) - g_1(b_1') = c_1 - g_1(f_1(a_1)) = c_1.$$

- "$\operatorname{im}\delta \subseteq \ker \bar{f}_2$": Let $c_1 \in C_1$ be arbitrary. Chose $(b_1, b_2, a_2) \in \text{chase}(c_1)$. This means in particular $d^B(b_1) = f_2(a_2)$. This implies

$$(\bar{f}_2 \circ \delta)(c_1) = \bar{f}_2([a_2]_A) = [f_2(a_2)]_B = [d^B(b_1)]_B = 0$$

- "$\ker \bar{f}_2 \subseteq \operatorname{im}\delta$": Let $[a_2]_A \in \ker \bar{f}_2$ be arbitrary. This implies

$$0 = \bar{f}_2([a_2]_A) = [f_2(a_2)]_B \Rightarrow \exists b_1 \in B_1 : b_2 := d^B(b_1) = f_2(a_2)$$

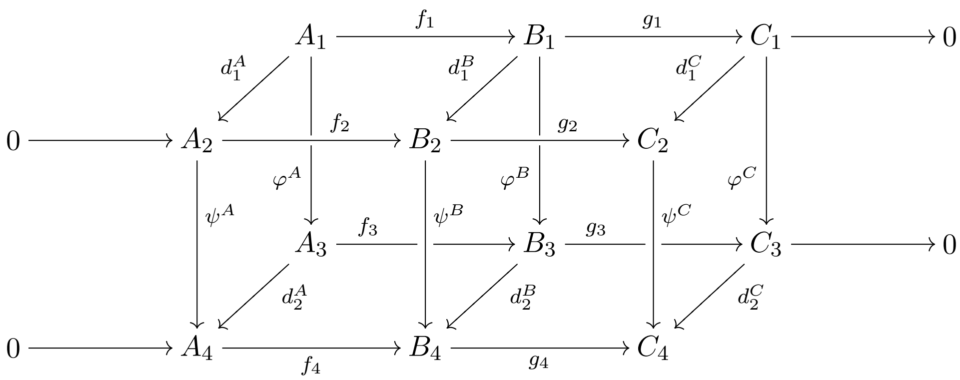
  Define $c_1 := g_1(b_1)$. It follows

$$d^C(c_1) = (d^C \circ g_1)(b_1) = (g_2 \circ d^B)(b_1) = (g_2 \circ f_2)(a_2) = 0,$$

  so $c_1 \in \ker d^C$. This implies that $(b_1, b_2, a_2) \in \text{chase}(c_1)$ so in particular $\delta(c_1) = [a_2]_A$.

1.1.6 Naturality

1.2 Theorem (Naturality)**.** Suppose we are given two Snake Lemma Diagrams and a morphism between them, i.e. a commutative diagram



Then there is a morphism between the induced exact sequences, i.e. a commutative diagram

$$\begin{array}{ccccccccccc}
\ker d_1^A & \xrightarrow{\hat{f}_1} & \ker d_1^B & \xrightarrow{\hat{g}_1} & \ker d_1^C & \xrightarrow{\delta_1} & \operatorname{coker} d_1^A & \xrightarrow{\bar{f}_2} & \operatorname{coker} d_1^B & \xrightarrow{\bar{g}_2} & \operatorname{coker} d_1^C \\
\downarrow{\scriptstyle \hat{\varphi}^A} & & \downarrow{\scriptstyle \hat{\varphi}^B} & & \downarrow{\scriptstyle \hat{\varphi}^C} & & \downarrow{\scriptstyle \bar{\psi}^A} & & \downarrow{\scriptstyle \bar{\psi}^B} & & \downarrow{\scriptstyle \bar{\psi}^C} \\
\ker d_2^A & \xrightarrow{\hat{f}_3} & \ker d_2^B & \xrightarrow{\hat{g}_3} & \ker d_2^C & \xrightarrow{\delta_2} & \operatorname{coker} d_2^A & \xrightarrow{\bar{f}_4} & \operatorname{coker} d_2^B & \xrightarrow{\bar{g}_4} & \operatorname{coker} d_2^C
\end{array}$$

Here the maps $\hat{\varphi}^A$,$\hat{\varphi}^B$,$\hat{\varphi}^C$ are just the restriction of φ^A,φ^B,φ^C and the maps $\bar{\psi}^A$,$\bar{\psi}^B$,$\bar{\psi}^C$ are the induced maps of ψ^A,ψ^B,ψ^C on the quotient.

Proof. We index the inclusions and projections generated by the top snake (1.1.(ii)) whith a 1 and the ones generated by the bottom snake with a 2.
Step 1 (Existence of $\hat{\varphi}$): We are already given a map $\varphi^A : A_1 \to A_3$ and claim that its restriction to the kernel is the desired map $\hat{\varphi}^A$: Let $a_1 \in \ker d_1^A$ be arbitrary. By commutativity

$$(d_2^A \circ \varphi^A)(a_1) = (\psi^A \circ d_1^A)(a_1) = 0$$

thus $\varphi^A(a_1) \in \ker d_2^A$. So $\hat{\varphi}^A := \varphi^A|_{\ker d_1^A}$ is a map $\ker d_1^A \to \ker d_2^A$. In the same manner we define $\hat{\varphi}^B$ and $\hat{\varphi}^C$.
Step 2 (Existence of $\bar{\psi}$): We are given a map $\psi^A : A_2 \to A_4$. Define $\bar{\psi}^A : \operatorname{coker} d_1^A \to \operatorname{coker} d_2^A$ by

$$\bar{\psi}^A([a_2]) := [\psi^A(a_2)]$$

Or in other words, we would like to define this map by $\bar{\psi}^A \circ \pi_1^A = \pi_2^A \circ \psi^A$. We have to check, that this is well defined: So let $a_1 \in A_1$ be arbitrary and $a_2 := d_1^A(a_1)$. By commutativity

$$\psi^A(a_2) = (\psi^A \circ d_1^A)(a_1) = (d_2^A \circ \varphi^A)(a_1) \in \operatorname{im} d_2^A$$

thus $[\psi^A(a_2)]_{A_4} = 0$. The maps $\bar{\psi}^B$ and $\bar{\psi}^C$ are defined in the same manner.
Step 3 (Commutativity of the left squares): By hypothesis $\varphi^B \circ f_1 = f_3 \circ \varphi^A$. By the snake lemma (1.1.(iii)) $\hat{f}_1$ and $\hat{f}_3$ are just restrictions of f_1 resp. f_3 and by step 1 $\hat{\varphi}^A$ and $\hat{\varphi}^B$ are just restrictions of φ^A resp. φ^B as well. Thus $\hat{\varphi}^B \circ \hat{f}_1 = \hat{f}_3 \circ \hat{\varphi}^A$ as well. The same argument holds for $\hat{\varphi}^C \circ \hat{g}_1 = \hat{g}_3 \circ \hat{\varphi}_B$.
Step 4 (Commutativity of the right squares): By hypothesis $\psi^B \circ f_2 = f_4 \circ \psi^A$. Thus by the snake lemma (1.1.(iii) and by definition

$$(\bar{\psi}^B \circ \bar{f}_2)([a_2]) = \bar{\psi}^B([f_2(a_2)]) = [\psi^B(f_2(a_2)] = [f_4(\psi^A(a_2))] = \bar{f}_4([\psi^A(a_2)]) = (\bar{f}_4 \circ \bar{\psi}^A)(a_2)$$

The same argument works for $\bar{\psi}^C \circ \bar{g}_2 = \bar{g}_4 \circ \bar{\psi}^B$.
Step 5 (Commutativity of the center square): Let $c_1 \in \ker d_1^C$ be arbitrary and chose $(b_1, b_2, a_2) \in \operatorname{chase}(c_1)$. We claim that

$$(\varphi^B(b_1), \psi^B(b_2), \psi^A(a_2)) \in \operatorname{chase}(\hat{\varphi}^C(c_1))$$

But indeed

$$\begin{aligned} g_3(\varphi^B(b_1)) &= (\varphi^C \circ g_1)(b_1) = \hat{\varphi}^C(c_1) \\ d_2^B(\varphi^B(b_2)) &= (\psi^B \circ d_1^B)(b_1) = \psi^B(b_2) \\ f_4(\psi^A(a_2)) &= (\psi^B \circ f_2)(a_2) = \psi^B(b_2) \end{aligned}$$

Thus by definition

$$(\bar{\psi}^A \circ \delta_1)(c_1) = (\bar{\psi}^A \circ \pi_1^A)(a_2) = (\pi_2^A \circ \psi^A)(a_2) = (\delta_2 \circ \hat{\varphi}^C)(c_1)$$

□

1.1.7 Long Exact Sequence

An immediate and extremly important application of the snake lemma is the following

1.3 Corollary (The long exact Sequence)**.** Every short exact sequence of chain complexes induces a long exact sequence in their homology. More precise: Let R be a ring and

$$0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$$

be a short exact sequence in the category of chain complexes of R-modules. Then there is a long exact sequence

$$\cdots \xrightarrow{\delta_{n-1}} H_n(A) \xrightarrow{[f_n]} H_n(B) \xrightarrow{[g_n]} H_n(C) \xrightarrow{\delta_n} H_{n-1}(A) \longrightarrow \cdots$$

Here $[f_n] = H_n(f)$, $[g_n] = H_n(g)$ and by slight abuse of notation the connection homomorphism δ_n can be calculated by

$$\forall c_n \in Z_n^C : \delta_n([c_n]) = [(f_{n-1}^{-1} \circ d_n^B \circ g_n^{-1})(c_n)].$$

Alternatively, if we define

$$\operatorname{chase}([c_n]) := \{(b_n, b_{n-1}, a_{n-1}) \in B_n \times B_{n-1} \times A_{n-1} \mid b_n \in g_n^{-1}(c_n), d_n^B(b_n) = b_{n-1}, f_{n-1}(a_{n-1}) = b_{n-1}\}$$

for any $c_n \in Z_n^C$, then

$$\delta([c_n]) = [a_2].$$

Moreover this long exact sequence is natural. This means that for any commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \xrightarrow{f} & B_* & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi^A} & & \downarrow{\scriptstyle \varphi^B} & & \downarrow{\scriptstyle \varphi^C} & & \\ 0 & \longrightarrow & A'_* & \xrightarrow{f'} & B'_* & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

in the category of short exact sequences of chain complexes of R modules, we get a morphism in the category of long exact sequences or R-modules, i.e. a commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{\delta_{n-1}} & H_n(A) & \xrightarrow{[f_n]} & H_n(B) & \xrightarrow{[g_n]} & H_n(C) & \xrightarrow{\delta_n} & H_{n-1}(A) & \longrightarrow & \cdots \\ & & {\scriptstyle [\varphi^A_n]}\downarrow & & {\scriptstyle [\varphi^B_n]}\downarrow & & {\scriptstyle [\varphi^C_n]}\downarrow & & {\scriptstyle [\varphi^A_{n-1}]}\downarrow & & \\ \cdots & \xrightarrow{\delta'_{n-1}} & H_n(A') & \xrightarrow{[f'_n]} & H_n(B') & \xrightarrow{[g'_n]} & H_n(C') & \xrightarrow{\delta'_n} & H_{n-1}(A') & \longrightarrow & \cdots \end{array}$$

Proof. By definition for every $n \in \mathbb{Z}$

$$\begin{array}{ccccccccc} & & A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} & \longrightarrow & 0 \\ & & {\scriptstyle d^A_{n+1}}\downarrow & & {\scriptstyle d^B_{n+1}}\downarrow & & {\scriptstyle d^C_{n+1}}\downarrow & & \\ 0 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & & \end{array}$$

satisfies the hypothesis of the Snake Lemma 1.1. Thus in the following diagram (which we regard to be the diagram for the index n)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z^A_{n+1} & \longrightarrow & Z^B_{n+1} & \longrightarrow & Z^C_{n+1} & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} & \longrightarrow & 0 \\ & & {\scriptstyle d^A_{n+1}}\downarrow & & {\scriptstyle d^B_{n+1}}\downarrow & & {\scriptstyle d^C_{n+1}}\downarrow & & \\ 0 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \frac{A_n}{d^A_{n+1}A_{n+1}} & \longrightarrow & \frac{B_n}{d^B_{n+1}B_{n+1}} & \longrightarrow & \frac{C_n}{d^C_{n+1}C_{n+1}} & \longrightarrow & 0 \end{array} \tag{1.2}$$

the top and the bottom row are part of a leS. Applying (1.2) to the top row and the index $n-1$ and to the bottom row for the index n, we obtain that the diagram

$$\begin{array}{ccccccccc} & & \frac{A_n}{d^A_{n+1}A_{n+1}} & \xrightarrow{\bar{f}_n} & \frac{B_n}{d^B_{n+1}B_{n+1}} & \xrightarrow{\bar{g}_n} & \frac{C_n}{d^C_{n+1}C_{n+1}} & \longrightarrow & 0 \\ & & {\scriptstyle \bar{d}^A_n}\downarrow & & {\scriptstyle \bar{d}^B_n}\downarrow & & {\scriptstyle \bar{d}^C_n}\downarrow & & \\ 0 & \longrightarrow & Z^A_{n-1} & \xrightarrow{\hat{f}_{n-1}} & Z^B_{n-1} & \xrightarrow{\hat{g}_{n-1}} & Z^C_{n-1} & & \end{array}$$

has exaxt rows. Here maps $\bar{f}_n$,$\bar{g}_n$,$\hat{f}_{n-1}$,$\hat{g}_{n-1}$ are induced as in the Snake Lemma (1.1.(iii)). The map $\bar{d}^A_n$ is induced via d^A_n and the universal property of the quotient

$$\begin{array}{ccc} A_n & \xrightarrow{d^A_n} & Z^A_{n-1} \\ \downarrow & \nearrow{\scriptstyle \bar{d}^A_n} & \\ \frac{A_n}{d^A_{n+1}A_{n+1}} & & \end{array}$$

and the maps $\bar{d}_n^B$,$\bar{d}_n^C$ likewise. The diagram is commutative since f and g are morphisms of chain complexes. So we may apply the snake lemma again, to obtain a morphism δ_n such that

$$0 \longrightarrow \ker \bar{d}_n^A \longrightarrow \ker \bar{d}_n^B \longrightarrow \ker \bar{d}_n^C \xrightarrow{\delta_n} \operatorname{coker} \bar{d}_n^A \longrightarrow \operatorname{coker} \bar{d}_n^B \longrightarrow \operatorname{coker} \bar{d}_n^C \longrightarrow 0$$

is exact. We claim that in fact this is the desired sequence. Notice that

$$H_n(A) = \frac{\ker(d_n^A : A_n \to Z_{n-1})}{\operatorname{im}(d_{n+1}^A : A_{n+1} \to A_n)} = \ker\left(\bar{d}_n^A : \frac{A_n}{d_{n+1}^A A_n} \to Z_{n-1}\right),$$

since

$$\forall a A_n : 0 = \bar{d}_n^A([a]) = d_n(a) \Leftrightarrow [a] \in H_n(A).$$

The same holds for B,C as well. For the cokernels we obtain

$$H_{n-1}(A) = \frac{\ker(d_{n-1}^A : A_{n-1} \to Z_{n-2})}{\operatorname{im}(d_n^A : A_n \to B_{n-1}^A)} = \operatorname{coker}\left(\bar{d}_n^A : \frac{A_n}{d_{n+1}^A A_{n+1}} \to Z_{n-1}^A\right),$$

since

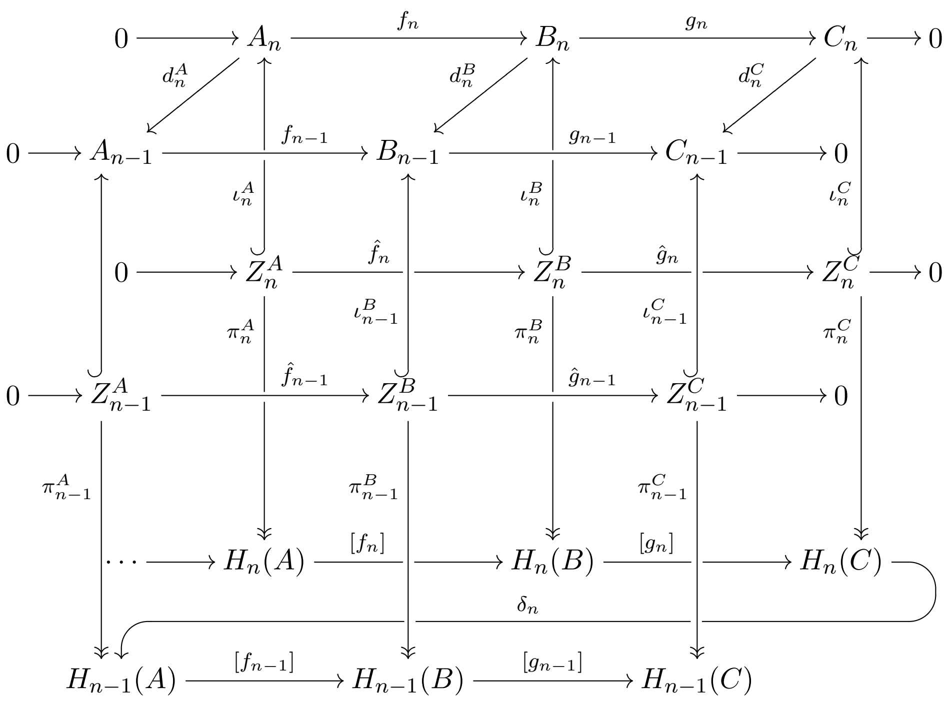
$$\forall a \in A_n : \bar{d}_n^A([a]) = d_n(a) \in d_n A_n.$$

The snake lemma (1.1(iii)) also states that the map $\ker \bar{d}_n^A \to \ker \bar{d}_n^B$ is just the restriction

$$\bar{f}_n|_{\ker \bar{d}_n^A} = H_n(f) = [f_n]$$

and similar we identify the map $\ker \bar{d}_n^B \to \ker \bar{d}_n^C$ as $[g_n]$. Also by 1.1.(iii) the map $\operatorname{coker} \bar{d}_n^A \to \operatorname{coker} \bar{d}_n^B$ is the map induced by $\bar{f}_{n-1}$ on the quotient and thus identical to $H_{n-1}(f) = [f_{n-1}]$. The same holds for $\operatorname{coker} \bar{d}_n^B \to \operatorname{coker} \bar{d}_n^C$. The formula for the connection morphism δ_n is also directly derived from 1.1(iv). By pasting the upper sequences together for all $n \in \mathbb{Z}$ we obtain the desired long exact sequence. The naturality statement is a direct application of Theorem 1.2. □

1.4 Remark. The construction of the long exact sequence from the Snake Lemma may be visualized in its finest beauty by:



1.5 Corollary (leS of the Pair)**.** Let (X, A) be a pair of spaces. Then there is a natural leS

$$\cdots \xrightarrow{\delta_{n+1}} H_n(A) \xrightarrow{[\iota]} H_n(X) \xrightarrow{[\pi]} H_n(X, A) \xrightarrow{\delta_n} H_{n-1}(A) \longrightarrow \cdots$$

where H_* denotes the Singular Homology functor, $\iota : A \to X$ the canonical inclusion and $\pi_* : C_*(X) \to C_*(X, A)$ the canonical projection. Using the fact that for any $\bar{z} \in C_n(X, A)$

$$\bar{z} \in Z_q(X, A) \Leftrightarrow \exists a \in C_n(A) : \exists x \in C_{n+1}(X) : \bar{z} = a + d^X_{n+1}(x)$$

the connection morphism δ can be described as

$$\delta([z]) = [d^A_n(a)]$$

This sequence is natural in (X, A).

Proof. There is a seS

$$0 \longrightarrow C_*(A) \xrightarrow{\iota_*} C_*(X) \xrightarrow{\pi_*} C(X, A) \longrightarrow 0$$

of chain complexes, where C_* denotes the singular chain functor. Thus the statement follow from 1.3. □

1.6 Corollary (leS of the Triple)**.** Let (X, Y, Z) be a triple of spaces. Denote by $i : (Y, Z) \to (X, Z)$ and $j : (X, Z) \to (X, Y)$ the canonical inclusions. Then there is a leS

$$\cdots \xrightarrow{\delta_{q-1}} H_q(Y, Z) \xrightarrow{[i_q]} H_q(X, Z) \xrightarrow{[j_q]} H_q(X, Y) \xrightarrow{\delta_q} H_{q-1}(Y, Z) \longrightarrow \cdots$$

The connection homomorphism can be described as follows: For any $c \in Z_q(X, Y)$

$$\delta([c]) = [(i^{-1}_{q-1} \circ d^{X,Z}_q \circ j^{-1}_q)(c)] = [d^{X,Z}_q(c)] \in Z_{q-1}(Y, Z)$$

Proof. It is more or less obvious, that

$$0 \longrightarrow C_*(Y, Z) \xrightarrow{i_*} C_*(X, Z) \xrightarrow{j_*} C_*(X, Y) \longrightarrow 0$$

is short exact. So the statement follows from 1.3.
To see exactness, chose any $q \in \mathbb{N}$, $y \in C_q(Y)$ and denote by $\lceil y \rceil^Z \in C_q(Y, Z)$ the corresponding equivalence class. We have

$$i_q(\lceil y \rceil^Z) = 0 \Rightarrow y \in Z \Rightarrow \lceil y \rceil^Z = 0$$

thus i_* is injective. For any $\lceil x \rceil^Y \in C_*(X, Y)$, we have

$$j_q(\lceil y \rceil^Z) = \lceil y \rceil^Y$$

by definition. Thus j_* is surjective. Clearly $j_* \circ i_* = 0$, thus $\operatorname{im} i_* \subseteq \ker j_*$. Conversely

$$\lceil x \rceil^Z \in \ker j_q \Rightarrow j_q(\lceil x \rceil^Z) = \lceil x \rceil^Y = 0 \Rightarrow x \in Y$$

Thus $i_q(\lceil x \rceil^Z) = \lceil x \rceil^Z$. □

1.2 The Five Lemma

1.7 Lemma. Let R be a ring and assume we are given the following commutative diagram in the category of R-modules:

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\
\big\downarrow\!\!\downarrow \varphi_1 & & \sim \big\downarrow \varphi_2 & & \big\downarrow \varphi_3 & & \big\downarrow \varphi_4 & & \big\hookdownarrow \varphi_5 \\
B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5
\end{array}$$

Let both rows be exact and φ_2, φ_4 be isomorphisms. If φ_1 is an epimorphism, then φ_3 is a monomorphism. If φ_5 is a monomorphism, then φ_3 is an epimorphism.

1.8 Remark. One usually memorizes this as: If the outer four maps $\varphi_1, \varphi_2, \varphi_4, \varphi_5$ are isomorphisms, so is the inner one φ_3.

Proof.
Injectivity: We show, that $\ker \varphi_3$ is trivial. So assume $x \in \ker \varphi_3$. This implies

$$(\varphi_4 \circ \alpha_3)(x) = (\beta_3 \circ \varphi_3)(x) = 0$$

due to commutativity of the diagram. Since φ_4 is injective

$$x \in \ker \alpha_3 = \operatorname{im} \alpha_2 \Longrightarrow \exists a_2 \in A_2 : \alpha_2(a_2) = x$$

This implies

$$(\beta_2 \circ \varphi_2)(a_2) = (\varphi_3 \circ \alpha_2)(a_2) = \varphi_3(x) = 0$$

Thus

$$\varphi_2(a_2) \in \ker \beta_2 = \operatorname{im} \beta_1 \Longrightarrow \exists b_1 \in B_1 : \beta_1(b_1) = \varphi_2(a_2)$$

Since φ_1 is surjective by hypothesis, there exists $a_1 \in A_1$ such that $\varphi_1(a_1) = b_1$. It follows

$$(\varphi_2 \circ \alpha_1)(a_1) = (\beta_1 \circ \varphi_1)(a_1) = \beta_1(b_1) = \varphi_2(a_2)$$

Since φ_2 is injective

$$\alpha_1(a_1) = a_2 \Longrightarrow a_2 \in \operatorname{im} \alpha_1 = \ker \alpha_2 \Longrightarrow x = \alpha_2(a_2) = 0$$

Surjectivity: Let $b_3 \in B_3$ be arbitrary. Since φ_4 is surjective

$$\exists a_4 \in A_4 : \varphi_4(a_4) = \beta_3(b_3)$$

We obtain

$$(\varphi_5 \circ \alpha_4)(a_4) = (\beta_4 \circ \varphi_4)(a_4) = (\beta_4 \circ \beta_3)(b_3) = 0$$

Since φ_5 is injective

$$\alpha_4(a_4) = 0 \Longrightarrow a_4 \in \ker \alpha_4 = \operatorname{im} \alpha_3 \Longrightarrow \exists a_3 \in A_3 : \alpha_3(a_3) = a_4$$

This impiles

$$(\beta_3 \circ \varphi_3)(a_3) = (\varphi_4 \circ \alpha_3)(a_3) = \varphi_4(a_4) = \beta_3(b_3)$$

and thus

$$\varphi_3(a_3) - b_3 \in \ker \beta_3 = \operatorname{im} \beta_2 \Longrightarrow \exists b_2 \in B_2 : \beta_2(b_2) = \varphi_3(a_3) - b_3$$

Since φ_2 is surjective there is a $a_2 \in A_2$ such that $\varphi_2(a_2)$, which implies

$$(\varphi_3 \circ \alpha_2)(a_2) = (\beta_2 \circ \varphi_2)(a_2) = \beta_2(b_2) = \varphi_3(a_3) - b_3 \Longleftrightarrow b_3 = \varphi_3(a_3 - \alpha_2(a_2))$$

$\square$

2 Resolutions

2.1 Definition (projective)**.** An R-Module P is projective, if

$$\begin{array}{ccc} & & P \\ & \swarrow^{\exists\beta} & \downarrow \gamma \\ B & \underset{\pi}{\twoheadrightarrow} & C \end{array}$$

for every other R-Modules B, C and every morphism $\gamma : B \to C$ and every surjective morphism $\pi : B \to C$ there exists a $\beta : P \to B$ such that $\pi \circ \beta = \gamma$.

2.1 The Horseshoe Lemma

2.2 Lemma. Let R be a ring. Assume we are given a short exact sequence

$$0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$$

in the category of R-modules. Let $(P'_*, p'_*, \varepsilon')$ be a projective resolution of A' and $(P''_*, p''_*, \varepsilon'')$ be a projective resolution of A''. Then the $P_n := P'_n \oplus P''_n$ can be connected by morhpisms to a projective resolution of A and there is a short exact sequence

$$0 \longrightarrow P' \xrightarrow{\iota} P \xrightarrow{\pi} P'' \longrightarrow 0$$

in the category of chain complexes of R-modules. The map $\iota_n : P'_n \to P_n$ is the canonical inclusion, the map $\pi_n : P_n \to P''_n$ the canonical projection.

Proof. We construct the augmentation map first: Since g is surjective and P''_0 is projective

$$\begin{array}{ccccc} P'_0 & & P_0 & & P''_0 \\ \downarrow \varepsilon' & & \vdots\, \varepsilon & \overset{\beta}{\swarrow} & \downarrow \varepsilon'' \\ A' & \xrightarrow{f} & A & \xrightarrow{g} & A'' \end{array}$$

there exists $\beta : P''_0 \to A$ such that $g \circ \beta = \varepsilon''$. Define $\varepsilon : P_0 \to A$ by $\varepsilon := (f \circ \varepsilon') \oplus \beta$. Now extend this diagram to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P'_0 & \xrightarrow{\iota_0} & P_0 & \xrightarrow{\pi_0} & P''_0 & \longrightarrow & 0 \\ & & \downarrow \varepsilon' & & \downarrow \varepsilon & & \downarrow \varepsilon'' & & \\ 0 & \longrightarrow & A' & \xrightarrow{f} & A & \xrightarrow{g} & A'' & \longrightarrow & 0 \end{array}$$

By construction both rows are exact. Furthermore

$$\varepsilon \circ \iota_0 = f \circ \varepsilon' \qquad g \circ \varepsilon = g \circ (f \circ \varepsilon' \oplus \beta) = 0 \oplus g \circ \beta = 0 \oplus \varepsilon'' = \pi_0 \circ \varepsilon''$$

thus both squares commute, i.e. this is a morphism of short exact sequences. The snake lemma 1.1 yields

$$\begin{array}{ccccccccc} & & \ker \varepsilon' & \longrightarrow & \ker \varepsilon & \longrightarrow & \ker \varepsilon'' & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P'_0 & \xrightarrow{\iota_0} & P_0 & \xrightarrow{\pi_0} & P''_0 & \longrightarrow & 0 \\ & & \downarrow \varepsilon' & & \downarrow \varepsilon & & \downarrow \varepsilon'' & & \\ 0 & \longrightarrow & A' & \xrightarrow{f} & A & \xrightarrow{g} & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & \operatorname{coker} \varepsilon & \longrightarrow & \operatorname{coker} \varepsilon'' & & \end{array}$$

where every square is commutative and the upper row together with the connection morphism $\delta : \ker \varepsilon'' \to \operatorname{coker} \varepsilon'$ and the bottom row is an exact sequence. Since $\varepsilon', \varepsilon''$ are both surjective by hypothesis their cokernels vanish. Exactness now implies $\operatorname{coker} \varepsilon = 0$, i.e. ε is surjective as well. This finishes the construction of the augmentation map.
The situation now is

$$\begin{array}{ccccccccc} & & P_1'' & & & & P_1' & & \\ & & \downarrow{\scriptstyle p_1'} & & & & \downarrow{\scriptstyle p_1'} & & \\ 0 & \longrightarrow & \ker \varepsilon' & \longrightarrow & \ker \varepsilon & \longrightarrow & \ker \varepsilon'' & \longrightarrow & 0 \end{array}$$

since the given projective resulutions are exact sequences by hypothesis. But this is precisely the same situation as in the beginning, just one index further above and with other names for the objects. So the filling of the horseshoe proceeds inductively by applying the procedure above again and again. □

2.2 The Fundamental Lemma

2.3 Theorem (Fundamental Lemma of Homological Algebra)**.** Let A,B be modules over R and (P_*, α), (Q_*, β) be projective resolutions of A and B resp. Then there exists isomorphisms

$$\operatorname{Hom}_R(A, B) \cong \operatorname{Hom}_R(H_0(P_*), H_0(Q_*)) \cong [P_*, Q_*]$$

where $[P_*, Q_*]$ denote the chain homotopy classes of chain maps $P_* \to Q_*$. The second isomorphism is induced by

$$(f : P_* \to Q_*) \mapsto H_0(f)$$

Proof. The first isomorphy is obtained by the following: By exactness of the projective resolution, by the surjectivity of α and by the universal property of the quotient, we obtain

$$\begin{array}{ccccccc} P_1 & \xrightarrow{\delta_1^P} & P_0 & \xrightarrow{\alpha} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_0^P} & \nearrow{\scriptstyle \bar{\alpha}} & & & \\ & & P_0/\ker \alpha & & & & \end{array}$$

Here π_0^P is the canonical projection and $\bar{\alpha}$ is the induced isomorphism satisfying $\bar{\alpha} \circ \pi_0^P = \alpha$. We obtain

$$A \cong P_0/\ker \alpha = P_0/\operatorname{im} \delta_1^P = H_0(P_*)$$

since the projective resolution complex

$$P_1 \xrightarrow{\delta_1^P} P_0 \longrightarrow 0$$

has homology

$$H_0(P_*) = \ker 0/\operatorname{im} \delta_1^P = P_0/\operatorname{im} \delta_1^P$$

Applying the same procedure to (Q_*, β) we obtain an analogue isomorphism $\bar{\beta} : H_0(Q_*) \to B$. Thus the maps

$$\begin{aligned} &\operatorname{Hom}_R(A, B) \to \operatorname{Hom}_R(H_0(P_*), H_0(Q_*)) && f \mapsto \bar{\beta} \circ f \circ \bar{\alpha} \\ &\operatorname{Hom}_R(H_0(P_*), H_0(Q_*)) \to \operatorname{Hom}_R(A, B) && g \mapsto \bar{\beta}^{-1} \circ g \circ \bar{\alpha}^{-1} \end{aligned}$$

yield the first isomorphy.
To proof the second isomorphy, we will show, that there exists a bijection

$$\Psi : [P_*, Q_*] \to \operatorname{Hom}_R(H_0(P_*), H_0(Q_*))$$

A chain homotopy class $[f] : P_* \to Q_*$ induces a map $H_0(f)$, wich is independent of the choice of representative f. So we obtain a well defined element

$$\Psi([f]) := H_0(f) \in \operatorname{Hom}_R(H_0(P_*), H_0(Q_*))$$

We claim, that Ψ is surjective: Let $f \in \operatorname{Hom}_R(H_0(P_*), H_0(Q_*))B$ be arbitrary. The projection map $\pi_0^Q : Q_0 \to H_0(Q_*)$ is surjective. So by projectivity of P_0

$$\begin{array}{ccc}
P_0 & \xrightarrow{\pi_0^P} & H_0(P_*) \\
{\scriptstyle \varphi_0} \downarrow & & \downarrow {\scriptstyle f} \\
Q_0 & \xrightarrow{\pi_0^Q} & H_0(Q_*)
\end{array}$$

there exists a map $\varphi_0 : P_0 \to Q_0$ such that

$$\pi_0^Q \circ \varphi_0 = f \circ \pi_0^P$$

We can now proceed by induction: Suppose we have already constructed $\varphi_0, \ldots, \varphi_{n-1}$. Then we obtain

$$\begin{array}{ccccc}
P_n & \xrightarrow{\delta_n^P} & P_{n-1} & \xrightarrow{\delta_{n-1}^P} & P_{n-2} \\
{\scriptstyle \varphi_n} \downarrow & & \downarrow {\scriptstyle \varphi_{n-1}} & & \downarrow {\scriptstyle \varphi_{n-2}} \\
Q_n & \xrightarrow{\delta_n^Q} & Q_{n-1} & \xrightarrow{\delta_{n-1}^Q} & Q_{n-2}
\end{array}$$

In case $n = 1$ we interpret $P_{-1} := H_0(P_*)$, $Q_{-1} := H_0(Q_*)$. By commutativiy we obtain

$$\delta_{n-1}^Q \circ \varphi_{n-1} \circ \delta_n^P = \varphi_{n-2} \circ \delta_{n-1}^P \circ \delta_n^P = 0$$

and thus

$$\operatorname{im} \varphi_{n-1} \circ \delta_n^P \subseteq \ker \delta_{n-1}^Q = \operatorname{im} \delta_n^Q$$

So we can regard $\varphi_{n-1} \circ \delta_n^P$ as a map $P_n \to \operatorname{im} \delta_n^Q$. Since $\delta_n^Q : Q_n \to \operatorname{im} \delta_n^Q$ is certainly surjective the projectivity of P_n yields a map $\varphi_n : P_n \to Q_n$ such that

$$\delta_n^Q \circ \varphi_n = \varphi_{n-1} \circ \delta_n^P$$

So we have constructed a chain map $\varphi_* : P_* \to Q_*$ with $\Psi([\varphi_*]) = f$.
We claim that Ψ is also injective: Suppose we have a map $f : H_0(P_*) \to H_0(Q_*)$ and two chain maps $\varphi_*, \psi_* : P_* \to Q_*$ such that $\Psi([(\varphi_*])) = \Psi([(\psi_*])) = f$. We have to construct a chain homotpy $\Sigma : P_* \to Q_{*+1}$. Consider

$$\begin{array}{ccccc}
P_1 & \xrightarrow{\delta_1^P} & P_0 & \xrightarrow{\pi_0^P} & H_0(P_*) \\
{\scriptstyle \psi_1} \downarrow \downarrow {\scriptstyle \varphi_1} & & {\scriptstyle \psi_0} \downarrow \downarrow {\scriptstyle \varphi_0} & & \downarrow {\scriptstyle f} \\
Q_1 & \xrightarrow[\delta_1^Q]{} & Q_0 & \xrightarrow[\pi_0^Q]{} & H_0(Q_*)
\end{array}$$

Since φ_0, ψ_0 both induce f we have

$$\pi_0^Q \circ (\psi_0 - \varphi_0) = \pi_0^Q \circ \psi_0 - \pi_0^Q \circ \varphi_0 = \pi_0^P \circ \varphi - \pi_0^P \circ \varphi = 0$$

Thus $\operatorname{im} \psi_0 - \varphi_0 \subseteq \ker \pi_0^Q = \operatorname{im} \delta_1^Q$. Since $\delta_1^Q : Q_1 \to \operatorname{im} \delta_1^Q$ is certainly surjective the projectiviy of P_0 yields a map $\Sigma_0 : P_0 \to Q_1$ such that

$$\delta_1^Q \circ \Sigma_0 = \psi_0 - \varphi_0$$

We continue now by induction: Suppose $\Sigma_0, \ldots, \Sigma_{n-1}$ are already defined such that

$$\forall 1 \leq i \leq n-1 : \delta_i^Q \circ \Sigma_i + \Sigma_{i-1} \circ \delta_i^P = \psi_i - \varphi_i$$

where we define $\Sigma_{-1} := 0$. Then we obtain again

$$\begin{CD}
P_{n+1} @>{\delta_{n+1}^P}>> P_n @>{\delta_n^P}>> P_{n-1} \\
@V{\psi_{n+1}}V{\varphi_{n+1}}V @V{\psi_n}V{\varphi_n}V @V{\psi_{n-1}}V{\varphi_{n-1}}V \\
Q_{n+1} @>>{\delta_{n+1}^Q}> Q_n @>>{\delta_n^Q}> Q_{n-1}
\end{CD}$$

Analogously we have:

$$\begin{aligned}
&\delta_n^Q \circ (\psi_n - \varphi_n - \Sigma_{n-1} \circ \delta_n^P) \\
&\quad = \delta_n^Q \circ \psi_n - \delta_n^Q \circ \varphi_n - \delta_n^Q \circ \Sigma_{n-1} \circ \delta_n^P \\
&\quad = \psi_{n-1} \circ \delta_n^P - \varphi_{n-1} \circ \delta_n^P - \delta_n^Q \circ \Sigma_{n-1} \circ \delta_n^P \\
&\quad = (\psi_{n-1} - \varphi_{n-1} - \delta_n^Q \circ \Sigma_{n-1}) \circ \delta_n^P \\
&\quad = \Sigma_{n-2} \circ \delta_{n-1}^P \circ \delta_n^P \\
&\quad = 0
\end{aligned}$$

So $\operatorname{im}(\psi_n - \varphi_n - \Sigma_{n-1} \circ \delta_n^P) \subseteq \ker \delta_n^Q = \operatorname{im} \delta_{n-1}^Q$. So again projectivity of P_{n+1} yields a map $\Sigma_n : P_n \to Q_{n+1}$ such that

$$\delta_{n+1} \circ \Sigma_n = \psi_n - \varphi_n - \Sigma_{n-1} \circ \delta_n^P$$

□

2.4 Definition. Let (P_*, α), be a projective resolution of M and (Q_*, β) be a projective resolution of N. Then we say P_* and Q_* are *resolutions over* $f \in \operatorname{Hom}_R(M, N)$ if they correspond to f under the isomorphisms the theorem above establishes.

2.5 Corollary. Let $(P_*, \alpha), (Q_*, \beta)$ be two projective resolutions of M. Then there exists a chain homotopy equivalence $\varphi : P_* \to Q_*$ such that

$$\begin{CD}
H_0(P_*) @>{\alpha}>> M \\
@V{H_0(\varphi)}VV @VV{\mathrm{id}}V \\
H_0(Q_*) @>{\beta}>> M
\end{CD}$$

commutes. Moreover φ is unique up to chain homotopy.

Proof. Especially $\mathrm{id}_M \in \operatorname{Hom}_R(M, M)$ and thus by the theorem above there are two chain maps $\varphi : P_* \to Q_*$ and $\psi : Q_* \to P_*$ over id_M. Consider

$$\begin{array}{ccccc}
P_* & \xrightarrow{\alpha} & M & \xleftarrow[\alpha]{} & P_* \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}} & & \Big| \\
Q_* & \xrightarrow{\beta} & M & & \Big|{\scriptstyle \mathrm{id}} \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
P_* & \xrightarrow{\alpha} & M & \xleftarrow[\alpha]{} & P_*
\end{array}$$

The identity id_P is a chain map $P_* \to P_*$ over id_M but $\psi \circ \varphi$ is also a chain map $P_* \to P_*$ over id_M. By the fundamental lemma $\psi \circ \varphi \simeq \mathrm{id}_P$. Analogously $\varphi \circ \psi \simeq \mathrm{id}_Q$. □

2.3 Ext

2.3.1 Properties of the Hom Functor

For the entire section let R be a Ring.

2.6 Definition (Hom Functor)**.** Let $A, B \in \mathrm{Mod}_R$. Then

$$\mathrm{Hom}_R(A, B) := \{\lambda : A \to B | \lambda \text{ is an R-Module homomorphism}\}$$

For any homomorphisms $f : A \to A'$, $g : B \to B'$ we define:

$$\mathrm{Hom}_R(f, g) : \mathrm{Hom}_R(A', B) \to \mathrm{Hom}_R(A, B')$$

by $\alpha \mapsto g \circ \alpha \circ f$. So

$$A \xrightarrow{f} A' \xrightarrow{\alpha} B \xrightarrow{g} B'$$

2.7 Lemma (Elemental Properties of Hom)**.**

(i) $\mathrm{Hom}_R(\_, \_)$ is a functor in two variables which is contravariant in the first variable and covariant in the second.

(ii) There exists a bijection $\mathrm{Hom}_R(\oplus_{i\in I} A_i, B) \cong \prod_{i\in I} \mathrm{Hom}_R(A_i, B)$

(iii) There exists a bijection $\mathrm{Hom}_R(A, \prod_{i\in I} B_i) \cong \prod_{i\in I} \mathrm{Hom}_R(A, B_i)$

Proof. (i) clear

(ii) Let $(\oplus_{i\in I} A_i, (\iota_{i\in I})$ be the direct sum of the A_i. Define $\varphi : \mathrm{Hom}_R(\oplus_{i\in I} A_i, B) \to \prod_{i\in I} \mathrm{Hom}_R(A_i, B)$ by sending $f \mapsto f_i$ where $f_i := f \circ \iota_i$. Conversely define $\psi : \prod_{i\in I} \mathrm{Hom}_R(A_i, B) \to \mathrm{Hom}_R(\oplus_{i\in I} A_i, B)$ by the following: Given a system of maps $(f_i)_{i\in I}$ the universal property of the direct sum

$$\begin{array}{ccc} A_i & \overset{\iota_i}{\hookrightarrow} & \oplus_{i\in I} A_i \\ {\scriptstyle f_i}\downarrow & \swarrow {\scriptstyle \exists! f} & \\ B & & \end{array}$$

yields precisely one $f =: \psi((f_i)_{i\in I}) : \oplus_{i\in I} A_i \to B$ such that for all $i \in I$ we have $f \circ \iota_i = f_i$. These mappings are obviously inverse to each other.

(iii) Let $(\prod_{i\in i} B_i, (\pi_i)_{i\in I})$ be the product of the B_i. Define $\psi : \mathrm{Hom}_R(A, \prod_{i\in I} B_i) \cong \prod_{i\in I} \mathrm{Hom}_R(A, B_i)$ by sending $f \mapsto f_i$ where $f_i := \pi_i \circ f$. Conversely define $\psi \prod_{i\in I} \mathrm{Hom}_R(A, B_i) \to \mathrm{Hom}_R(A, \prod_{i\in I} B_i)$ by the following: Given a system of maps $(f_i)_{i\in I}$ the universal property of the product

$$\begin{array}{ccc} B_i & \overset{\pi_i}{\twoheadleftarrow} & \prod_{i\in I} B_i \\ {\scriptstyle f_i}\uparrow & \nearrow {\scriptstyle \exists! f} & \\ A & & \end{array}$$

there exists precisely one $f =: \psi((f_i)_{i\in I})$ such that for all $i \in I$ we have $f_i = \pi_i \circ f$. These mappings are obviously inverse to each other.

□

2.8 Theorem (Exactness)**.**

(i) $\mathrm{Hom}_R(\_, B)$ is left exact, i.e. for any e.S.

$$A \xrightarrow{\alpha} A' \xrightarrow{\beta} A'' \longrightarrow 0$$

the sequence

$$0 \longrightarrow \mathrm{Hom}_R(A'', B) \xrightarrow{\beta^*} \mathrm{Hom}_R(A', B) \xrightarrow{\alpha^*} \mathrm{Hom}_R(A, B)$$

is also exact. Here $\beta^* := \mathrm{Hom}_R(\beta, \mathrm{id}_B)$, $\alpha^* := \mathrm{Hom}_R(\alpha, \mathrm{id}_B)$.

(ii) If the s.e.S.

$$0 \longrightarrow A \xrightarrow{\alpha} A' \xrightarrow{\beta} A'' \longrightarrow 0$$

splits, then

$$0 \longrightarrow \mathrm{Hom}_R(A'', B) \xrightarrow{\beta^*} \mathrm{Hom}_R(A', B) \xrightarrow{\alpha^*} \mathrm{Hom}_R(A, B) \longrightarrow 0$$

is also a s.e.S.

(iii) $\mathrm{Hom}_R(A, \_)$ is left exact, i.e. for any e.S.

$$0 \longrightarrow B \xrightarrow{\alpha} B' \xrightarrow{\beta} B''$$

the sequence

$$0 \longrightarrow \mathrm{Hom}_R(A, B) \xrightarrow{\alpha_*} \mathrm{Hom}_R(A, B') \xrightarrow{\beta_*} \mathrm{Hom}_R(A, B'')$$

is also exact. Here $\alpha_* := \mathrm{Hom}_R(\alpha, \mathrm{id}_A)$, $\beta_* := \mathrm{Hom}_R(\beta, \mathrm{id}_A)$.

(iv) If the s.e.S.

$$0 \longrightarrow B \xrightarrow{\alpha} B' \xrightarrow{\beta} B''$$

splits, then

$$0 \longrightarrow \mathrm{Hom}_R(A, B) \xrightarrow{\alpha_*} \mathrm{Hom}_R(A, B') \xrightarrow{\beta_*} \mathrm{Hom}_R(A, B'') \longrightarrow 0$$

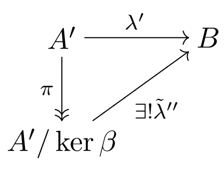
is also exact.

Proof.

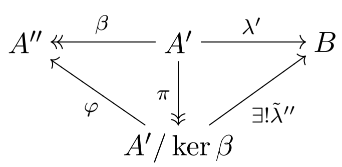
(i) β^* is injective: Let $\lambda'' \in \mathrm{Hom}_R(A'', B)$ such that

$$0 = \beta^*(\lambda'') = \lambda'' \circ \beta$$

Let $a'' \in A''$. By surjectivity of β there exists a $a' \in A'$ such that $a'' = \beta(a')$. So $\lambda''(a'') = \lambda''(\beta(a')) = (\lambda'' \circ \beta)(a') = 0$. And thus $\lambda'' = 0$.
$\mathrm{im}\,\beta^* \subseteq \ker \alpha^*$: By hypothesis $\beta \circ \alpha = 0$. So by functoriality $\beta^* \circ \alpha^* = 0$ as well and thus $\mathrm{im}\,\beta^* \subseteq \ker \alpha^*$.
$\ker \alpha^* \subseteq \mathrm{im}\,\beta^*$: Let $\lambda' \in \ker \alpha^*$, i.e. $\lambda' \circ \alpha = 0$. This means that $\lambda'|_{\mathrm{im}\,\alpha} = \lambda'|_{\ker \beta} = 0$. By the universal property of the quotient

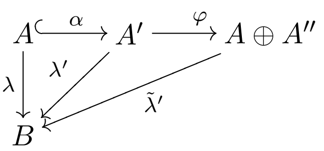


there exists precisely one $\tilde{\lambda}'' : A'/\ker \beta \to B$ such that $\tilde{\lambda}'' \circ \pi = \lambda'$. By exactness of (**??**) there exists an isomorphism $\varphi : A'/\ker \beta \to A''$. So defining $\lambda'' : A'' \to B$ by $\lambda'' := \tilde{\lambda}'' \circ \varphi^{-1}$ we obtain alltogether by the universal properties of both quotiens



that $\beta^*(\lambda'') = \lambda'' \circ \beta = \tilde{\lambda}'' \circ \varphi^{-1} \circ \beta = \tilde{\lambda}'' \circ \pi = \lambda'$.

(ii) One possible way to describe the splitting of a s.e.S. is to say that there exists an isomorphism $\varphi : A' \to A \oplus A''$. It remains to show, that $\alpha^* : \mathrm{Hom}_R(A', B) \to \mathrm{Hom}_R(A, B)$ is surjective. So let $\lambda \in \mathrm{Hom}_R(A, B)$ be arbitrary. Then we can construct a map



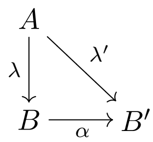
$\tilde{\lambda}' : A \oplus A'' \to B$ by setting $\tilde{\lambda}' := \lambda \oplus 0$ and then defining $\lambda' : A' \to B$ by $\lambda' := \tilde{\lambda}' \circ \varphi$. Alltogether this yields:

$$\alpha^*(\lambda') = \lambda' \circ \alpha = \tilde{\lambda}' \circ \varphi \circ \alpha = \lambda \circ \varphi \circ \alpha \oplus 0 = \lambda$$

(iii) α_* is injective: Let $\lambda \in \mathrm{Hom}_R(A, B)$ such that $0 = \alpha_*(\lambda) = \alpha \circ \lambda$. This implies

$$\forall a \in A : \lambda(a) \in \ker \alpha = \{0\}$$

since α is injective by hypothesis. So $\lambda = 0$.
$\mathrm{im}\, \alpha_* \subseteq \ker \beta_*$: Since $\beta \circ \alpha = 0$ by covariant functoriality $\beta_* \circ \alpha_* = 0$ as well and thus $\mathrm{im}\, \alpha_* \subseteq \ker \beta_*$.
$\ker \beta_* \subseteq \mathrm{im}\, \alpha_*$: Let $\lambda' \in \mathrm{Hom}_R(A, B')$ such that $0 = \beta_*(\lambda') = \beta \circ \lambda'$. This implies that $\mathrm{im}\, \lambda' \subseteq \ker \beta = \mathrm{im}\, \alpha$. So we can construct a map



$\lambda : A \to B$ by the following: Let $a \in A$, then $\lambda'(a) \in \mathrm{im}\, \lambda' \subseteq \mathrm{im}\, \alpha$ and so there exists $b \in B$ such that $\alpha(b) = \lambda'(a)$. Define $\lambda(a) := b$. We have to show, that λ is well defined. So let $b, b' \in B$ such that $\alpha(b) = \alpha(b') = \lambda'(a)$. This implies

$$0 = \alpha(b) - \alpha(b') = \alpha(b - b') \Rightarrow b - b' \in \ker \alpha = \{0\}$$

Thus $b = b'$ and $\lambda' = \alpha \circ \lambda = \alpha_*(\lambda)$.

(iv) It remains only to show, that β_* is surjective. Let the splitting be realised by a map $s : B'' \to B'$ such that $\beta \circ s = \mathrm{id}_{B''}$. If $\lambda'' \in \mathrm{Hom}_R(A, B'')$ define $\lambda' : A \to B'$ by $\lambda' := s \circ \lambda''$. Then by construction

$$\beta_*(\lambda') = \beta \circ \lambda' = \beta \circ s \circ \lambda'' = \mathrm{id}_{B''} \circ \lambda'' = \lambda''$$

□

2.9 Corollary. Let $A \subset X$ be a subspace then

$$0 \longrightarrow S_*(A) \longrightarrow S_*(X) \longrightarrow S_*(X, A) \longrightarrow 0$$

is a s.e.S. which splits. So

$$0 \longrightarrow S^*(X, A) \longrightarrow S^*(X) \longrightarrow S^*(A) \longrightarrow 0$$

is also a s.e.S.

2.3.2 Ext

2.10 Definition. Let A, B be R-modules. Let (P_*, ϵ) be a projective resolution of A. Then $\mathrm{Hom}_R(P^*, B)$ is a cochain complex. Define

$$\mathrm{Ext}^n_R(A, B) := H^n(\mathrm{Hom}_R(P_*, B)))$$

2.11 Theorem (Properties of Ext)**.** The Ext functor satisties the following

(i) $\mathrm{Ext}^0_R(A, B) \cong \mathrm{Hom}_R(A, B)$

(ii) The definition of Ext does not depend on the choice of the projective resolution (P_*)

(iii) If R is a PID, then $\mathrm{Ext}^n_R(A, B) = 0$, if $n \geq 2$.

(iv) So for abelian groups A, B we can define $\mathrm{Ext}(A, B) := \mathrm{Ext}^1_{\mathbb{Z}}(A, B)$.

(v) Ext^n is a functor in two variables, contravariant in the first entry and covariant in the second. So for any $f : A \to A'$, $g : B \to B'$ we have

$$\mathrm{Ext}^n_R(f, g) : \mathrm{Ext}^n_R(A', B) \to \mathrm{Ext}^n_R(A, B')$$

(vi) $\mathrm{Ext}(A, \prod_{i \in I} B_i) = \prod \mathrm{Ext}(A, B_i)$

(vii) $\mathrm{Ext}(\bigoplus_{i \in I} A_i, B) = \prod_{i \in I}(\mathrm{Ext}(A_i, B))$

Proof. (i) Let (P_*, ϵ) be a projective resolution of A. By definition the sequence

$$P_1 \xrightarrow{p} P_0 \xrightarrow{\epsilon} A \longrightarrow 0$$

is exact. Since $\mathrm{Hom}(\_, B)$ is left extact, it follows, that

$$0 \longrightarrow \mathrm{Hom}_R(A, B) \xrightarrow{\epsilon^*} \mathrm{Hom}_R(P_0, B) \xrightarrow{p^*} \mathrm{Hom}_R(P_1, B)$$

is also exact. So in particular ϵ^* is injective and thus

$$\ker p^* = \mathrm{im}\, \epsilon^* \cong \mathrm{Hom}_R(A, B)$$

By taking a look at the complex

$$0 \longrightarrow \mathrm{Hom}_R(P_0, B) \xrightarrow{p^*} \mathrm{Hom}_R(P_1, B) \longrightarrow \cdots$$

we see, that

$$H^0(\mathrm{Hom}_R(P_*, B)) = \ker p^* / \mathrm{im}\, 0 = \ker p^* \cong \mathrm{Hom}_R(A, B)$$

□

3 Epic Theorems of Homological Algebra

3.1 Universal Coefficient Theorems

3.1 Theorem (Universal Coefficient Theorem for Homology)**.** Let be (P_*, d) be a projective chain complex of R-modules, such that for every $n \in \mathbb{N}$ dP_n is projective as well. Then for every R-module M and every $n \in N$

$$0 \longrightarrow H_n(P_*) \otimes_R M \xrightarrow{\alpha_n} H_n(P_* \otimes_R M) \xrightarrow{\beta_n} \mathrm{Tor}^R_1(H_{n-1}(P_*), M) \longrightarrow 0$$

is a short exact sequence which is non canonically split. The maps α_*, β_* may be explicitely computed by [3]

$$\begin{aligned}
&\forall z_n \in Z_n(P_*) : \forall m \in M : \alpha_n([z_n] \otimes m) = [z_n \otimes m] \\
&\forall z_n \otimes m \in Z_n(P_* \otimes M) : \forall m \in M : \beta([z_n \otimes m]) = dz_n \otimes m
\end{aligned}$$

Proof. For every $n \in \mathbb{N}$ denote $Z_n := Z_n(P_*)$, $B_n := B_n(P_*)$, $B'_n := B_{n-1}$,$H_n := H_n(P_*)$ and let $\iota_n : Z_n \to P_n$ and $i_n : B_n \to Z_n$ be the canonical inclusions. We always write $\otimes := \otimes_R$ and $\mathrm{id} := \mathrm{id}_M$
Step 1 (Obtain a seS): The following sequence

$$0 \longrightarrow Z_* \xrightarrow{\iota} P_* \xrightarrow{d} B'_* \longrightarrow 0$$

is a seS of chain complexes, where all the induced differentials in Z_* and B'_* are zero. Since dP_n is projective

$$\begin{array}{ccc} & & dP_n \\ & \overset{s}{\swarrow} & \downarrow \mathrm{id} \\ P_n & \underset{d}{\twoheadrightarrow} & dP_n \end{array}$$

there exists a splitting $s : dP_n \to P_n$, $d \circ s = \mathrm{id}$. Since $\_ \otimes M$ is additive

$$0 \longrightarrow Z_* \otimes M \xrightarrow{\iota \otimes \mathrm{id}} P_* \otimes M \xrightarrow{d \otimes \mathrm{id}} B'_* \otimes M \longrightarrow 0$$

is a seS as well. Thus we obtain a leS in Homology

$$\cdots \xrightarrow{\delta_{n+1}} H_n(Z_* \otimes M) \xrightarrow{[\iota_n \otimes \mathrm{id}]} H_n(P_* \otimes M) \xrightarrow{[d_n \otimes \mathrm{id}]} H_n(B'_* \otimes M) \xrightarrow{\delta_n} \cdots$$

This implies that

$$0 \longrightarrow \frac{H_n(Z_* \otimes M)}{\ker[\iota_n \otimes \mathrm{id}]} \xrightarrow{\overline{\iota_n \otimes \mathrm{id}}} H_n(P_* \otimes M) \xrightarrow{[d_n \otimes \mathrm{id}]} \mathrm{im}[d \otimes \mathrm{id}]_n \longrightarrow 0$$

is short exact, where $\overline{\iota_n \otimes \mathrm{id}}$ denotes the map induced by $[\iota_n \otimes \mathrm{id}]$ on the quotient (via universal property). In the following steps we will take a closer look at this sequence and proof that in fact this already is the sequence we are looking for.
Step 2 (Analyze the modules): By construction all the differentials in Z_* are identically zero and so are the differentials in $Z_* \otimes M$ and $\mathrm{id} := \mathrm{id}_M$ Thus

$$H_n(Z_* \otimes M) = Z_* \otimes M.$$

The differentials in B'_* and thus the ones in $B'_* \otimes M$ vanish as well and thus

$$H_n(B'_* \otimes M) = B'_* \otimes M.$$

Furthermore since the original sequence is long exact, we have

$$\mathrm{im}[d_n \otimes \mathrm{id}] = \ker \delta_n \qquad \frac{H_n(Z_* \otimes M)}{\ker[\iota_n \otimes \mathrm{id}]} = \frac{Z_* \otimes M}{\mathrm{im}\, \delta_{n+1}} = \mathrm{coker}\, \delta_{n+1}$$

Step 3 (Analyze the connection morphism): The explicit characterization of the connection homomorphism δ from Corollary 1.3 states, that in this particular case

$$\delta_{n+1} = [\iota_n \otimes \mathrm{id} \circ (d_{n+1} \otimes \mathrm{id}) \circ (d_{n+1} \otimes \mathrm{id})^{-1}] = [i_n \otimes \mathrm{id}]$$

[3]Notice carefully, that β is <u>not</u> the zero map! The decicive point is that there are $z_n \otimes m \in P_n \otimes M$ such that $(d_n \otimes \mathrm{id})(z_n \otimes m) = d_n(z_n) \otimes m = 0$, but $d_n(z_n) \neq 0$.

where as usual "$(d_{n+1}\otimes \mathrm{id})^{-1}$" has to be read as "take an arbitrary preimage" and the brackets indicate "take represented class in homology". Thus together with step 2 we obtain in particular

$$\ker \delta_n = \ker[i_{n-1}\otimes \mathrm{id}] = \ker i_{n-1}\otimes \mathrm{id} \qquad \operatorname{coker} \delta_{n+1} = \operatorname{coker}[i_n \otimes \mathrm{id}] = \operatorname{coker} i_n \otimes \mathrm{id}$$

Step 4 (Calculating $\ker \delta_n$): We claim that

$$0 \longrightarrow dP_n \xrightarrow{i_{n-1}} Z_{n-1} \xrightarrow{\pi_{n-1}} H_{n-1} \longrightarrow 0$$

is an augmented projective resolution of H_{n-1}: The sequence is exact by construction and dP_n is projective by hypothesis. The split seS from step 1 implies that $P_{n-1} \cong Z_{n-1} \oplus dP_n$. Since P_{n-1} is projective by hypothesis, there exists a free module F_{n-1} and a submodule $M_{n-1} \subset F_{n-1}$ such that

$$F_{n-1} \cong P_{n-1} \oplus M_{n-1} \cong Z_{n-1} \oplus dP_n \oplus M_{n-1}$$

thus Z_{n-1} is also a direct summand of a free module, hence projective. Consequently the first Homology Group of the complex

$$(*) \qquad 0 \longrightarrow dP_n \otimes M \xrightarrow{i_{n-1}\otimes \mathrm{id}} Z_{n-1}\otimes M \longrightarrow 0$$

yields to

$$\operatorname{Tor}_1^R(H_{n-1}, M) = H_1(*) = \ker i_{n-1}\otimes \mathrm{id} = \ker \delta_n$$

Step 5 (Calculating $\operatorname{coker} \delta_{n+1}$): Using the same sequence as above one degree higher, we obtain the seS

$$0 \longrightarrow B_n \xrightarrow{i_n} Z_n \xrightarrow{\pi_n} H_n \longrightarrow 0$$

Since $\_ \otimes M$ is right exact

$$B_n \otimes M \xrightarrow{i_n \otimes \mathrm{id}} Z_n \otimes M \xrightarrow{\pi_n \otimes \mathrm{id}} H_n \otimes M \longrightarrow 0$$

is exact as well and thus

$$\operatorname{coker} \delta_{n+1} = \operatorname{coker} i_n \otimes \mathrm{id} = \frac{Z_n \otimes M}{\operatorname{im} i_n \otimes \mathrm{id}} = \frac{Z_n\otimes M}{\ker \pi_n \otimes \mathrm{id}} \xrightarrow[\sim]{\overline{\pi_n \otimes \mathrm{id}}} H_n \otimes M$$

where $\overline{\pi_n \otimes \mathrm{id}}^{-1}$ is the induced isomorphism on the quotient. Thus our desired map is

$$\alpha_n : H_n \otimes M \to H_n(P_* \otimes M), \alpha := \overline{\iota \otimes \mathrm{id}} \circ \overline{\pi_n \otimes \mathrm{id}}^{-1}$$

which maps precisely as claimed.

Step 6: Inserting the results of step 2-5 the seS of step 1 proves the claim.

Construction of the splitting: In step 1 we already constructed a splitting $s_n : dP_n \to P_n$ for

$$0 \longrightarrow Z_n(P_*) \xrightarrow{\iota} P_n \xrightarrow{d} dP_n \longrightarrow 0$$

Functoriality implies

$$d_n \circ s_n = \mathrm{id}_{dP_n} \Rightarrow d_n \otimes \mathrm{id} \circ s_n \otimes \mathrm{id} = \mathrm{id}_{dP_n \otimes M} \Rightarrow [d_n \otimes \mathrm{id}] \circ [s_n \otimes \mathrm{id}] = \mathrm{id}_{H_n(dP_n \otimes M)}$$

□

3.2 Lemma (Integers and Rings)**.**

- For every ring R, there exists a unique ring homomorphism $\varphi_R : \mathbb{Z} \to R$. So $\mathbb{Z}$ is initial object in the category of rings.

- Every ring R is a $\mathbb{Z}$ - Algebra.
- Every R-module is a $\mathbb{Z}$ - module.

Proof. By definition a ring homomorphism $f : R \to L$ between two rings R, L with unit is a map satisfying

$$(1) : \forall a, b \in R : f(a+b) = f(a) + f(b) \qquad (2) : \forall a, b \in R : f(ab) = f(a)f(b) \qquad (3) : f(1_R) = 1_L$$

Condition (1) immediately implies

$$f(0_R) = 0_L \qquad\qquad \forall a \in R : f(-a) = -f(a)$$

So if $\varphi : \mathbb{Z} \to R$ is any ring homomorphism, we have

$$\varphi(0) = 0 \qquad \forall n \geq 1 : \varphi(n) = \varphi\left(\sum_{i=1}^{n} 1_{\mathbb{Z}}\right) \overset{(1)}{=} \sum_{i=1}^{n} \varphi(1_{\mathbb{Z}}) \overset{(3)}{=} \sum_{i=1}^{n} 1_R \qquad \forall n \leq -1 : \varphi(n) = -\varphi(-n)$$

So φ is unique and using these relations to define φ_R we obtain the desired homomorphism.
Every ring R already has an additive and multiplicative structure. The scalar multiplication is defined by $\mathbb{Z} \times R \to R$, $(n, r) \mapsto \varphi_R(n)r$.
Given any R-module M, we obtain the $\mathbb{Z}$-module structure by defining scalar multiplication by $\mathbb{Z} \times M \to M$, $(n, m) \mapsto \varphi_R(n)m$. □

3.3 Corollary (Universal Coefficient Theorem for Singular Homology)**.** Let X be a topological space and R be any commutative ring with unit. Then there is a seS

$$0 \longrightarrow H_q(X) \otimes_{\mathbb{Z}} R \longrightarrow H_q(X, R) \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(H_{q-1}, R) \longrightarrow 0\,,$$

which splits non-canonically.

Proof. By the lemma above, we interpret R and $C_q(X, R)$ as a $\mathbb{Z}$-modules and drop φ_R in notation. We define $\varphi : C_q(X) \times R \to C_q(X, R)$, $(\sum_i n_i \sigma_i, r) \mapsto \sum_i n_i r \sigma_i$. This is bilinear and thus descends to a linear map $\bar{\varphi} : C_q(X) \otimes_{\mathbb{Z}} R \to C_q(X, R)$. We claim that this is an isomorphism.
It is injective since

$$0 = \bar{\varphi}\left(\sum_i n_i \sigma_i \otimes r\right) = \sum_i n_i r \sigma_i$$

implies that for every i $n_i r = 0$ (since the σ_i are linearly independent). This implies

$$\left(\sum_i n_i \sigma_i \otimes r\right) = \left(\sum_i n_i r \sigma_i\right) \otimes 1 = 0$$

To proof surjectivity let $\sum_i r_i \sigma_i \in C_q(X, R)$ be arbitrary. Clearly

$$\bar{\varphi}\left(\sum_i \sigma_i \otimes r_i\right) = \sum_i \varphi(\sigma_i, r_i) = \sum_i r_i \sigma_i$$

This means, we can interpret

$$H_q(X, R) = H_q(C_*(X, R)) = H_q(C_*(X) \otimes_{\mathbb{Z}} R)$$

and thus the statement follows from 3.1 □

3.4 Theorem (Universal Coefficient Theorem for Cohomology)**.** Let R be a PID and C_* be a projective chain complex. Then there exists a split exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_R(H_{n-1}(C_*), M) \xrightarrow{\alpha} H^n(\operatorname{Hom}_R((C_*), M)) \xrightarrow{\beta} \operatorname{Hom}_R(H_n(C_*), M) \longrightarrow 0$$

which is natural in C_* and M.
But be warned, my fellow: Folks in town say, the splitting is not natural.

Proof. Define $C^* := \operatorname{Hom}_R(C_*, M)$ with differentials $\delta^n = \operatorname{Hom}_R(d^n, M)$.
Construction of β: For any $n \in \mathbb{N}$ define $\beta_n : H^n(\operatorname{Hom}_R((C_*), M)) \to \operatorname{Hom}_R(H_n(C_*), M)$ by the following: Remember that

$$H^n(\operatorname{Hom}_R(C_*, M)) = \frac{Z^n(\operatorname{Hom}_R(C_*, M)}{B^n(\operatorname{Hom}_R(C_*, M))} = \frac{\ker \delta^n}{\operatorname{im} \delta^{n-1}}$$

An arbitrary cohomology class $[\lambda] \in H^n(C^*)$ has a representative $\lambda \in Z^n$. For any $c \in C_{n+1}$ we obtain $\lambda(\delta_{n+1}(c)) = \delta^n(\lambda) = 0$. So $\operatorname{im} \delta_{n+1} = B_n \subset \ker \lambda$. By the universal property of the quotient

$$\begin{array}{ccc} Z_n & \xrightarrow{\lambda} & B \\ {\scriptstyle \pi_n}\downarrow & \nearrow_{\bar{\lambda}} & \\ \frac{Z_n}{B_n} & & \end{array}$$

we obtain a well defined $\bar{\lambda} : H_n \to B$. Define $\beta_n([\lambda]) := \bar{\lambda}$. In order to check that this definition does not depend on the choice of representative λ of $[\lambda]$ we have to check, that

$$[\lambda] = 0 \Rightarrow \bar{\lambda} = 0$$

But $[\lambda] = 0$ implies the existence of a $\mu \in \operatorname{Hom}_R(C_{n-1}, M)$ such that $\lambda = \delta^{n-1}(\mu) = \mu \circ \delta_n$. So $\lambda|_{Z_n} = 0$ and thus $\bar{\lambda} = 0$.
Construction of a splitting for β: Given any $\tau \in \operatorname{Hom}_R(H_n, M)$ we have to construct a $[\lambda] \in H^n(\operatorname{Hom}_R(C_*, M))$ such that $\beta([\lambda]) = \tau$. We want to extend τ

$$\begin{array}{ccccc} Z_n & \xrightarrow{\pi_n} & H_n & \xrightarrow{\tau} & M \\ \downarrow & & & \nearrow_{\lambda} & \\ C_n & & & & \end{array}$$

to the entire C_n. In order to archieve this consider the sequence

$$0 \longrightarrow Z_n \xrightarrow{i_n} C_n \xrightarrow{\delta_n} B_{n-1} \longrightarrow 0$$

It is obviously exact and since R is a PID these are all projective modules. So the sequence splits non canonically to $C_n = Z_n \oplus B'_{n-1}$. Define $\lambda := \tau \circ \pi_n \oplus 0$. We have to show that λ represents a cohomology class. But this is clear since

$$\delta^n(\lambda) = \lambda \circ \delta_{n+1} = \tau \circ \pi_n \circ \delta_{n+1} = 0$$

So we have constructed a well defined $\gamma : \operatorname{Hom}_R(H_n(C_*), M) \to H^n(\operatorname{Hom}_R(C_*, M))$. It follows from the definitions, that

$$(\beta \circ \gamma)(\tau) = \beta(\tau \circ \pi \oplus 0) = \beta(\tau \circ \pi) = \tau$$

This proves that γ is a splitting and also that β is surjective.
Construction of α: We need to construct an $\alpha : \mathrm{Ext}^1_R(H_{n-1}(C_*), M) \to H^n(\mathrm{Hom}_R(C_*, M))$. As in the discussion above we obtain a s.e.S. of chain complexes

$$0 \longrightarrow Z_* \xrightarrow{\iota} C_* \xrightarrow{\delta} B_{*-1} \longrightarrow 0$$

which splits since R is a PID. The differentials of Z_* and B_{*-1} are just the zero maps. The splitting implies that

$$0 \longrightarrow \mathrm{Hom}_R(B_{*-1}, M) \xrightarrow{d^*} \mathrm{Hom}_R(C_*, M) \xrightarrow{\iota^*} \mathrm{Hom}_R(Z_*, M) \longrightarrow 0$$

is a s.e.S. as well. The snake lemma for co-chain complexes yields a long exact sequence in cohomology with connecting homomorphisms δ^n :

$$\cdots \longrightarrow \mathrm{Hom}_R(Z_{n-1}, M) \xrightarrow{\delta^{n-1}} \mathrm{Hom}_R(B_{n-1}, M) \xrightarrow{H^n(\delta_*)} H^n(\mathrm{Hom}_R(C_*, M))$$

If you paint the diagram for the snake lemma in this particular situation you see, that $\delta^n = j^n$ where $j^n : \mathrm{Hom}_R(Z_n, M) \to \mathrm{Hom}_R(B_m, M)$ is just the inclusion induced by $j_n : B_n \to Z_n$.
Next we consider

$$0 \longrightarrow B_n \xrightarrow{j_n} Z_n \xrightarrow{\pi_n} H_n \longrightarrow 0$$

and regard this as a projective resolution of H_n. Applying the Hom functor we obtain

$$0 \longrightarrow \mathrm{Hom}_R(Z_n, M) \xrightarrow{j_n^*} \mathrm{Hom}_R(B_n, M) \longrightarrow 0$$

and thus

$$\mathrm{Ext}^1_R(H_n, B) = \frac{\mathrm{Hom}_R(B_n, B)}{\mathrm{im}\, j_n^*}$$

Consider

$$\begin{array}{ccccc} \mathrm{Hom}_R(Z_{n-1}, M) & \xrightarrow{j^{n-1} = \delta^{n-1}} & \mathrm{Hom}_R(B_{n-1}, M) & \xrightarrow{H^n(\delta_*)} & H^n(\mathrm{Hom}_R((C_*), M)) \\ & & \downarrow & \overset{\alpha}{\nearrow} & \\ & & \mathrm{Ext}^1_R(H_{n-1}, M) & & \end{array}$$

The upper row is exact since it is taken from the long exact sequence, so

$$\mathrm{im}\, \delta^{n-1} = \mathrm{im}\, j^{n-1} = \ker H^n(\delta_*)$$

Thus the existence of α follows from the universal property of the quotient.
α is injective by construction since we factored out the kernel.

$\square$

3.2 Künneth Formulae

3.2.1 Bicomplexes

3.5 Definition (Bicomplex)**.** Let R be a ring. A family $\{C_{p,q}\}_{p,q\in\mathbb{Z}}$ of R-Modules together with maps $d^h : C_{p,q} \to C_{p-1,q}$ and $d^v : C_{p,q} \to C_{p,q-1}$

$$\begin{array}{ccccccc} & & \cdots & & \cdots & & \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & C_{p,q} & \xrightarrow{d^h_{p,q}} & C_{p-1,q} & \longrightarrow & \cdots \\ & & {\scriptstyle d^v_{p,q}}\downarrow & & \downarrow{\scriptstyle d^v_{p-1,q}} & & \\ \cdots & \longrightarrow & C_{p,q-1} & \xrightarrow[d^h_{p,q-1}]{} & C_{p-1,q-1} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ & & \cdots & & \cdots & & \end{array}$$

is a *double complex* or *bicomplex*, if

$$d^h \circ d^h = d^v \circ d^v = d^v \circ d^h + d^h \circ d^v = 0$$

A *morphism of bicomplexes* $f_{p,q} : (C_{p,q}, d^{C,v}, d^{C,h}) \to (D_{p,q}, d^{D,v}, d^{D,h})$ is a famliy of morphisms in Mod_R such that

$$d^{D,v} \circ f = f \circ d^{C,v} \qquad\qquad d^{D,h} \circ f = f \circ d^{C,h}$$

This defines the *category of bicomplexes* $\mathrm{BiCh}(\mathrm{Mod}_R)$.

3.6 Definition (Totalization)**.** Let $(C_{p,q}, d^C)$ be a bicomplex. Define

$$\mathrm{Tot}(C)_n := \bigoplus_{p+q=n} C_{p,q} \qquad\qquad d^{\mathrm{Tot}}_n := \bigoplus_{p+q=n} d^{C,h} + \bigoplus_{p+q=n} d^{C,v}$$

Then we call $(\mathrm{Tot}(C), d^{\mathrm{Tot}}$ the *totalized chain complex.*
Given a morphism of bicomplexes $f_{p,q} : (C_{p,q}, d^{C,v}, d^{C,h}) \to (D_{p,q}, d^{D,v}, d^{D,h})$ we define $\mathrm{Tot}(f)_n : \mathrm{Tot}(C)_n \to \mathrm{Tot}(D)_n$ by

$$\mathrm{Tot}(f)_n := \bigoplus_{p+q=n} f_{p,q}$$

3.7 Lemma (Totalization)**.** Totalization is a functor

$$\mathrm{Tot} : \mathrm{BiCh}(\mathrm{Mod}_R) \to \mathrm{Ch}(\mathrm{Mod}_R)$$

3.8 Definition (Tensorization)**.** If (A, d^A), (B, d^B) are chain complexes of R-modules we obtain a canonical *tensorized bi-complex* $(C_{p,q}, d^C)$ by defining

$$C_{p,q} := A_p \otimes B_q \qquad\qquad d^{C,v}_{p,q} := d^A_p \otimes \mathrm{id}_q \qquad\qquad d^{C,h}_{p,q} := (-1)^p \,\mathrm{id}_p \otimes d^B_q$$

3.9 Lemma (Tensorization Functor)**.** Tensorization is a functor

$$\bigotimes : \mathrm{Ch}(\mathrm{Mod}_R) \times \mathrm{Ch}(\mathrm{Mod}_R) \to \mathrm{BiCh}(\mathrm{Mod}_R)$$

3.10 Remark. By forgetting about all the differentials, we can interpret these functors as

$$\mathrm{Tot} : \mathrm{Mod}_R^{\mathbb{Z}\times\mathbb{Z}} \to \mathrm{Mod}_R^{\mathbb{Z}} \qquad\qquad \bigotimes : \mathrm{Mod}_R^{\mathbb{Z}} \times \mathrm{Mod}_R^{\mathbb{Z}} \to \mathrm{Mod}_R^{\mathbb{Z}\times\mathbb{Z}}$$

3.11 Theorem (Künneth Formula)**.** Let (P_*, d^P), (Q_*, d^Q) be two chain complexes of R-modules, such that every P_n and every dP_n is projective. Then there exists a short exact sequence

$$0 \longrightarrow \mathrm{Tot}(H_*(P) \otimes_R H_*(Q))_n \longrightarrow H_n(\mathrm{Tot}(P_* \otimes_R Q_*)) \longrightarrow \mathrm{Tot}(\mathrm{Tor}_1^R(H_*(P), H_*(Q)))_{n-1} \longrightarrow 0$$

If $R = \mathbb{Z}$ and P is a complex of free abelian groups, this sequence is non-canonically split.

Proof. The proof will be long and painful, but analogous to the universal coefficient theorem of homology (Theorem 3.1) you should definitvely read first. Define $Z_n := Z_n(P_*) := \ker d_n^P$, $B_n := B_n(P_*) := d_{n+1}^P$, $B'_n := B_{n-1} = \mathrm{im}\, d_n^P P_n \subset P_{n-1}$. Denote by $\iota_n : Z_n \to P_n$ and $i_n : B_n \to Z_n$ the canonical inclusions. We always write $\otimes := \otimes_R$ and we denote by $d^\otimes$ the differential in $\mathrm{Tot}(P_* \otimes Q_*)$.

Step 1 (Producing a seS): Since B' is projective by hypothesis, we have a split exact sequence of chain complexes

$$0 \longrightarrow Z_* \xrightarrow{\iota} P_* \xrightarrow{d^P} B'_* \longrightarrow 0$$

Since every $\_ \otimes Q_l$ is additive, we obtain in particular split exact sequences

$$0 \longrightarrow Z_k \otimes Q_l \xrightarrow{\iota_k \otimes \mathrm{id}_l} P_k \otimes Q_l \xrightarrow{d_k^P \otimes \mathrm{id}_l} B'_k \otimes Q_l \longrightarrow 0$$

where $\mathrm{id}_l := \mathrm{id}_{Q_l}$ and $k, l \in \mathbb{Z}$ such that $k + l = n$. Thus their direct sums assemble to short exact sequences

$$0 \longrightarrow \bigoplus_{k+l=n} Z_k \otimes Q_l \xrightarrow{I_n} \bigoplus_{k+l=n} P_k \otimes Q_l \xrightarrow{\Delta_n} \bigoplus_{k+l=n} B'_k \otimes Q_l \longrightarrow 0$$

where $I_n := \bigoplus_{k+l=n} \iota_k \otimes \mathrm{id}_l$, $\Delta_n := \bigoplus_{k+l=n} d_k^P \otimes \mathrm{id}_l$ and $n \in \mathbb{Z}$. Thus we obtain one short exact sequence of chain complexes

$$0 \longrightarrow \mathrm{Tot}(Z_* \otimes Q_*) \xrightarrow{I} \mathrm{Tot}(P_* \otimes Q_*) \xrightarrow{\Delta} \mathrm{Tot}(B' \otimes Q_*) \longrightarrow 0$$

This seS induces a leS in their homology (by 1.3)

$$\ldots H_n(\mathrm{Tot}(Z \otimes Q)) \xrightarrow{[I_n]} H_n(\mathrm{Tot}(P \otimes Q)) \xrightarrow{[\Delta_n]} H_n(\mathrm{Tot}(B' \otimes Q)) \xrightarrow{\delta_n} H_{n-1}(\mathrm{Tot}(Z \otimes Q)) \ldots$$

In particular for every $n \in \mathbb{Z}$ the sequence

$$0 \longrightarrow \frac{H_n(\mathrm{Tot}(Z \otimes Q))}{\ker[I_n]} \xrightarrow{\overline{I_n}} H_n(\mathrm{Tot}(P \otimes Q)) \xrightarrow{[\Delta_n]} \mathrm{im}[\Delta_n] \longrightarrow 0$$

where $\overline{I}$ is the map induced by $[I]$ on the quotient, is short exact. By exactness of the long sequence this seS is identical to:

$$0 \longrightarrow \mathrm{coker}\, \delta_{n+1} \xrightarrow{\overline{I_n}} H_n(\mathrm{Tot}(P \otimes Q)) \xrightarrow{[\Delta]} \ker \delta_n \longrightarrow 0$$

Step 2 (Analyze the Modules): Since homology is additive, since the differentials in Z_* and B'_* are identically zero and since Z_* and B'_* are both flat, we obtain from the universal coefficient theorem (c.f. 3.1)

$$H_n(\mathrm{Tot}(Z_* \otimes Q_*)) = H_n\left(\bigoplus_{k+l=*} Z_k \otimes Q_l\right) \cong \bigoplus_{k+l=n} H_l(Z_k \otimes Q_*) \cong \bigoplus_{k+l=n} Z_k \otimes H_l(Q_*)$$

and likewise

$$H_n(\mathrm{Tot}(B'_* \otimes Q_*)) = H_n\left(\bigoplus_{k+l=*} B'_k \otimes Q_l\right) \cong \bigoplus_{k+l=n} H_l(B'_k \otimes Q_*) \cong \bigoplus_{k+l=n} B'_k \otimes H_l(Q_*)$$
$$= \bigoplus_{k+l=n-1} B_k \otimes H_l(Q_*)$$

Step 3 (Analyze the Connection Morphism): By the explicit formula (c.f. 1.3) for the connection homomorphism δ and Step 2, we obtain for any $p_k \in P_k$, $b'_k := d^P_k p_k \in B'_k$, $q_l \in Z_l(Q_*)$

$$\begin{aligned}\delta_n(b'_k \otimes [q_l]) &= [(I_n^{-1} \circ d^{\otimes} \circ \Delta_n^{-1})(d^P_k p_k \otimes [q_l])] \\ &= [(I_n^{-1}(d^{\otimes}(p_k \otimes q_l)] \\ &= [(I_n^{-1}(d^P_k p_k \otimes q_l + (-1)^k (p_k \otimes d^Q_l q_l))] \\ &= J_n(b'_k \otimes [q_l])\end{aligned}$$

where $J_n := \bigoplus_{k+l=n-1} i_k \otimes [\mathrm{id}_l]$.
Step 4 (Calculate $\ker \delta_n$): For every k the sequence

$$0 \longrightarrow dP_{k+1} \xrightarrow{i_k} Z_k(P) \xrightarrow{\pi_k} H_k(P_*) \longrightarrow 0$$

is exact and thus a projective [4] resolution of $H_k(P_*)$. Consequently the torsion of $\_ \otimes H_l(Q_*)$ may be calculated by taking the first homology of the complex

$$0 \longrightarrow dP_{k+1} \otimes H_l(Q_*) \xrightarrow{i_k \otimes [\mathrm{id}_l]} Z_k(P_*) \otimes H_l(Q_*) \longrightarrow 0$$

Thus

$$\mathrm{Tor}_1^R(H_k(P), H_l(Q_*)) = \ker i_k \otimes [\mathrm{id}_l]$$

which implies

$$\bigoplus_{k+l=n-1} \mathrm{Tor}_1^R(H_k(P), H_l(Q_*)) = \bigoplus_{k+l=n-1} \ker i_k \otimes [\mathrm{id}_l] = \ker J_n = \ker \delta_n$$

Step 5 (Calculate $\mathrm{coker}\, \delta_n$): Consider again the sequence from step 4

$$0 \longrightarrow dP_{k+1} \xrightarrow{i_k} Z_k(P) \xrightarrow{\pi_k} H_k(P) \longrightarrow 0$$

Since $\_ \otimes H_l(Q)$ is right exact

$$dP_{k+1} \otimes H_l(Q) \xrightarrow{i_k \otimes [\mathrm{id}_l]} Z_k(P) \otimes H_l(Q) \xrightarrow{\pi_k \otimes [\mathrm{id}_l]} H_k(P) \otimes H_l(Q) \longrightarrow 0$$

is exact as well. Consequently

$$\mathrm{coker}\, i_k \otimes [\mathrm{id}_l] = \frac{Z_k(P) \otimes H_l(Q)}{\mathrm{im}\, i_k \otimes [\mathrm{id}_l]} = \frac{Z_k(P) \otimes H_l(Q)}{\ker \pi_k \otimes [\mathrm{id}_l]} \cong H_k(P) \otimes H_l(Q)$$

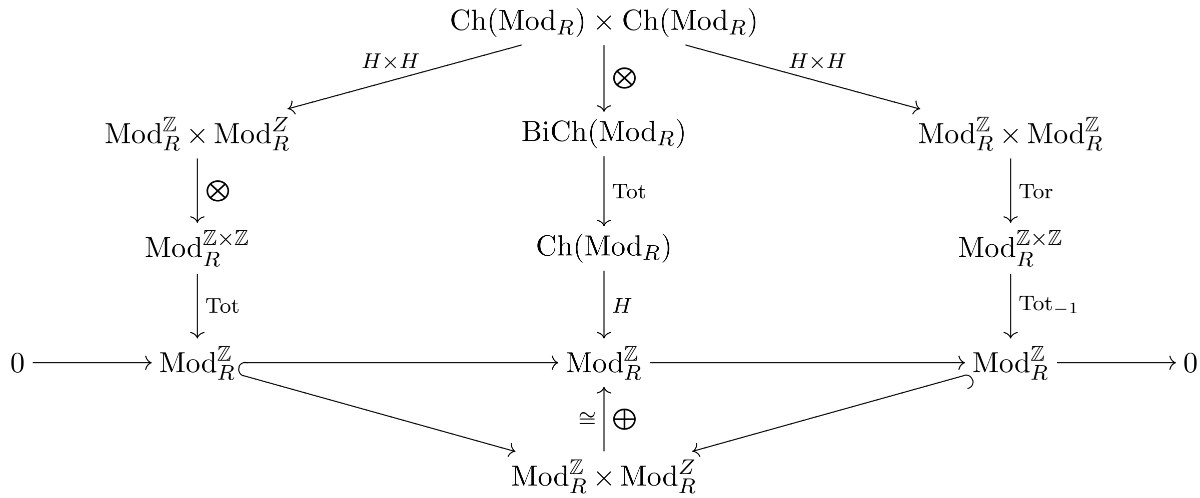
Thus

$$\mathrm{coker}\, \delta_{n+1} = \mathrm{coker}\, J_{n+1} \cong \bigoplus_{k+l=n} \mathrm{coker}\, i_k \otimes [\mathrm{id}_l] \cong \bigoplus_{k+l=n} H_k(P) \otimes H_l(Q)$$

□

[4] The fact that $Z_k(P)$ is projective follows as in the proof of the Universal Coefficient Theorem from the fact that the sequence in step 1 splits and thus $P_k = Z_k \oplus dP_{k+1}$. Since P_k is projective is it a direct summand of a free module and thus Z_k is as well.

3.12 Remark. To appreciate the statement of the Künneth formula better, it may be helpful to view the Künneth Theorem from a more categorical or algorithmical point of view: Its input are two chain complexes, i.e. an object in $\mathrm{Ch}(\mathrm{Mod}_R) \times \mathrm{Ch}(\mathrm{Mod}_R)$ and its output are three different objects in $\mathrm{Mod}_R^{\mathbb{Z}}$ related to each other by an exact sequence. These three objects are obtained by totally different compositions of functors:



The bottom is supposed to visualize the splitting.

3.3 Method of Acyclic Models

An important topological application of the Künneth Formula is the Product Theorem for Singular Homology, which relates $H_*(X \times Y)$ to $H_*(X) \otimes H_*(Y)$. In order to prove this, we need to establish the theorem of Eilenberg-Zilber, which relates $H_*(X \times Y)$ to $H_*(C_*(X) \otimes C_*(Y)$ and in order to prove this, we need the following rather abstract concept of acyclic models.

3.13 Definition (Natural Equivalence Classes)**.** Let $\mathcal{C}$ be an arbitrary category. Consider two arbitrary but fixed functors $F_*, G_* : \mathcal{C} \to \mathrm{Ch}_{\geq 0}(\mathrm{Mod}_R)$. Two natural transformations $\varphi_*, \psi_* : F_* \rightrightarrows G_*$ are *naturally chain homotopic* if for every object $X \in \mathcal{C}$ there exists a family of natural transformations $D_n^X : F_n(X) \rightrightarrows G_{n+1}(X)$, $n \in \mathbb{N}$, such that D_*^X is a chain homotopy between $\varphi_*^X, \psi_*^X : F_*(X) \to G_*(X)$. This defines an equivalence relation on the set of natural transformations $F_* \rightrightarrows G_*$ and the set of equivalence classes is denoted by

$$\pi(F_*, G_*).$$

3.14 Remark. It may be useful to write down the relations implied above somwhat more explicity. So let's take two functors $F_*, G_* : \mathcal{C} \to \mathrm{Ch}_{\geq 0}(\mathrm{Mod}_R)$ and consider two objects $X, Y \in \mathcal{C}$ as well as a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

(i) For φ_* to be a natural transformation $F \rightrightarrows G$, we require in particular, that is a morphism of chain complexes, i.e.

$$\varphi_*^X \in \mathrm{Hom}_{\mathrm{Ch}_{\geq 0}(\mathrm{Mod}_R)}(F(X), G(X))$$

which means by definition, that

$$\forall n \in \mathbb{N} : d_n^{G(X)} \circ \varphi_n^X = \varphi_{n-1}^X \circ d_n^{F(X)}$$

where $d_*^{F(X)}$ denotes the differential in the chain complex $F_*(X)$ and $d_*^{G(X)}$ denotes the differential in $G_*(X)$.

(ii) By definition of a natural transformation, this diagram commutes:

$$\begin{array}{ccc} F_*(X) & \xrightarrow{F_*(f)} & F_*(Y) \\ \downarrow{\scriptstyle \varphi_*^X} & & \downarrow{\scriptstyle \varphi_*^Y} \\ G_*(X) & \xrightarrow{G_*(f)} & G_*(Y) \end{array}$$

(iii) Contrary to (i), the D_n is only required to be a natural transformation $F_n \rightrightarrows G_n$, i.e.

$$D_n^X \in \mathrm{Hom}_{\mathrm{Mod}_R}(F_n(X), G_n(X))$$

and the diagram

$$\begin{array}{ccc} F_n(X) & \xrightarrow{F_n(f)} & F_n(Y) \\ \downarrow{\scriptstyle D_n^X} & & \downarrow{\scriptstyle D_n^Y} \\ G_{n+1}(X) & \xrightarrow{G_{n+1}(f)} & G_{n+1}(Y) \end{array}$$

commutes for every $n \in \mathbb{N}$. But the map D_* is <u>not</u> a chain map $F_*(X) \to G_{*+1}(X)$, since this would force the relation $D_{n-1}^X \circ d_n^F = d_n^G \circ D_n^X$ which in gereral conflicts with (iv).

(iv) For D_* to be a homotopy between φ_* and ψ_* we require

$$d_{n+1}^{G(X)} \circ D_n^X + D_{n-1}^X \circ d_n^{F(X)} = \varphi_n^X - \psi_n^X$$

For simplicity one can drop the superscript X or the subscript n if it is clear. But remember that φ and D of course depend on both. If you are confused over all these depencencies, first think of two chain complexes F and G and two maps φ, ψ between them and a homotopy D between φ and ψ. If you 'parametrize" this situation by objects in $\mathcal{C}$, you get precisely what we have just defined. The diagram looks like

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & F_{n+1}(X) & \xrightarrow{d_{n+1}^{F(X)}} & F_n(X) & \xrightarrow{d_n^{F(X)}} & F_{n-1}(X) & \longrightarrow & \cdots \\ & & {\scriptstyle \psi_{n+1}^X} \downarrow \downarrow {\scriptstyle \varphi_{n+1}^X} & \swarrow {\scriptstyle D_n^X} & {\scriptstyle \psi_n^X} \downarrow \downarrow {\scriptstyle \varphi_n^X} & \swarrow {\scriptstyle D_{n-1}^X} & {\scriptstyle \psi_{n-1}^X} \downarrow \downarrow {\scriptstyle \varphi_{n-1}^X} & & \\ \cdots & \longrightarrow & G_{n+1}(X) & \xrightarrow{d_{n+1}^{G(X)}} & G_n(X) & \longrightarrow & G_{n-1}(X) & \xrightarrow{d_n^{G(X)}} & \cdots \end{array}$$

3.15 Definition (Model Category)**.** A functor $F_n : \mathcal{C} \to \mathrm{Mod}_R$ is *free*, if there exists an index set J and a set of objects $\{M_{n,j} \in \mathcal{C}\}_{j \in J}$, called *models*, and a set of elements $\{u_{n,j} \in F_n(M_{n,j})\}_{j \in J}$, called *base generators*, such that for every $X \in \mathcal{C}$ the set

$$\{F_n(f)(u_{n,j}) \in F_n(X) | j \in J, f \in \mathrm{Hom}_{\mathcal{C}}(M_{n,j}, X)\}$$

is a basis for $F_n(X)$.
A functor $F_* : \mathcal{C} \to \mathrm{Ch}_{\geq 0}(\mathrm{Mod}_R)$ is *free* if for every $n \in \mathbb{N}$ the functor F_n is free.
We define $\mathcal{M}$ to be the subcategory of $\mathcal{C}$ whose objects are given by all the $M_{n,j}$ and whose morphisms are the same as in $\mathcal{C}$. This category is called the *model category of* F and its objects are the *models for* F. Of course there is a canonical inclusion functor $i : \mathcal{M} \to \mathcal{C}$.

3.16 Example. For every $n \in \mathbb{N}$, the singular chain complex functor $C_n : Top \to \mathrm{Mod}_R$ is free. By definition $C_n(X)$ the R-module freely generated by all continuous maps $f : \Delta^n \to X$, where Δ^n, the standard n-simplex. So take $M_{n,0} := \Delta^n$ as the only model. The only base generator is $u_{n,0} := \mathrm{id}^n : \Delta^n \to \Delta^n$. Then for any $f : \Delta^n \to X$, we have by definition $C_n(f)(u_{n,0}) = f \circ \mathrm{id} = f$.

3.17 Lemma (Uniqueness of natural transformations)**.**

(i) Let $F_n, G_n : \mathcal{C} \to \mathrm{Mod}_R$ be functors and let F be free. For any prescribed set of objects $\{v_{n,j} \in G_n(M_{n,j})\}_{j\in J}$, there exists a unique natural transformation $\varphi_n : F_n \rightrightarrows G_n$ such that

$$\forall j \in J : \varphi_n(u_{n,j}) = v_{n,j}$$

(ii) Let $F_*, G_* : \mathcal{C} \to \mathrm{Ch}_{\geq 0}(\mathrm{Mod}_R)$, let F_* be free and $\{v_{n,j} \in G_n(M_{n,j})\}_{n\in\mathbb{N}, j\in J_n}$ be any prescribed set of objects. Then any natural transformation $\varphi_* : F_* \rightrightarrows G_*$ satisfying

$$\forall n \in \mathbb{N} : d_n^G \circ \varphi_n = \varphi_{n-1} \circ d_n^F$$

and

$$\forall n \in \mathbb{N} : \forall j \in J_n : \varphi(u_{n,j}) = v_{n,j}$$

is unique.

Proof.

(i) We show uniqueness first: Since F_n is free, any $x \in F_n(X)$ has a unique representation

$$x = \sum_{j,f} \lambda_{j,f} F_n(f)(u_{n,j})$$

Since φ_n is natural

$$\varphi_n(x) = \sum_{j,f} \lambda_{j,f} \left(\varphi_n \circ F_n(f)\right)(u_{n,j}) = \sum_{j,f} \lambda_{j,f} \left(G_n(f) \circ \varphi_n\right)(u_{n,j})$$

So the $\varphi_n(u_{n,j})$ uniquely determine φ_n.
On the other hand we can simply define φ_n by this formula to show existence.

(ii) By (i) all the φ_n are unique and the compatibility condition ensures that φ_* is a chain map.

□

3.18 Definition (acyclic)**.** Let $G_* : \mathcal{C} \to \mathrm{Ch}_{\geq 0}(\mathrm{Mod}_R)$ be a functor and $\mathcal{M} \subset \mathcal{C}$ be any subcategory. Then we call G_* *acyclic* with respect to $\mathcal{M}$, if

$$\forall M \in \mathcal{M} : \forall i > 0 : H_i(G_*(M)) = 0$$

Note carefully that $H_0(G_*(M)) \neq 0$ is allowed.

3.19 Example. The singular chain complex functor is also acylic with respect to the model category $\mathcal{M} := \{\Delta^n | n \in \mathbb{N}\}$.

Notice, that the following theorem is a massive generalization of the Fundamental Theorem of Homological Algebra (2.3).

3.20 Theorem (Method of acyclic Models)**.** Let $F_*, G_* : \mathcal{C} \to \mathrm{Ch}_{\geq 0}(\mathrm{Mod}_R)$ be functors, let F_* be free with model category $\mathcal{M}$ and let G_* be acyclic with respect to $\mathcal{M}$. Denote by $H_0^{\mathcal{M}}(F_*, G_*)$ the set of natural transformations $H_0 \circ F_* \circ i \rightrightarrows H_0 \circ G_* \circ i$. Then there is a set bijection

$$\pi(F_*, G_*) \to H_0^{\mathcal{M}}(F_*, G_*)$$

induced by

$$(\varphi_* : F_* \rightrightarrows G_*) \mapsto (H_0 \circ \varphi_0 \circ i : H_0 \circ F_0 \circ i \rightrightarrows H_0 \circ G_0 \circ i)$$

Proof. As in Definition 3.15 denote by

$$\mathcal{M} := \{M_{n,j} \in \mathcal{C}\}_{n\in\mathbb{N}, j\in J_n}$$

the model category of F_* and by

$$\{u_{n,j} \in F_n(M_{n,j})\}_{n\in\mathbb{N}, j\in J_n}$$

the set of base generators.

Surjectivity: Let

$$\varphi : H_0 \circ F_* \circ i \rightrightarrows H_0 \circ G_* \circ i$$

be any natural transformation. For every $j \in J_0$, there exists $v_{0,j} \in G_0(M_{0,j})$ such that

$$[v_{0,j}] = \varphi([u_{0,j}]) \in H_0(G_*(M_{0,j}))$$

We define

$$\varphi_0(u_{0,j}) := v_{0,j}$$

and obtain a natural transformation $\varphi_0 : F_0 \rightrightarrows G_0$ satisfying

$$H_0 \circ \varphi_0 \circ i = \varphi$$

by Lemma 3.17. We will inductively extend φ_0 to a natural transformation $F_* \rightrightarrows G_*$. We will show by induction, that there are natural transformations

$$\forall 0 \leq i < n : \varphi_i : F_i \rightrightarrows G_i$$

satisfying the relations

$$\forall 0 \leq i < n : d_i^G \circ \varphi_i = \varphi_{i-1} \circ d_i^F$$

By defining $\varphi_{-1} := 0$ the construction of φ_0 above can be interpreted as the induction start $n = 0$. For the induction step $n-1 \to n$ assume the maps φ_i described above are already constructed. We will construct the map φ_n as follows:

$$\begin{array}{ccccc}
F_n(M_{n,j}) & \xrightarrow{d_n^F} & F_{n-1}(M_{n,j}) & \xrightarrow{d_{n-1}^F} & F_{n-2}(M_{n,j}) \\
\downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle \varphi_{n-2}} \\
G_n(M_{n,j}) & \xrightarrow{d_n^G} & G_{n-1}(M_{n,j}) & \xrightarrow{d_{n-1}^G} & G_{n-2}(M)
\end{array}$$

If $n = 1$ and $j \in J_1$

$$[\varphi_{n-1}(d_n^F(u_{n,j}))] = [\varphi_0]([d_1^F(u_{1,j})]) = [\varphi](0) = 0 \Longrightarrow \varphi_{n-1}(d_n^F(u_{n,j})) \in B_{n-1}^G(M_{n,j})$$

If $n > 1$ and $j \in J_n$, the induction hypothesis guarantees

$$d_{n-1}^G(\varphi_{n-1}(d_n^F(u_{n,j}))) = (d_{n-1}^G \circ \varphi_{n-1} \circ d_n^F)(u_{n,j}) = (\varphi_{n-2} \circ d_{n-1}^F \circ d_n^F)(u_{n,j}) = 0$$

and thus $\varphi_{n-1}(d_n^F(u_{n,j})) \in Z_{n-1}^G(M_{n,j})$. Since $n-1 > 0$ and G_* is acyclic by hypothesis, we obtain in any case

$$\varphi_{n-1}(d_n^F(u_{n,j})) \in B_{n-1}^G(M_{n,j})$$

Consequently

$$\exists v_{n,j} \in G_n(M_{n,j}) : d_n^G(v_{n,j}) = \varphi_{n-1}(d_n^F(u_{n,j}))$$

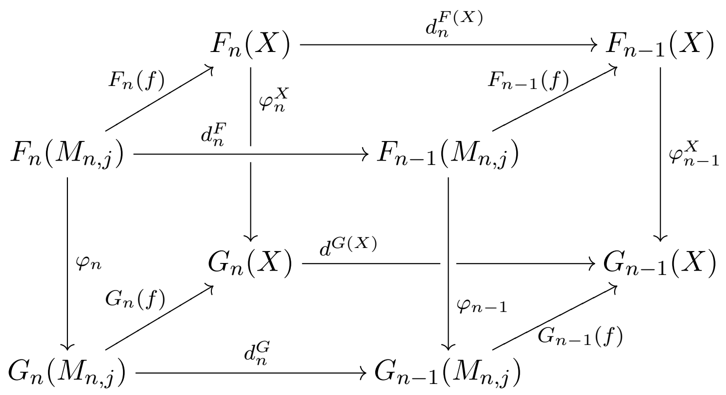
Define

$$\varphi_n(u_{n,j}) := v_{n,j}$$

Again by Lemma 3.17 this defines a unique natural transformation $\varphi_n : F_n \rightrightarrows G_n$ which satisfies

$$d_n^G \circ \varphi_n(u_{n,j}) = \varphi_{n-1} \circ d_n^F(u_{n,j})$$

by construction. We will show that this implies $d_n^{G(X)} \circ \varphi_n^X = \varphi_{n-1}^X \circ d_n^{F(X)}$ for every $X \in \mathcal{C}$ using this commutative cube:



Since F_* is free by hypothesis, we may calculate

$$\begin{aligned}(d_n^{G(X)} \circ \varphi_n^X)(x) &= (d_n^{G(X)} \circ \varphi_n^X)\left(\sum_{j,f} \lambda_{j,f} F_n(f)(u_{n,j})\right) = \sum_{j,f} \lambda_{j,f}(d_n^{G(X)} \circ \varphi_n^X \circ F_n(f))(u_{n,j}) \\ &= \sum_{j,f} \lambda_{j,f}(G_{n-1}(f) \circ d_n^G \circ \varphi_n)(u_{n,j}) = \sum_{j,f} \lambda_{j,f}(\varphi_{n-1}^X \circ F_{n-1}(f) \circ d_n^F)(u_{n,j}) \\ &= \sum_{j,f} \lambda_{j,f}(\varphi_{n-1}^X \circ d_n^{F(X)} \circ F_n(f))(u_{n,j}) = \varphi_{n-1}^X \circ d_n^{F(X)} \left(\sum_{j,f} \lambda_{j,f} F_n(f)(u_{n,j})\right) \\ &= (\varphi_{n-1}^X \circ d_n^{F(X)})(x)\end{aligned}$$

Thus $\varphi_* : F_* \rightrightarrows G_*$ is the desired natural transformation.

Injectivity: Suppose $\varphi_*, \psi_* : F_* \rightrightarrows G_*$ are both natural transformations such that

$$\forall M \in \mathcal{M} : [\varphi_0] = [\psi_0] : H_0(F(M)) \to H_0(G(M))$$

We have to show that φ_* is naturally chain homotopic to ψ_*. The hypothesis implies, that in particular

$$[\varphi_0](u_{0,j}) = [\psi_0](u_{0,j}) \Rightarrow [\varphi_0(u_{0,j}) - \psi_0(u_{0,j})] = 0 \Rightarrow \exists w_{1,j} \in G_1(M_{0,1}) : \varphi_0(u_{0,j}) - \psi_0(u_{0,j}) = d_1^G(w_{1,j})$$

Define

$$D_0(u_{0,j}) := w_{1,j}$$

and again use Lemma 3.15 to obtain a unique natural transformation $D_0 : F_0 \rightrightarrows G_1$ such that

$$d_1^G \circ D_0 = \varphi_0 - \psi_0$$

We will again proceed by induction and show, that there are natural transformations $D_n : F_n \rightrightarrows G_{n+1}$ such that

$$\forall 0 \leq i < n : d_{i+1}^G \circ D_i + D_{i-1} \circ d_i^F = \varphi_i - \psi_i$$

By setting $D_{-1} := 0$ this hast just been accomplished for $n = 1$. For the induction step $n - 1 \to n$ consider

$$\bar{u}_{n,j} := (\varphi_n - \psi_n - D_{n-1} \circ d_n^F)(u_{n,j}) \in G_n(M_{n,j})$$

Since

$$d_n^G \bar{u}_{n,j} = d_n^G \circ (\varphi_n - \psi_n - D_{n-1} \circ d_n^F)(u_{n,j}) = (d_n^G \circ \varphi_n - d_n^G \circ \psi_n - d_n^G \circ D_{n-1} \circ d_n^F)(u_{n,j})$$
$$= (\varphi_{n-1} \circ d_n^F - \psi_{n-1} \circ d_n^F - (\varphi_{n-1} - \psi_{n-1} - D_{n-2} \circ d_{n-1}^F) \circ d_n^F)(u_{n,j}) = 0$$

we have $\bar{u}_{n,j} \in Z_n^G(M_{n,j})$. Since G is acyclic, this implies $\bar{u}_{n,j} \in B_n^G(M_{n,j})$ and thus

$$\exists w_{n,j} \in G_{n+1}(M_{n,j}) : d_{n+1}^G(w_{n,j}) = \bar{u}_{n,j}$$

Setting $D_n(u_{n,j}) := w_{n,j}$, this again defines a unique natural transformation $D_n : F_n \rightrightarrows G_{n+1}$ such that $d_{n+1}^G \circ D_n = \varphi_n - \psi_n - D_{n-1} \circ d_n^F$. □

3.3.1 Applications

The method of acyclic models has some immediate powerful applications.

3.21 Corollary (Eilenberg-Zilber)**.** Let X, Y be two topological spaces. Then there are natural chain homotopy equivalences

$$\alpha : \mathrm{Tot}(C_*(X) \otimes C_*(Y))_* \leftrightarrows C_*(X \times Y) : \beta$$

In particular

$$H_*(\mathrm{Tot}(C_*(X) \otimes C_*(Y))_*) \cong H_*(X \times Y)$$

Proof. Define $\mathcal{C} := \mathbf{Top} \times \mathbf{Top}$ to be the product category between **Top** and **Top**, i.e. the objects are tupels (X, Y) where $X, Y \in \mathbf{Top}$ and morphisms $f = (f_1, f_2) : (X_1, Y_1) \to (X_2, Y_2)$ where $f_1 \in \mathrm{Hom}_{\mathbf{Top}}(X_1, X_2)$ and $f_2 \in \mathrm{Hom}_{\mathbf{Top}}(Y_1, Y_2)$.
Construction of β: Define functors $F_*, G_* : \mathcal{C} \to \mathrm{Ch}(\mathrm{Mod}_R)$ by

$$F_*(X, Y) := C_*(X \times Y) \qquad\qquad G_*(X, Y) := \mathrm{Tot}(C_*(X) \otimes C_*(Y))$$

Choose models $M_n := M_{n,0} := (\Delta^n, \Delta^n)$ and declare the generators $u_n := u_{n,0} : \Delta^n \to \Delta^n \times \Delta^n$ to be the diagonal maps $x \mapsto (x, x)$. Clearly $\mathcal{M} \subset \mathcal{C}$ and $u_n \in F_n(M_n)$. To see that F_* really is free on these models, just notice that by the universal property of the product any pair $(f_1, f_2) \in \mathrm{Hom}_{\mathcal{C}}((\Delta^n, \Delta^n), (X, Y))$ defines a unique map $f : \Delta^n \to X \times Y$ and that $f(x) = (f_1(x), f_2(x)) = (f_1, f_2)(u_n(x))$. Conversely every map $\Delta^n \to X \times Y$ is of that form. Thus by definition

$$F_n(X, Y) = C_n(X \times Y) = \mathbb{Z}[\{f : \Delta^n \to X \times Y | f \in \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, X \times Y)\}]$$
$$= \mathbb{Z}[\{(f_1, f_2) \circ u_n : \Delta^n \to X \times Y | (f_1, f_2) \in \mathrm{Hom}_{\mathcal{C}}((\Delta^n, \Delta^n, (X \times Y))\}]$$

So F is free. Since all the simplices are convex

$$H_q(G_*(\Delta^n, \Delta^n)) = H_q(\mathrm{Tot}(C_*(\Delta^n) \otimes C_*(\Delta^n))_*) \cong \mathrm{Tot}(H_*(\Delta^n) \otimes H_*(\Delta^n))_q) = \delta_{q,0}\mathbb{Z}$$

by the Künneth Formula (3.11) (here $\delta_{0,q}$ is just the Kronecker delta). Thus G_* is acyclic. Consequently the method of acyclic models (3.20) is applicable. We calculate

$$H_q(F_*(\Delta^n, \Delta^n)) = H_q(\Delta^n \times \Delta^n) = \delta_{q,0}\mathbb{Z}$$

Thus

$$H_0^{\mathcal{M}}(F_*, G_*) \cong \{\pm \mathrm{id}_{\mathbb{Z}}\}$$

Chose β_* to be the natural transformation $F_* \rightrightarrows G_*$ corresponding to $\mathrm{id}_{\mathbb{Z}}$.
Construction of α: Redefine functors $F_*, G_* : \mathcal{C} \to \mathrm{Ch}(\mathrm{Mod}_R)$

$$F_*(X, Y) := \mathrm{Tot}(C_*(X) \otimes C_*(Y)) \qquad\qquad G_*(X, Y) := C_*(X \times Y)$$

Choose new models $M_{n,j} := (\Delta^j, \Delta^{n-j})$, $0 \leq j \leq n$ and generators $u_{n,j} := \mathrm{id}_j \otimes \mathrm{id}_{n-j}$, where $\mathrm{id}_j := \mathrm{id}_{\Delta^j} : \Delta^j \to \Delta^j$. By definition

$$C_j(X) = \mathbb{Z}[\{C_*(f)(\mathrm{id}_j) : \Delta^j \to X | f \in \mathrm{Hom}_{\mathbf{Top}}(\Delta^j, X)\}]$$

thus F_* is free. Again G_* is acyclic with $H_0(G_*(\Delta^j \times \Delta^{n-j})) \cong \mathbb{Z} \cong H_0(F_*(\Delta^j \times \Delta^{n-j}))$. Define α_* to be the natural transformation $F_* \rightrightarrows G_*$ corresponding to $\mathrm{id}_{\mathbb{Z}}$.
Then $\beta \circ \alpha$ is a natural transformation $F_* \rightrightarrows F_*$ as well as $\mathrm{id} \in \mathrm{Ch}(\mathrm{Mod}_R)$. Since they both agree in zero homology, they are naturally chain homotopic. The same holds for $\alpha \circ \beta$. □

References

[1] Hüttenhain, Jesko: *Homologietheorie ohne Löcher - Ein topologischer Sommersemestertraum*, http://www.uni-bonn.de/~rattle/works/homologie.ohne.loecher.pdf