Spherical Voronoi Diagrams

A PyData London story unfolded

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1. Planar Voronoi Diagrams
   - Example: Traveling in London
   - Formal description
   - Realization in Python

2. Spherical Voronoi Diagrams
   - Example: Traveling around the world
   - Problem description
   - PyData story
   - Realization in Python
   - Applications

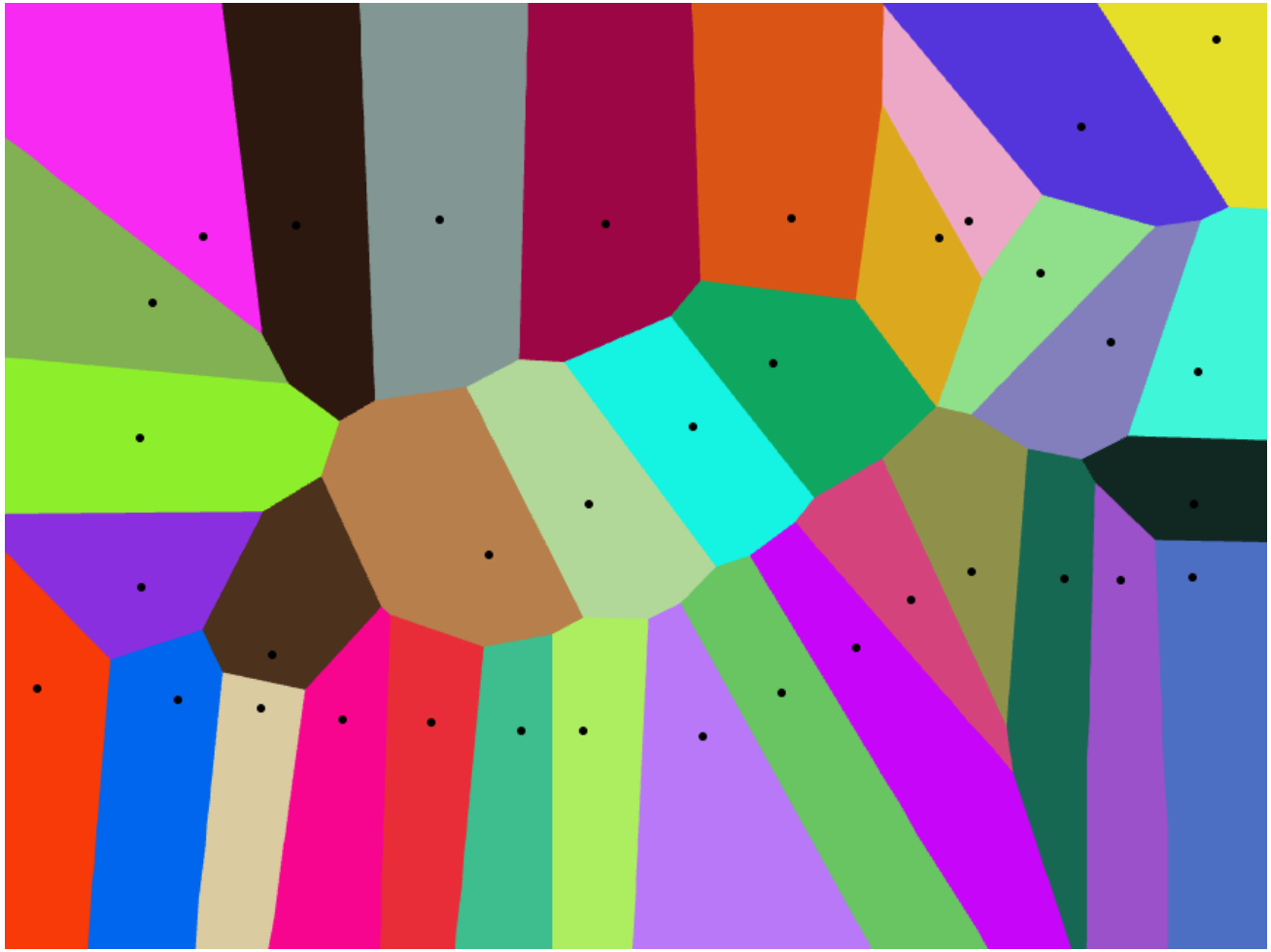
3. Appendix

Outline

Voronoi diagrams in the plane

Input: Generator Points

Voronoi diagrams in the plane



http://alexbeutel.com/webgl/voronoi.html

Output: Voronoi Regions (coloured)

Mathematical definition

Definition (Voronoi diagram)

Let (X, d) be a metric space (for instance $\mathbf{R}^2$) and $p = (p_1, \ldots, p_K)$ be a finite collection of K distinct points called *generators*.

1. For each $1 \leq k \leq K$, the set

$$R_k := \{x \in X \mid \forall j \neq k : d(x, p_k) \leq d(x, p_j)\}$$

   is the *k-th Voronoi region*.

2. The collection $(R_k)_{1 \leq k \leq K}$ is the *Voronoi diagram* defined by p.

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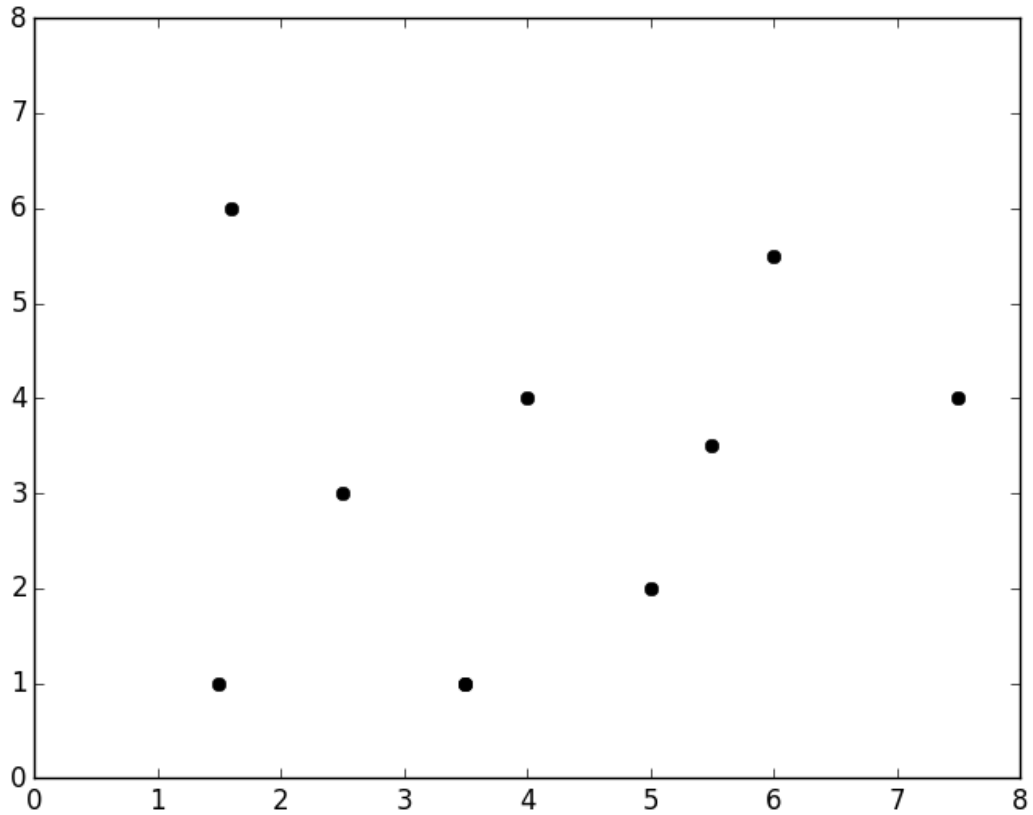
2. The collection $(R_k)_{1 \leq k \leq K}$ is the *Voronoi diagram* defined by p.

Theorem

If (X, d) is $\mathbf{R}^2$ with the Euclidean distance function, then every Voronoi region of every Voronoi diagram is a convex polytope.

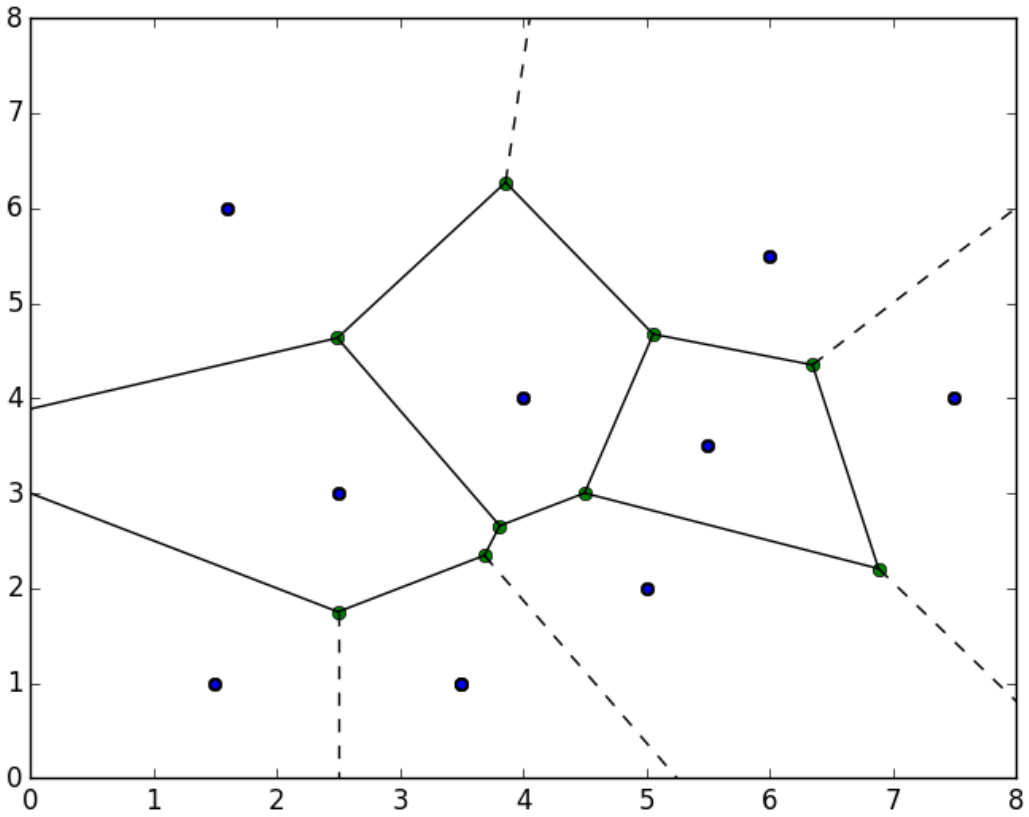
Python: Plotting the generator points

```
points = numpy.array([[1.5, 1.], ...])
import matplotlib.pyplot as plt
plt.plot(points[:, 0], points[:, 1], 'o', color='black')
plt.show()
```



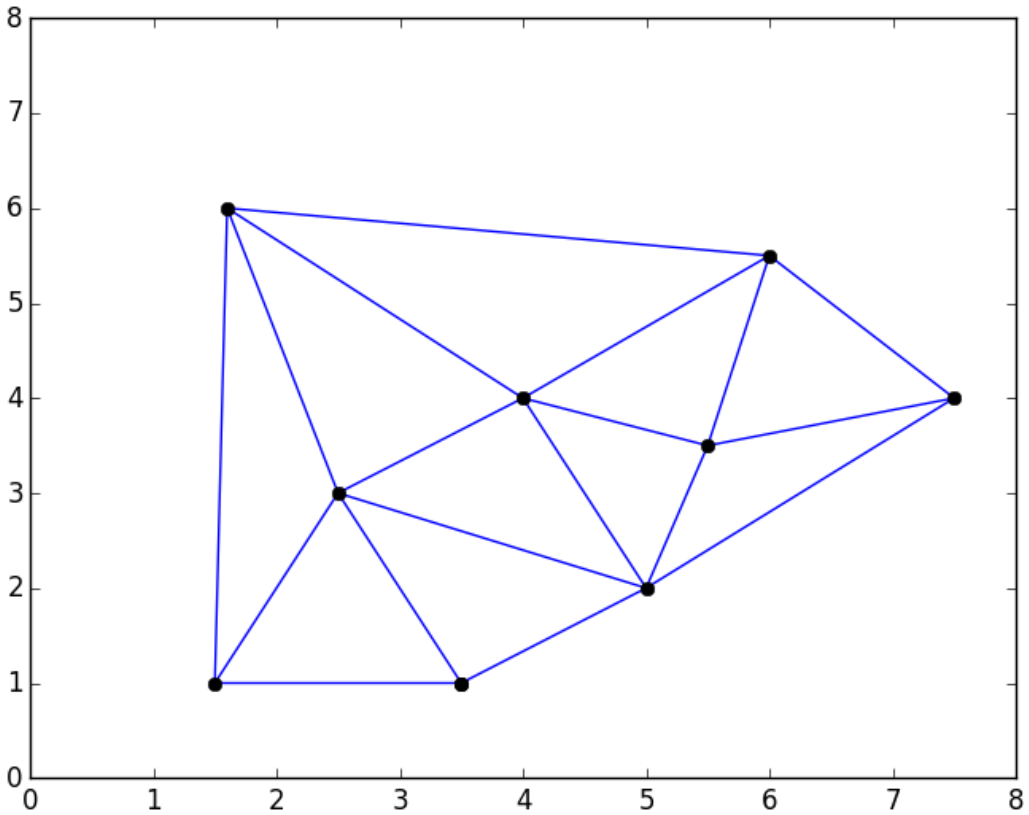
Python: Calculating Voronoi diagram

```
from scipy.spatial import Voronoi, voronoi_plot_2d
vor = Voronoi(points)
voronoi_plot_2d(vor, ax)
plt.show()
```



Python: Delaunay triangulation

```
from scipy.spatial import Delaunay
tri = Delaunay(points)
ax.triplot(points[:,0], points[:,1], tri.simplices.copy(), color='blue')
plt.show()
```



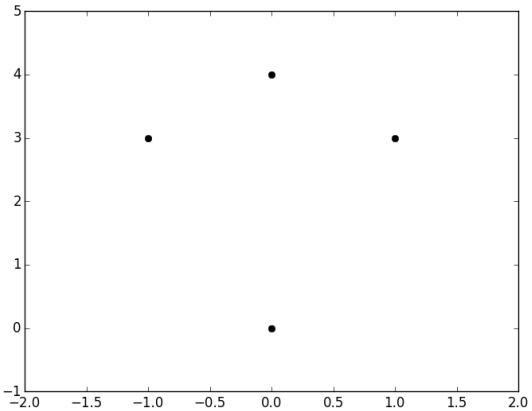
Delaunay triangulation

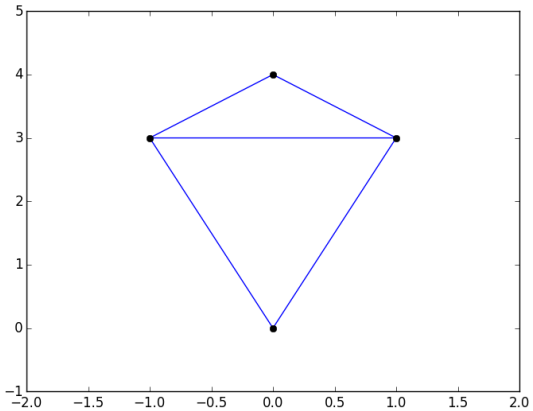
Definition (Delaunay triangulation)

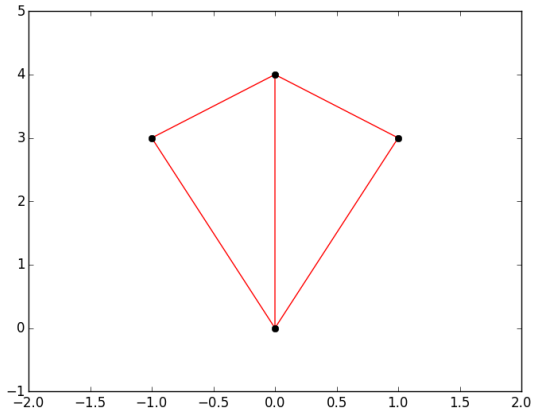
Let P be a set of points in the d-dimensional Euclidean space $\mathbf{R}^d$. A triangulation $T(P)$ of P is a *Delaunay triangulation* if the circum-hyperspheres of any simplex in $T(P)$ contain no point of P.

Theorem

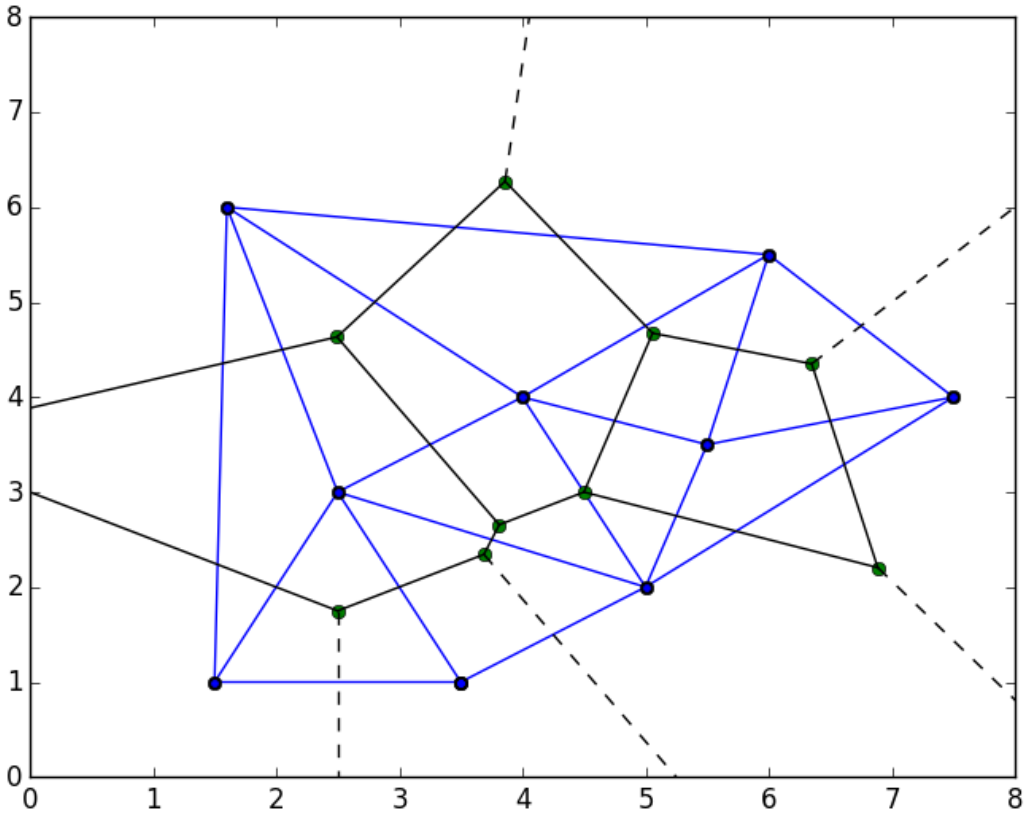
For any discrete set P in $\mathbf{R}^d$ in *general position*, there exists a unique Delaunay triangulation.





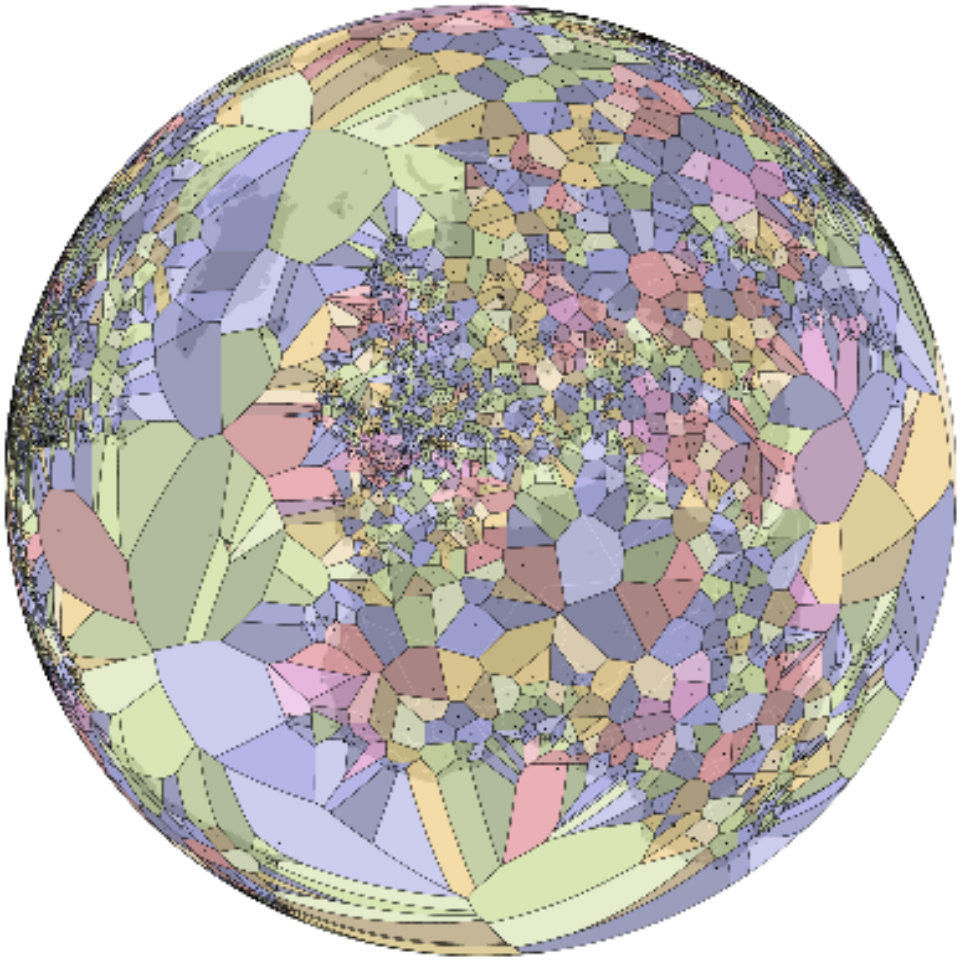


Python: Delaunay triangulation and Voronoi diagram



Outline

World Aiport Map



https://www.jasondavies.com/maps/voronoi/airports/

A long time ago (PyData London 2015)...

https://www.youtube.com/watch?v=gxNa9BD5CnQ

Spherical Voronoi Diagrams

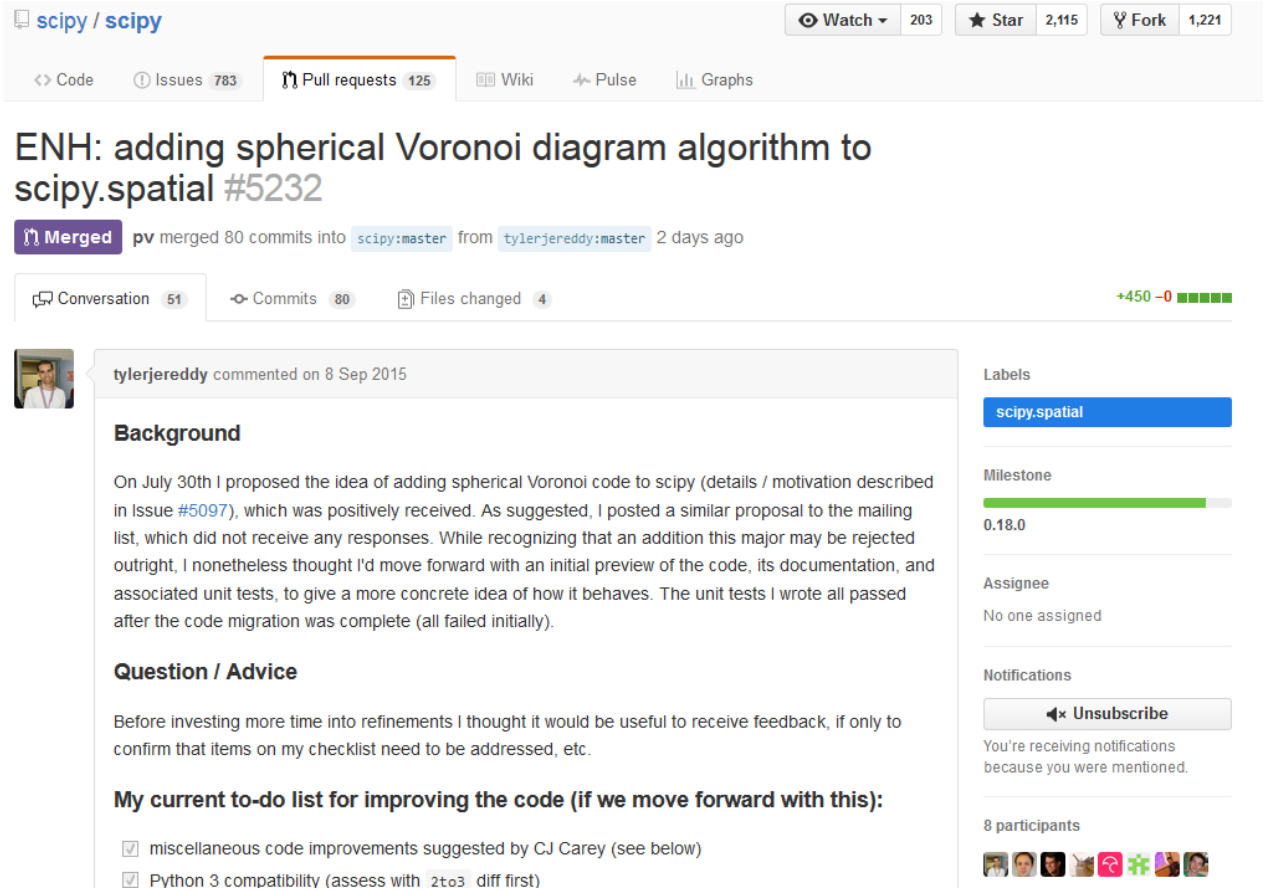
- Exactly the same problem as planar Voronoi diagram, just on a sphere instead of a plane.
- Many theoretical papers exist.
- Found no ready-to-go implementation in Python.

PyData London 2015 initiated quite a collaboration between...

- Tyler Reddy,
- Nikolai Nowaczyk,
- Ross Hemsley,
- Edd Edmondson,
- Joe Pitt-Francis,
- Mark Sansom,
- Ralf Gommers,
- Pauli Virtanen,
- Juan Luis Cano Rodríguez,
- CJ Carey,
- ...

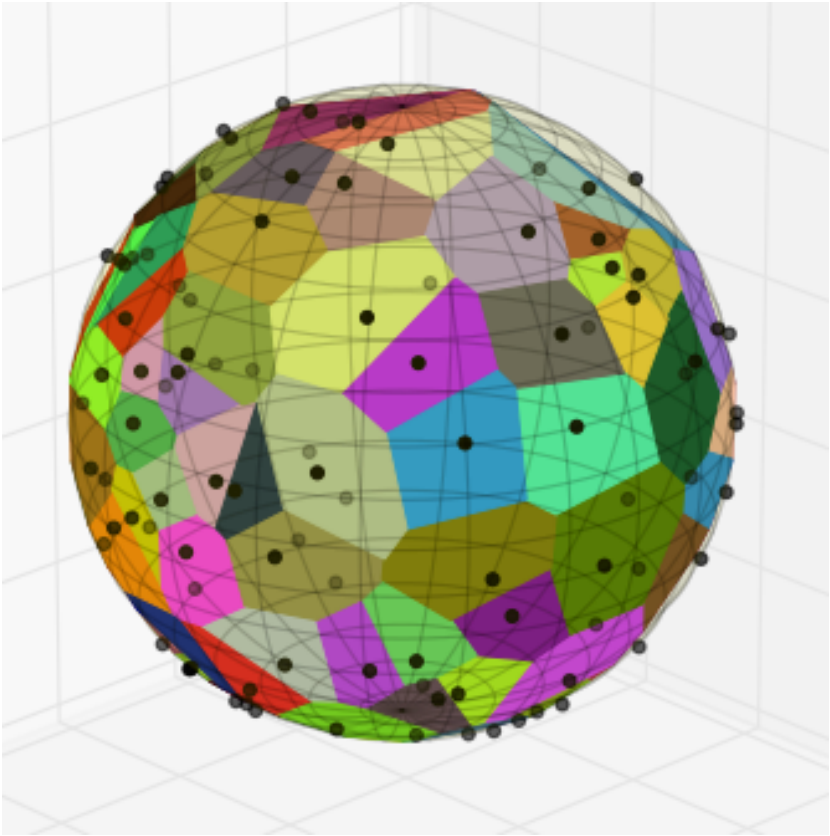
Result of PyData 2015: scipy.spatial.SphericalVoronoi

https://github.com/scipy/scipy/pull/5232



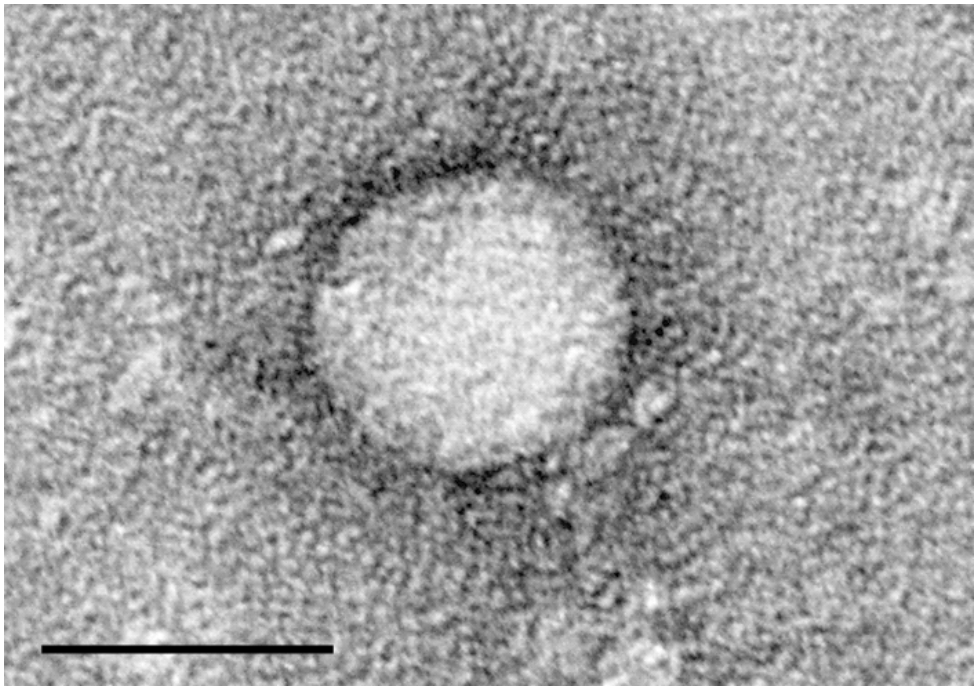
Will be in the 0.18.0 release of scipy.

PyData London 2016: scipy.spatial.SphericalVoronoi



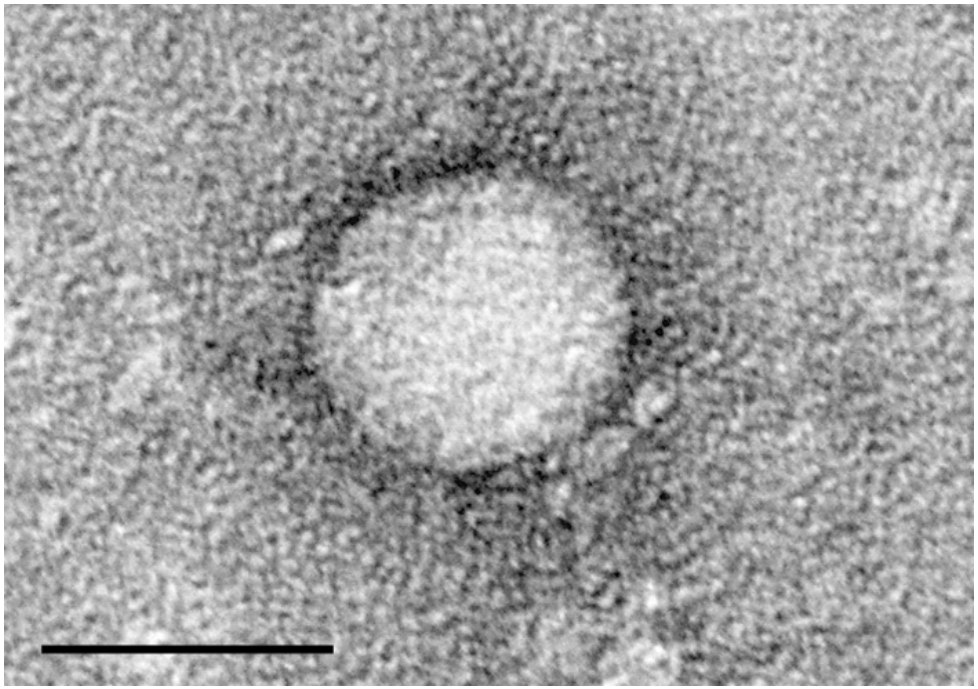
```
>>> points = np.array([...])
>>> from scipy.spatial import
      SphericalVoronoi
>>> sv = SphericalVoronoi(points)
```

Possible Application: Hepatitis C



- 200mio people infected worldwide
- can cause liver disease, cirrhosis, liver cancer
- 350.000 deaths per year
- new medication available (more successful, but expensive)

Possible Application: Hepatitis C



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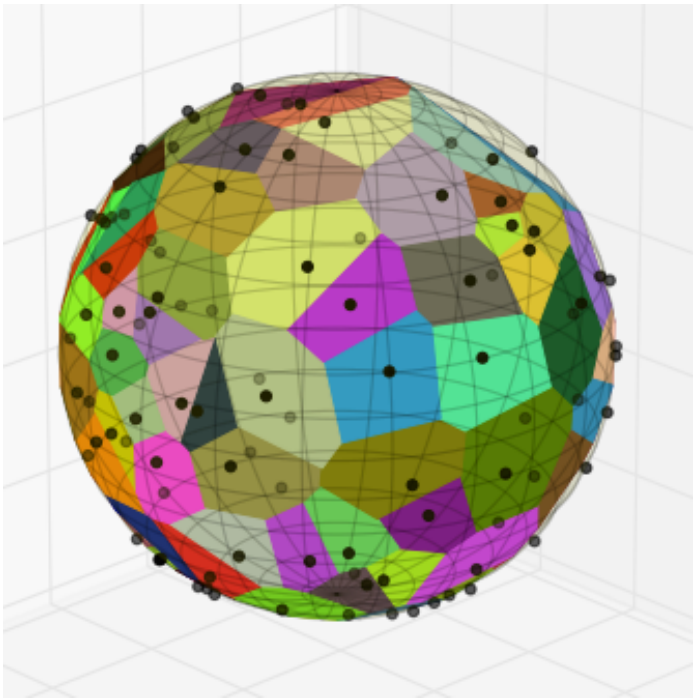
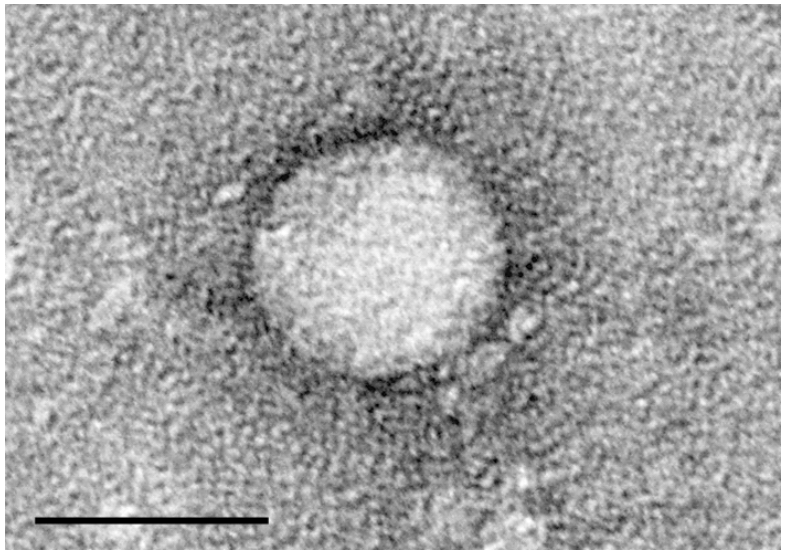
- new medication available (more successful, but expensive)

no vaccine

Modelling Hepatitis C



This could be a virus!



Hepatitis C virus

- Computational virology tries to understand a virus through computer simulations.
- Calculating spherical Voronoi diagrams is an important tool.

Maybe more collaboration coming up...

- Help with plotting:
  https://github.com/matplotlib/matplotlib/issues/5294
- Help with surface area:
  https://github.com/scipy/scipy/issues/6069

Questions, Comments? Get in touch!

- Nikolai Nowaczyk
  - mail@nikno.de
  - https://github.com/niknow
- Tyler Reddy
  - tyler.reddy@bioch.ox.ac.uk
  - https://github.com/tylerjereddy
- How to further improve `scipy.spatial.SphericalVoronoi` ?
- Other applications of spherical Voronoi diagrams?
- Get the code examples: https://github.com/niknow/pydata-london-2016-voronoi

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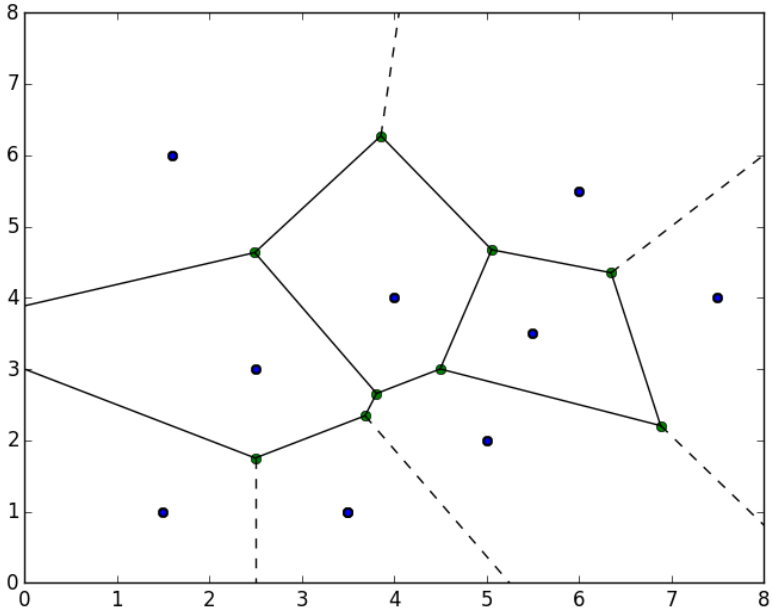
THANK YOU!

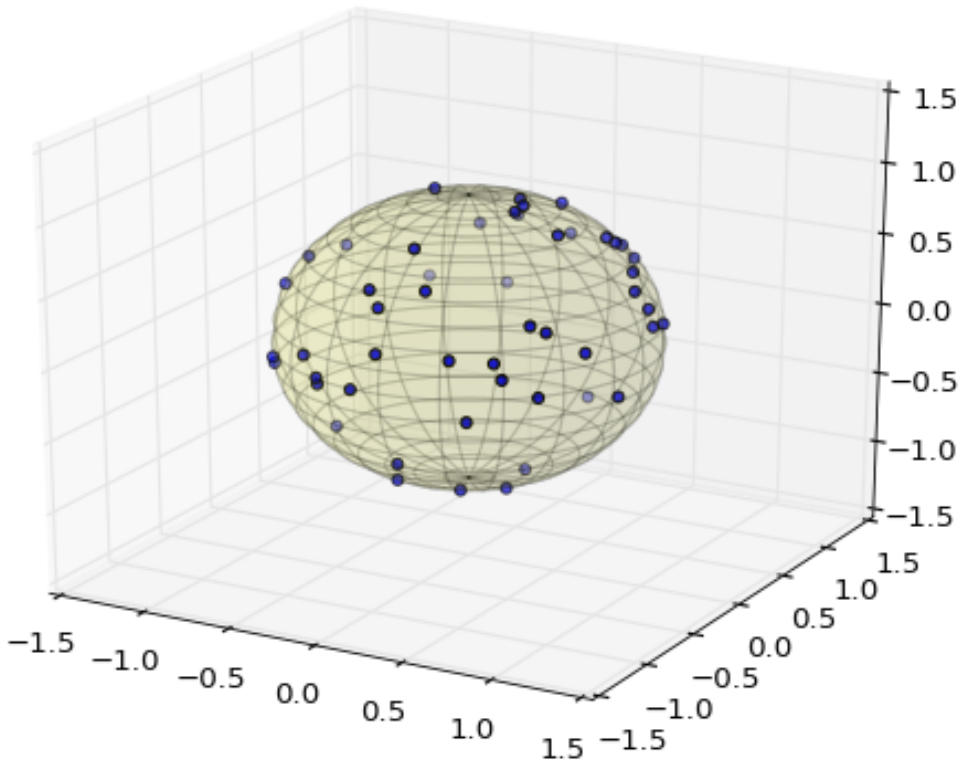
Outline

Python: A closer look at the data structures



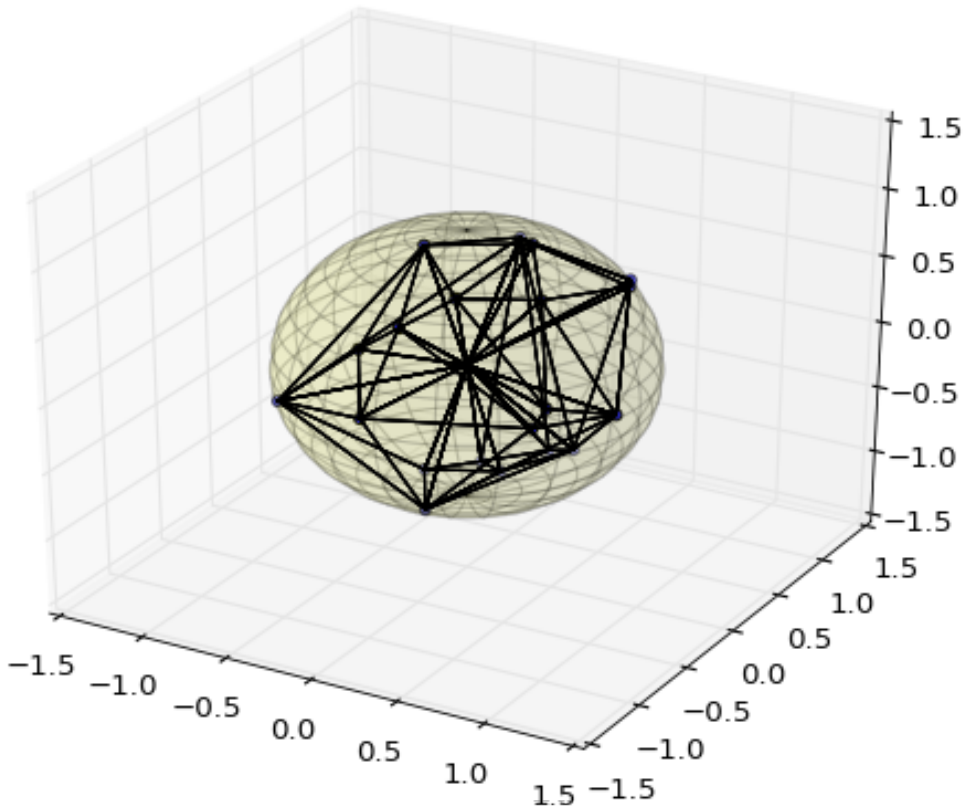

```
>>> points
[[1.5 , 1.0], [3.5 , 1.0], [5.0 ,
      2.0], [2.5 , 3.0], [3.5 , 1.0],
       [4.0 , 4.0], [5.5 , 3.5],
      [6.0 , 5.5], [7.5 , 4], [1.6 ,
      6.0]]
>>> points.shape
(10L, 2L)
vor = Voronoi(points)
>>> vor.vertices
[[-1.11, 3.55], [2.49 , 4.63],
      [3.81 , 2.65625], [3.86 ,
      6.27], [6.87 , 2.20], [4.5 ,
      3.], [ 6.35 , 4.35], [5.06 ,
      4.67], [2.5 , 1.75], [3.69 ,
      2.34]]
>>> vor.vertices.shape
(10L, 2L)
>>> vor.regions
[[-1, 0, 1, 3], [7, 5, 4, 6], [7,
      3, 1, 2, 5], [7, 3, -1, 6],
      [9, -1, 8], [], [-1, 4, 6],
      [9, 2, 5, 4, -1], [8, 0, -1],
       [9, 2, 1, 0, 8]]
# list of tuples of integers. The
      tuple at the k-th position
      contains the indices of the
      Voronoi vertices of the
      Voronoi region surrounding
      the k-th generator.
```

Algorithm:



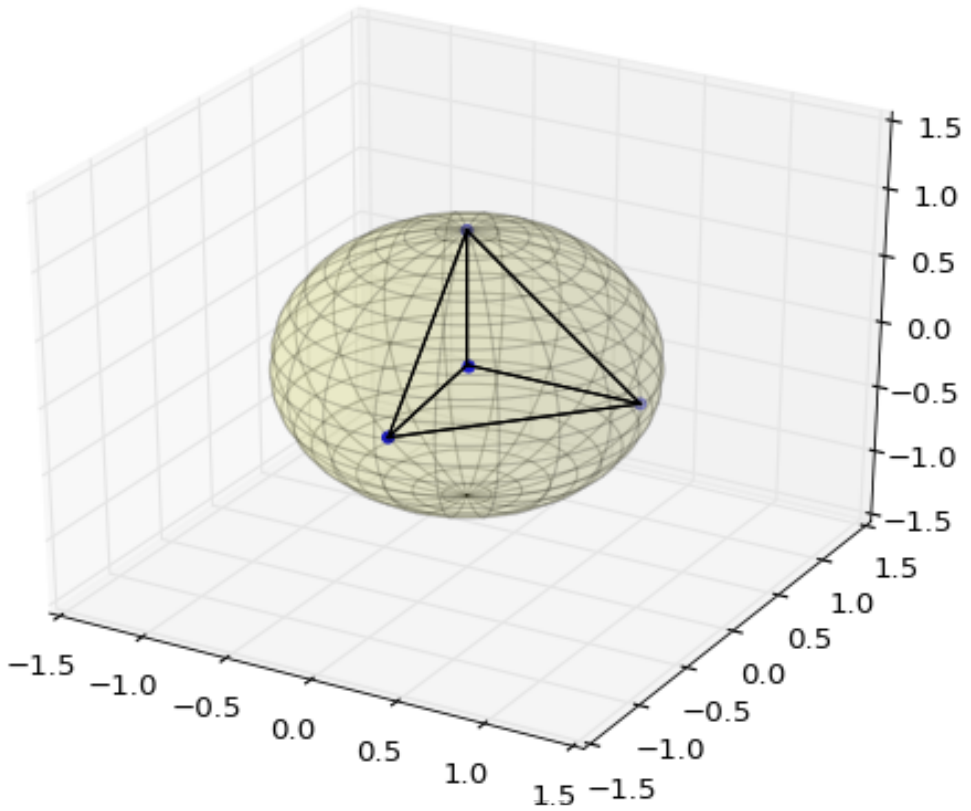
Generator points

Algorithm:



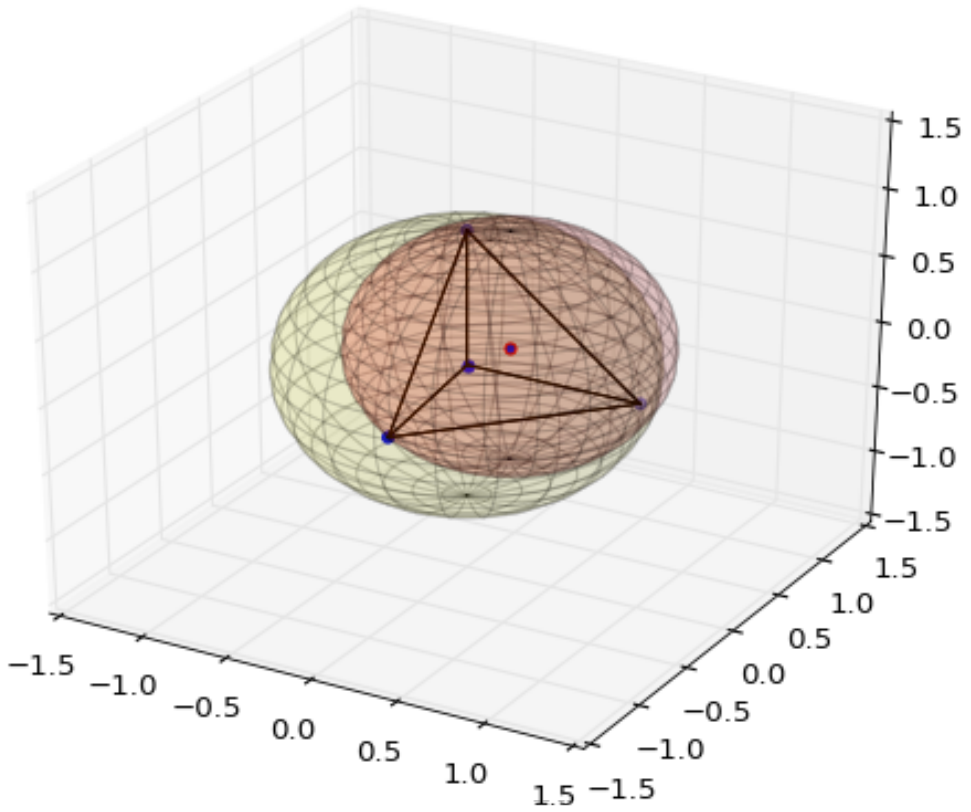
Calculate triangulation

Algorithm:



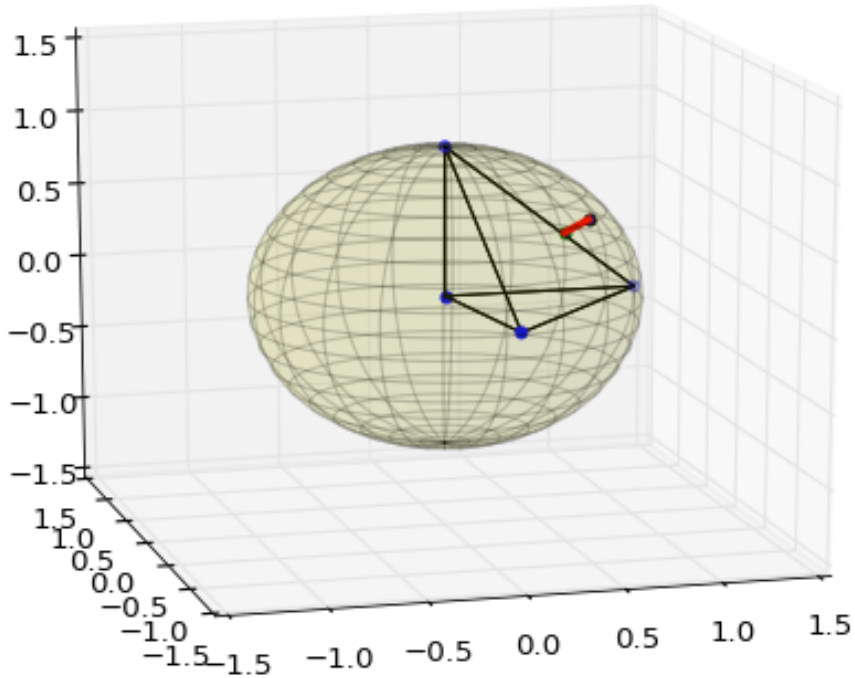
For each tetrahedron in the triangulation...

Algorithm:



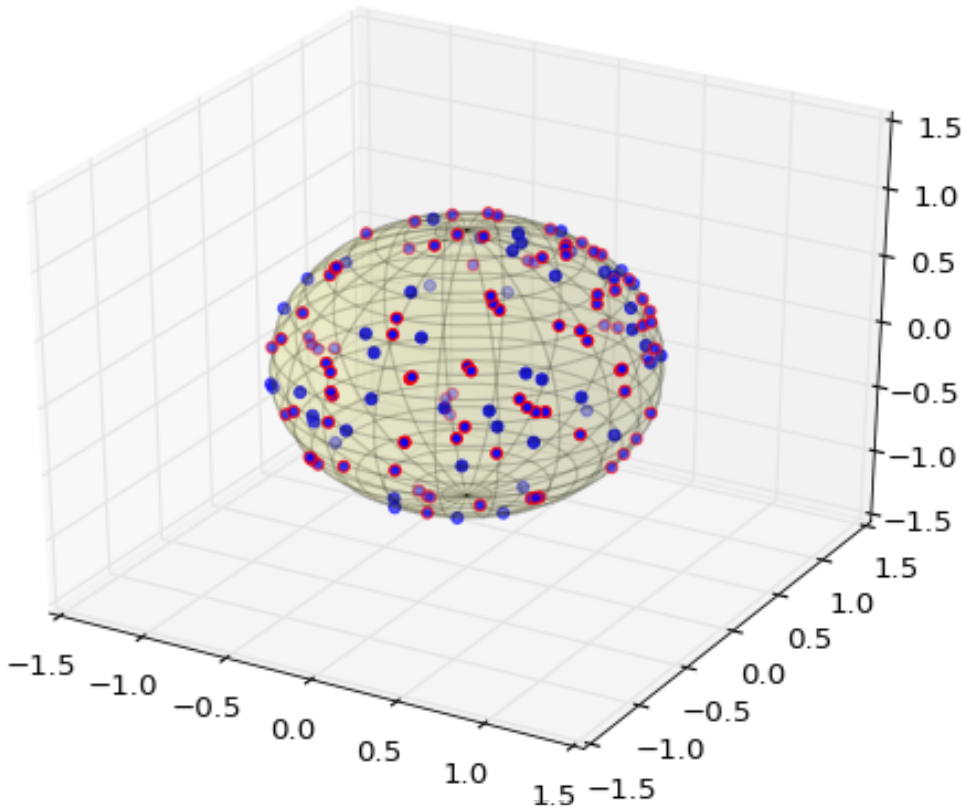
...compute circumcenter of circumsphere.

Algorithm:



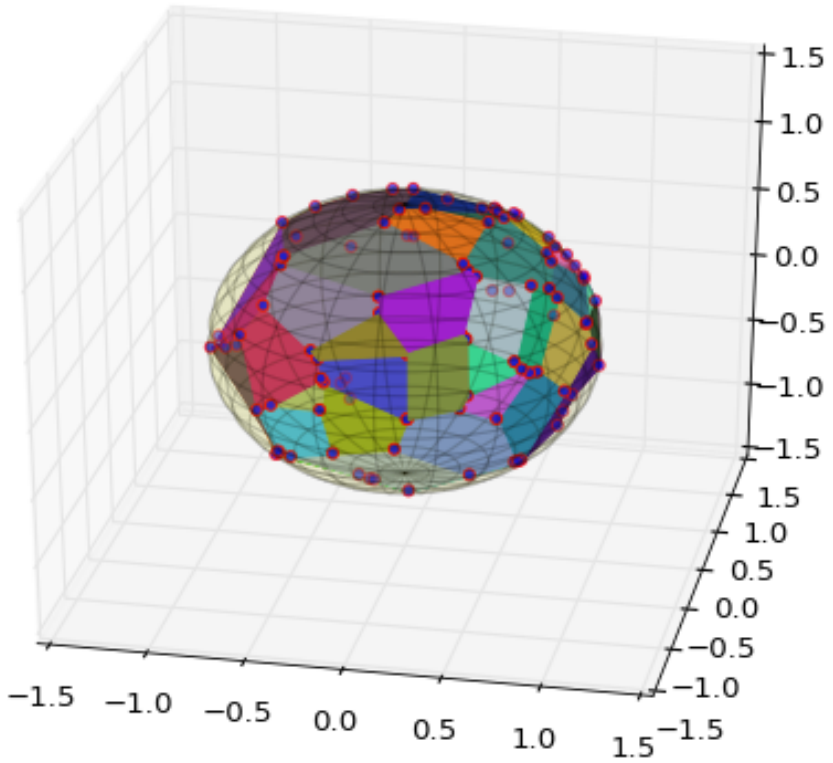
...and project onto sphere.

Algorithm:



Get lots of points on the sphere.

Algorithm:



These points are the Voronoi vertices!

SphericalVoronoi Algorithm: Summary

1. Calculate 3D Delaunay triangulation of the generator points.
2. Add the center of the sphere to the triangulation and get a tetrahedralization.
3. Calculate the circumcenter of the circumsphere of each tetahedron.
4. Projet circumcenters to the sphere to get the Voronoi vertices.
5. Use the neighbourhood information from the triangulation to infer, which vertices belong to which region.