Global Analysis II

Nikolai Nowaczyk

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"And the evil that was once vanquished shall rise anew. Wrapped in the guise of man shall he walk amongst the innocent and terror shall consume they that dwell among the earth. The sky shall rain fire and the seas will become as blood. The righteous shall fall before the wicked and all creation shall tremble before the burning of standards of hell."

MEPHISTO, 1264

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2. PDO: Partial Differential Operators

"Lucky for you I'm a freaking genius."

SEAMUS ZELAZNY HARPER, 10087 CY

This chapter is designed to give an introduction to the theory of partial differential operators ("PDO"). The treatment of Pseudo-Differential Operators ("ΨDO") later is even more technical than the one of partial differential operators. So in order to understand the ideas behind the ΨDO theory, the PDOs are helpful. Of course they are also useful and nice in their own right.

2.1. Local PDOs and their Symbols

First we have to establish the local theory on an open subset $U \subset \mathbb{R}^n$, which is then generalized to operators on manifolds.

2.1.1 Convention. We denote the partial derivative on $\mathbb{R}^n$ in direction $1 \leq j \leq n$ by ∂_j. For any multi-index α, we define

$$\partial^\alpha := \partial_1^{\alpha_1} \circ \ldots \circ \partial_n^{\alpha_n}.$$

For reasons that will become apparent later, we define $D_j := \frac{1}{i}\partial_j$ (here $i \in \mathbb{C}$ is the imaginary unit and not an index) and consequently

$$D^\alpha := i^{-|\alpha|}\partial^\alpha.$$

For any vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$

$$x^\alpha := \prod_{i=1}^n x_i^{\alpha_i}.$$

2.1.2 Definition (Differential Operator)**.** Let $U \subset \mathbb{R}^n$ be open. A *complex (partial) differential operator on U of order k*, $k \in \mathbb{N}$, (a "PDO") is a $\mathbb{C}$-linear map $P : \mathcal{C}^\infty(U, \mathbb{C}^r) \to \mathcal{C}^\infty(U, \mathbb{C}^s)$ such that for every $\alpha \in \mathbb{N}^n$, $0 \leq |\alpha| \leq k$, there exist $P_\alpha \in \mathcal{C}^\infty(U, \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^r, \mathbb{C}^s))$ such that

$$P = \sum_{|\alpha| \leq k} P_\alpha D^\alpha.$$

A real differential operator is defined analogously. The set of all such operators is denoted by

$$\mathrm{Diff}^k(U, \mathbb{C}^r, \mathbb{C}^s)$$

and we set

$$\mathrm{Diff}(U, \mathbb{C}^r, \mathbb{C}^s) := \bigcup_{k \in \mathbb{N}} \mathrm{Diff}^k(U, \mathbb{C}^r, \mathbb{C}^s).$$

If all the P_α are compactly supported, we write $P \in \mathrm{Diff}_c^k(U, \mathbb{C}^r, \mathbb{C}^s)$. The quantity

$$\min(k \in \mathbb{N} \mid P \in \mathrm{Diff}^k(U, \mathbb{C}^r, \mathbb{C}^s))$$

is the *minimal order of P*.

2.1.3 Remark.

(i) Of course we assume that $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^r, \mathbb{C}^s)$ is given the smooth structure obtained by identifying it with $\mathbb{C}^{s\times r}$.

(ii) The set $\mathrm{Diff}^k_{\mathbb{C}}(U, \mathbb{C}^r, \mathbb{C}^s)$ itself is canonically a module over $\mathcal{C}^\infty(U)$ and a $\mathbb{C}$-vector space.

(iii) Chosing a bases $\{E_\mu\}$ of $\mathbb{C}^r$ and $\{F_\nu\}$ of $\mathbb{C}^s$, we can fully expand P in coordinates as

$$Ps = \sum_{|\alpha|\leq k} \sum_{\nu=1}^{s} \sum_{\mu=1}^{r} (P^\alpha)^\nu_\mu D^\alpha(s^\mu) e_\nu. \tag{2.1}$$

(iv) It is clear that any $P \in \mathrm{Diff}^k(U, \mathbb{C}^r, \mathbb{C}^s)$ may be applied to a function $f \in \mathcal{C}^k(U)$. But since we want to work on smooth manifolds, things will as usual get easier, if we simply work in the smooth world.

(v) We will occasionally drop the $_{\mathbb{C}}$ in notation, since we always work with complex numbers. Of course one could also use real PDOs.

(vi) We explicitely allow the case $k = 0$. An operator $P \in \mathrm{Diff}^0(U, \mathbb{C}^r, \mathbb{C}^s)$ is still called a "differential operator" although it does not differentiate anything.

2.1.4 Definition (symbol)**.** Let $P \in \mathrm{Diff}^k_{\mathbb{C}}(U, \mathbb{C}^r, \mathbb{C}^s)$ be an operator

$$P = \sum_{|\alpha|\leq k} P_\alpha D^\alpha.$$

Define $\Sigma_P, \sigma_P : U \times \mathbb{R}^n \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^r, \mathbb{C}^s)$ by

$$\Sigma_P(x,\xi) := \sum_{|\alpha|\leq k} P_\alpha(x)\xi^\alpha, \qquad \sigma_P(x,\xi) := \sum_{|\alpha|=k} P_\alpha(x)\xi^\alpha.$$

We say Σ is the *full symbol of* P and σ is the *principal symbol of* P.

2.1.5 Remark.

(i) Since we are working on an open subset of $\mathbb{R}^n$ at the moment, it is clear that we may identify differential operators with their full symbols ($\xi_j \leftrightarrow D_j$).

(ii) The principal symbol will have a coordinate invariant meaning on manifolds. Therefore the term "symbol" is often used in the literature to refer to the pricipal symbol. The term "full symbol" is sometimes used for Σ_P. We will establish all the local theory for the full symbol as well.

2.2. Diffeomorphism Invariance

If there is any chance of defining a PDO calculus on manifolds, the property of being a PDO has to be invariant under diffeomorphisms.

2.2.1 Definition (Push-forward of operators)**.** Let $V, \tilde{V} \subset \mathbb{R}^n$ be open, $P \in \mathrm{Diff}^k(V, \mathbb{C}^r, \mathbb{C}^s)$ be a PDO and $F : V \to \tilde{V}$ be a smooth diffeomorphism. Then the map $\tilde{P} := F_*P : \mathcal{C}^\infty(\tilde{V}, \mathbb{C}^r) \to \mathcal{C}^\infty(\tilde{V}, \mathbb{C}^s)$ defined by

$$\tilde{s} \mapsto P(\tilde{s} \circ F) \circ F^{-1}$$

is the *push-forward of* P *along* F.

2.2.2 Lemma (Diffeomorphism invariance)**.** With the notation of Definition 2.2.1 above: Let $\alpha \in \mathbb{N}^n$, $|\alpha| = k \geq 1$, be a multi-index and $P := D^\alpha \in \mathrm{Diff}^k(V, \mathbb{C}^r, \mathbb{C}^r)$. Then $\tilde{P} := F_*(D^\alpha) \in \mathrm{Diff}^k(\tilde{V}, \mathbb{C}^r, \mathbb{C}^r)$, thus there exist $\tilde{P}_\alpha \in \mathcal{C}^\infty(\tilde{V}, \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^r))$ such that

$$\forall \tilde{s} \in \mathcal{C}^\infty(\tilde{V}, \mathbb{C}^r, \mathbb{C}^r) : \tilde{P}(\tilde{s}) = F_*(D^\alpha)(\tilde{s}) = D^\alpha(\tilde{s} \circ F) \circ F^{-1} = \sum_{|\beta| \leq k} \tilde{P}_\beta D^\beta(\tilde{s}).$$

Denote by $I_r \in \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^r)$ the identity and let $A := \nabla F \circ F^{-1} \in \mathcal{C}^\infty(\tilde{V}, \mathbb{R}^{n \times n})$. Then the symbols can be expressed by

$$\sigma_{D^\alpha}(x, \xi) = I_r \xi^\alpha, \qquad \sigma_{F_*(D^\alpha)}(\tilde{x}, \xi) = \sum_{|\beta| = k} \tilde{P}_\beta(\tilde{x}) \xi^\beta = I_r (A^t(\tilde{x})\xi)^\alpha,$$

Proof. We will show the statement by induction over k.
Step 1 ($k = 1$): This implies that $\alpha = e_j$ for some $1 \leq j \leq n$. The chain rule for total derivatives states

$$\nabla(\tilde{s} \circ F) = \nabla \tilde{s} \circ F \cdot \nabla F,$$

which implies

$$\partial_j(\tilde{s} \circ F) = \nabla \tilde{s} \circ F \cdot \partial_j F.$$

Consequently by definition

$$\begin{aligned} F_*(\partial^\alpha)(\tilde{s}) &= F_*(\partial_j)(\tilde{s}) = \partial_j(\tilde{s} \circ F) \circ F^{-1} = \nabla \tilde{s} \cdot (\partial_j F \circ F^{-1}) \\ &= \sum_{i=1}^n (\partial_j F^i \circ F^{-1}) \partial_i \tilde{s} = I_r \sum_{i=1}^n A^i_j \partial_i \tilde{s}. \end{aligned}$$

By multiplying with $-i$, this shows $F_*(D^\alpha) \in \mathrm{Diff}^1(\tilde{V}, \mathbb{C}^r, \mathbb{C}^r)$. The symbols are given by

$$\sigma_{\partial_j}(x, \xi) = I_r \xi_j, \qquad \sigma_{F_*(\partial_j)}(x, \xi) = I_r \sum_{i=1}^n A^i_j(x) \xi_i = I_r (A^t(x)\xi)_j = I_r (A^t(x)\xi)^\alpha.$$

Step 2 ($k \to k+1$): If $|\alpha| = k + 1$, there exists $\hat{\alpha} \in \mathbb{N}^n$, $|\hat{\alpha}| = k$, and $1 \leq j \leq n$ such that $\alpha = \hat{\alpha} + e_j$. By induction hypothesis, for any $\beta \leq \alpha$ the operator $F_*(D^\beta)$ is a PDO. Consequently

$$\forall \beta \leq \alpha : \exists p^\beta_\gamma \in \mathcal{C}^\infty(U, \mathbb{C}^{r \times r}) : F_*(D^\beta) = \sum_{|\gamma| \leq |\beta|} P^\beta_\gamma D^\gamma \tag{2.2}$$

and

$$\forall \beta \leq \alpha : \sigma_{F_*(D^\beta)} = \sum_{|\gamma| = |\beta|} p^\beta_\gamma \xi^\gamma = I_r (A^t \xi)^\beta. \tag{2.3}$$

We calculate

$$\begin{aligned} F_*(\partial^\alpha)(\tilde{s}) &= \partial^\alpha(\tilde{s} \circ F) \circ F^{-1} = \partial^{\hat{\alpha}} \partial_j (\tilde{s} \circ F) \circ F^{-1} = \partial^{\hat{\alpha}} \Big(\sum_{i=1}^n \partial_j F^i \cdot \partial_i \tilde{s} \circ F \Big) \circ F^{-1} \\ &= \Big(\sum_{\beta \leq \hat{\alpha}} \sum_{i=1}^n \binom{\hat{\alpha}}{\beta} \partial^{\hat{\alpha} - \beta} \partial_j F^i \cdot \partial^\beta (\partial_i \tilde{s} \circ F) \Big) \circ F^{-1} \\ &= \sum_{\beta \leq \hat{\alpha}} \sum_{i=1}^n \binom{\hat{\alpha}}{\beta} \partial^{\hat{\alpha} - \beta} \partial_j F^i \circ F^{-1} \cdot F_*(\partial^\beta)(\partial_i \tilde{s}) \\ &\overset{(2.2)}{=} \sum_{\beta \leq \hat{\alpha}} \sum_{i=1}^n \sum_{|\gamma| \leq |\beta|} P^\beta_\gamma \binom{\hat{\alpha}}{\beta} \partial^{\hat{\alpha} - \beta} A^i_j \partial^\gamma \partial_i (\tilde{s}). \end{aligned}$$

By multiplying with $(-i)^{k+1}$, this shows $F_*(D^\alpha) \in \operatorname{Diff}^{k+1}(\tilde{V}, \mathbb{C}^r, \mathbb{C}^r)$. We analyse the highest order terms. These occure precisely, if $|\gamma + e_i| = k+1 \Leftrightarrow |\gamma| = k$. Since $|\gamma| \leq |\beta|$ and $\beta \leq \hat{\alpha}$, this can only happen, if $\beta = \hat{\alpha}$ and $|\gamma| = k$.

$$\begin{aligned}\sigma_{F_*(D^\alpha)}(x,\xi) &= \sum_{|\gamma|=k} P_\gamma^{\hat{\alpha}}(x)\Big(\sum_{i=1}^n A_j^i(x)\xi_i\Big)\xi^\gamma = \sum_{|\gamma|=k} P_\gamma^{\hat{\alpha}}(x)(A^t(x)\xi)_j\xi^\gamma \\ &= (A^t(x)\xi)_j \sum_{|\gamma|=k} P_\gamma^{\hat{\alpha}}(x)\xi^\gamma \overset{(2.3)}{=} I_r(A^t(x)\xi)_j(A^t(x)\xi)^{\hat{\alpha}} = I_r(A^t(x)\xi)^\alpha.\end{aligned}$$

□

2.2.3 Theorem (Diffeomorphism Invariance)**.** With the notation of Definition 2.2.1 we claim: $\tilde{P} = F_*(P) \in \operatorname{Diff}^k(\tilde{V}, \mathbb{C}^r, \mathbb{C}^s)$, i.e. there exist $\tilde{P}_\alpha$ such that

$$\forall \tilde{s} \in \mathcal{C}^\infty(\tilde{V}, \mathbb{C}^r, \mathbb{C}^s) : \tilde{P}(\tilde{s}) = F_*(P)(\tilde{s}) = \sum_{|\alpha|\leq k} \tilde{P}_\alpha D^\alpha.$$

Moreover the symbol has a representation

$$\sigma_{\tilde{P}}(\tilde{x},\xi) = \sum_{|\alpha|=k} \tilde{P}_\alpha(\tilde{x})\xi_\alpha = \sum_{|\alpha|=k} (P_\alpha \circ F^{-1})(\tilde{x})(A^t(\tilde{x})\xi)^\alpha = \sigma_P(F^{-1}(\tilde{x}), A^t(\tilde{x})\xi),$$

where $A := \nabla F \circ F^{-1}$.

Proof. By definition we obtain

$$\tilde{P}(\tilde{s}) = F_*(P)(\tilde{s}) = \Big(\sum_{|\alpha|\leq k} P_\alpha D^\alpha(s \circ F)\Big) \circ F^{-1} = \sum_{|\alpha|\leq k} P_\alpha \circ F^{-1} F_*(D^\alpha)(\tilde{s}).$$

By applying the first part of Lemma 2.2.2, we conclude $\tilde{P} \in \operatorname{Diff}^k(\tilde{V}, \mathbb{C}^r, \mathbb{C}^s)$. By applying the second part and analyzing the highest order terms, we conclude that the symbol satisfies

$$\sigma_{\tilde{P}}(\tilde{x},\xi) = \sum_{|\alpha|=k} (P_\alpha \circ F^{-1})(\tilde{x})\sigma_{F_*(D^\alpha)}(\tilde{x},\xi) = \sum_{|\alpha|=k} (P_\alpha \circ F^{-1})(\tilde{x})(A^t(\tilde{x})\xi)^\alpha.$$

□

2.2.4 Remark. One might be tempted to look for a transformation formula for the lower order terms as well. This is extremely difficult and unneccessary for our purposes. The problem is that there is no really good chain rule for arbitrary partial differential operatos, i.e. expressing $\partial^\alpha(\tilde{s} \circ F)$ in terms of $\partial^\beta \tilde{s}$ and $\partial^\gamma F$ is not so easy. This problem is known as *Faá di Bruno's formula*, but has been stated initially by Arbogast in 1800. For some special cases there exists a formula that is sometimes useful. For our purposes here it is not.

2.2.1. Global PDOs and their Symbols

Let M be a smooth manifold of dimension m. The suitable setting for PDOs is to let them operate between the sections of two smooth complex vector bundles over the same manifold (notice already that this means we will be able to speak about the exterior differential as a PDO).

2.2.5 Definition (complex vector bundle)**.** A map $\pi : E \to M$ is a *smooth complex vector bundle of rank* r if the following conditions are satisfied:

(i) E is a smooth manifold.

(ii) The map π is smooth and surjective.

(iii) For all $p \in M$ *fibre over* p, $E_p := \pi^{-1}(p)$, is endowed with a complex vector space structure of complex dimension k.

(iv) For every $p \in M$ there exists an open neighbourhood $U \subset M$ of p and a *local trivialization*, i.e. a diffeomorphism $\Phi : E_U := \pi^{-1}(U) \to U \times \mathbb{C}^r$ such that $\mathrm{pr} \circ \Phi = \mathrm{id}_U$, where $\mathrm{pr} : U \times \mathbb{C}^r \to U$ is the canonical projection, and for every $q \in U$ the restriction $\Phi : E_q \to \{q\} \times \mathbb{C}^r \cong \mathbb{C}^r$ is a complex vector space isomorphism.

2.2.6 Definition (section)**.** If $\pi : E \to M$ is a complex vector bundle, a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$ is a *section in* E *over* M. The space of all such sections is denoted by $\Gamma(M, E)$.

2.2.7 Definition (frame)**.** Let $\pi : E \to M$ be a smooth complex vector bundle of rank r. Let $U \mathring{\subseteq} M$, $E_1, \dots, E_r \in \Gamma(U, E)$ such that for any $p \in U$, $(E_1|_p, \dots, E_r|_p)$ is a basis for E_p. Then $(E_1, \dots, E_r)$ is a *local frame for* E.

2.2.8 Lemma. Let $\pi : E \to M$ be a vector bundle of rank r. For any local frame $E_1, \dots, E_r$ on U of E the map

$$\begin{aligned} \Phi : \pi^{-1}(U) &\to U \times \mathbb{C}^r \\ v = \textstyle\sum_{j=1}^r v^i E_i &\mapsto \left(\pi(v), \textstyle\sum_{j=1}^r v^i e_i\right) \end{aligned}$$

provides a local trivialization on U for E. Conversely, for any local trivialization Ψ, the maps

$$E_i|_p := \Psi^{-1}(p, e_i)$$

provide a local frame $E_1, \dots, E_r$ on U for E. Thus local frames and local trivializations are in a one-to-one correspondence.

2.2.9 Definition (associated pushforwards)**.** Let $\pi : E \to M$ be a complex vector bundle of rank r and $\Phi : E_U \to U \times \mathbb{C}^r$ be a local trivialization. Denote by $\mathrm{pr}_2 : U \times \mathbb{C}^r \to \mathbb{C}^r$ the canonical projection. We obtain the *pushforward* $\Phi_* : \Gamma(U, E) \to \mathcal{C}^\infty(U, \mathbb{C}^r)$ defined by

$$s \mapsto \mathrm{pr}_2 \circ \Phi \circ s = \Phi_2 \circ s$$

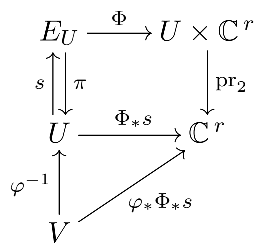
and for any chart $\varphi : U \to V$ of M the *pushforward* $\varphi_* : \mathcal{C}^\infty(U, \mathbb{C}^r) \to \mathcal{C}^\infty(V, \mathbb{C}^r)$

$$f \mapsto f \circ \varphi^{-1}.$$

By composing we obtain a map $\varphi_* \Phi_* := \varphi_* \circ \Phi_* : \Gamma(U, E) \to \mathcal{C}^\infty(V, \mathbb{C}^r)$.

2.2.10 Lemma.

(i) By construction the following diagram commutes:



Notice that $\varphi_* \Phi_*$ transports a local section $s : U \to E_U$ to a function $V \to \mathbb{C}^r$.

(ii) The map φ_* is bijective with inverse $(\varphi_*)^{-1} = (\varphi^{-1})_* : \mathcal{C}^\infty(V, \mathbb{C}^r) \to \mathcal{C}^\infty(U, \mathbb{C}^r)$.

(iii) The map Φ_* is bijective with inverse $\Phi_*^{-1} : \mathcal{C}^\infty(U, \mathbb{C}^r) \to \Gamma(U, E)$, $f \mapsto \Phi^{-1} \circ \mathrm{id}_U \times f$.

(iv) The map $\varphi_*\Phi_* : \Gamma(U, E) \to \mathcal{C}^\infty(V, \mathbb{C}^r)$, $s \mapsto \Phi_2 \circ s \circ \varphi^{-1}$, is bijective with inverse $\Phi_*^{-1} \circ \varphi_*^{-1} : \mathcal{C}^\infty(V, \mathbb{C}^r) \to \Gamma(U, E)$, $f \mapsto \Psi^{-1} \circ \mathrm{id} \times f \circ \varphi$.

Proof. The first two statements are clear. To see the third one, remember that any local trivialization can be written as

$$\Phi = (\Phi_1, \Phi_2) = (\mathrm{id}_U, \Phi_2) = \mathrm{id}_U \times \Phi_2.$$

Therefore we obtain

$$\begin{aligned} \forall s \in \Gamma(U, E) : (\Phi_*^{-1} \circ \Phi_*)(s) &= \Phi_*^{-1}(\Phi_2 \circ s) = \Phi^{-1} \circ \mathrm{id}_U \times \Phi_2 \circ s \\ &= \Phi^{-1} \circ \Phi \circ s = s \end{aligned}$$

and

$$\forall f \in \mathcal{C}^\infty(U, \mathbb{C}^r) : (\Phi_* \circ \Phi_*^{-1})(f) = \Phi_*(\Phi^{-1} \circ \mathrm{id}_U \times f) = \mathrm{pr}_2 \circ \Phi \circ \Phi^{-1} \circ \mathrm{id}_U \times f = f.$$

□

2.2.11 Definition (Differential operators between vector bundles)**.** Let E, F be smooth complex vector bundles over M of rank r and s. A $\mathbb{C}$-linear map $P : \Gamma(M, E) \to \Gamma(M, F)$ is a *differential operator of rank k*, if for any $p \in M$ there exists a chart $\varphi : U \to V$, $p \in U$, and local trivializations $\Phi : E_U \to U \times \mathbb{C}^r$ and $\Psi : F_U \to U \times \mathbb{C}^s$, there exists $D \in \mathrm{Diff}^k(V; \mathbb{C}^r, \mathbb{C}^s)$, called a *local representation of P*, such that

$$\begin{array}{ccc} \Gamma_c(U, E) & \xrightarrow{P} & \Gamma_c(U, F) \\ \downarrow{\scriptstyle \varphi_*\Phi_*} & & \downarrow{\scriptstyle \varphi_*\Psi_*} \\ \mathcal{C}_c^\infty(V, \mathbb{C}^r) & \xrightarrow{D} & \mathcal{C}_c^\infty(V, \mathbb{C}^s) \end{array}$$

commutes, i.e.

$$\varphi_*\Psi_* \circ P \circ (\varphi_*\Phi_*)^{-1} = D.$$

We say P *satisfies the PDO property on U with respect to* φ, Φ, Ψ. The set of all differential operators of order k between E and F is denoted by

$$\mathrm{Diff}^k(M; E, F).$$

Analogously we set

$$\mathrm{Diff}(M; E, F) = \bigcup_{k \in \mathbb{N}} \mathrm{Diff}^k(M; E, F).$$

2.2.12 Lemma (local PDOs)**.** Let E, F be trivial vectors bundles over $U \subset M$ with trivializations Φ, Ψ. Let $E_1, \dots E_r$, $F_1, \dots, F_s$ be the associated local trivializations (c.f. 2.2.8).

(i) Assume there exists a chart $\varphi : U \to V$. For any multi-index α, $|\alpha| = k$,

$$D^\alpha := D^\alpha_{\varphi,\Phi} := (\varphi_*\Phi_*)^{-1} \circ D^\alpha \circ \varphi_*\Phi_* \in \mathrm{Diff}^k(U; E, E) \tag{2.4}$$

and we may calculate

$$\forall s \in \Gamma(U, E) : D^\alpha_{\varphi,\Phi}(s) = \sum_{\mu=1}^{r} D^\alpha_\varphi(s^\mu) E_\mu, \tag{2.5}$$

where $D^\alpha_\varphi \in \mathrm{Diff}^k(U; \mathbb{C}, \mathbb{C})$ is given by:

$$\forall f \in \Gamma(U; \mathbb{C}, \mathbb{C}) = \mathcal{C}^\infty(U, \mathbb{C}) : D^\alpha_\varphi(f) := D^\alpha(f \circ \varphi^{-1}) \circ \varphi.$$

(ii) A linear map $P : \Gamma(U, E) \to \Gamma(U, F)$ satisfies $P \in \mathrm{Diff}^k(U; E, F)$ if and only if there are $P_\alpha \in \Gamma(U, \mathrm{Hom}(E, F))$ such that

$$P = \sum_{|\alpha| \leq k} P_\alpha D^\alpha_{\varphi,\Phi}. \tag{2.6}$$

This operator acts on local sections by

$$\forall s \in \Gamma(U, E) : P(s) = \sum_{|\alpha| \leq k} \sum_{\nu=1}^{s} \sum_{\mu=1}^{r} (P_\alpha)^\nu_\mu D^\alpha(s^\mu) F_\nu, \tag{2.7}$$

where the $((P_\alpha)^\nu_\mu) \in \mathscr{C}^\infty(U, \mathbb{C}^{s \times r})$ are the coordinate matrices of P_α with respect to the local frames (notice the resemblence to 2.1.) Therefore P is a PDO if and only if there exist local frames and $((P_\alpha)^\nu_\mu) \in \mathscr{C}^\infty(U, \mathbb{C}^{s \times r})$ such that (2.7) holds.

(iii) The operators D^α satisfy the Leibniz rule:

$$\forall f \in \mathscr{C}^\infty(U, \mathbb{C}) : \forall s \in \Gamma(U, E) : D^\alpha_{\varphi,\Phi}(fs) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta_\varphi(f) D^{\alpha-\beta}_{\varphi,\Phi}(s). \tag{2.8}$$

Even more general: For any vector bundle homomorphism $\theta \in \Gamma(U, \mathrm{Hom}(E, F))$

$$\forall s \in \Gamma(U; E) : D^\alpha_{\varphi,\Psi}(\theta s) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta_{\varphi,\Phi,\Psi}(\theta) D^{\alpha-\beta}_{\varphi,\Phi}(s). \tag{2.9}$$

Proof.

(i) The facts that $D^\alpha_{\varphi,\Phi} \in \mathrm{Diff}^k(U; E, E)$, $D^\alpha_\varphi(U; \mathbb{C}, \mathbb{C})$ follow directly from the definition 2.2.11. By construction

$$\begin{aligned} D^\alpha_{\varphi,\Phi}(s) &= \sum_{\mu=1}^{r} D^\alpha_{\varphi,\Phi}(s^\mu E_\mu) = \sum_{\mu=1}^{r} (\varphi_*\Phi_*)^{-1}(D^\alpha(\varphi_*\Phi_*(s^\mu E_\mu))) \\ &= \sum_{\mu=1}^{r} (\varphi_*\Phi_*)^{-1}(D^\alpha(s^\mu \circ \varphi^{-1})) = \sum_{\mu=1}^{r} \Phi_*^{-1}((D^\alpha(s^\mu \circ \varphi^{-1})) \circ \varphi) \\ &= \sum_{\mu=1}^{r} D^\alpha_\varphi(s^\mu) E_\mu. \end{aligned}$$

(ii) By definition 2.2.11 P is a PDO if and only if there exists a $\tilde{P} \in \mathrm{Diff}^k(V; \mathbb{C}^r, \mathbb{C}^s)$ such that $\varphi_* \Psi_* \circ P \circ (\varphi_* \Phi_*)^{-1} = \tilde{P}$. Let

$$\tilde{P} = \sum_{|\alpha| \leq k} \tilde{P}_\alpha D^\alpha$$

and calculate

$$\begin{aligned} P &= (\varphi_* \Psi_*)^{-1} \sum_{|\alpha| \leq k} \tilde{P}_\alpha D^\alpha \circ (\varphi_* \Phi_*) \\ &= \sum_{|\alpha| \leq k} \underbrace{(\varphi_* \Psi_*)^{-1} \tilde{P}_\alpha \varphi_* \Phi_*}_{=: P_\alpha} D^\alpha_{\varphi, \Phi}. \end{aligned}$$

We calculate

$$\begin{aligned} P(s) &\overset{(2.6)}{=} \sum_{|\alpha| \leq k} P_\alpha D^\alpha_{\varphi,\Phi}(s) = \sum_{|\alpha| \leq k} \sum_{\mu=1}^{r} P_\alpha D^\alpha_{\varphi,\Phi}(s^\mu E_\mu) \overset{(2.4)}{=} \sum_{|\alpha| \leq k} \sum_{\mu=1}^{r} P_\alpha E_\mu D^\alpha_\varphi(s^\mu) \\ &= \sum_{|\alpha| \leq k} \sum_{\nu=1}^{s} \sum_{\mu=1}^{r} (P_\alpha)^\nu_\mu F_\nu D^\alpha_\varphi(s^\mu). \end{aligned}$$

(iii) To see the first equation, we just calculate

$$\begin{aligned} D^\alpha_{\varphi,\Phi}(fs) &\overset{(2.5)}{=} \sum_{\mu=1}^{r} D^\alpha_\varphi(fs) E_\mu = \sum_{\mu=1}^{r} D^\alpha(f \circ \varphi^{-1} s^\mu \circ \varphi^{-1})) \circ \varphi E_\mu \\ &\overset{\text{A.1.3}}{=} \sum_{\mu=1}^{r} \sum_{\beta \leq \alpha} (D^\beta(f \circ \varphi^{-1}) D^{\alpha-\beta}(s^\mu \circ \varphi^{-1})) \circ \varphi E_\mu \\ &= \sum_{\beta \leq \alpha} D^\beta_\varphi(f) \sum_{\mu=1}^{r} D^{\alpha-\beta}_\varphi(s^\mu) E_\mu \overset{(2.5)}{=} \sum_{\beta \leq \alpha} D^\beta_\varphi(f) D^{\alpha-\beta}_{\varphi,\Phi}(s). \end{aligned}$$

To see the second equation, first notice that $\mathrm{Hom}(E, F)$ is a bundle of rank rs. Using the local frames $E_1, \ldots, E_r$ and $F_1, \ldots, F_s$, we obtain a local trivialization (Φ, Ψ) of $\mathrm{Hom}(E, F)$ by mapping any element $\theta(p) \in \mathrm{Hom}_p(E, F)$ to its coordinate matrix $\theta(p)^\nu_\mu$ with respect to these frames. Therefore, we may calculate analogously

$$\begin{aligned} D^\alpha_{\varphi,\Psi}(\theta s) &\overset{(2.5)}{=} \sum_{\nu=1}^{s} D^\alpha_\varphi((\theta s)^\nu) F_\nu = \sum_{\nu=1}^{s} \sum_{\mu=1}^{r} D^\alpha_\varphi(\theta^\nu_\mu s^\mu) F_\nu \overset{(2.8)}{=} \\ &= \sum_{\nu=1}^{s} \sum_{\mu=1}^{r} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta_\varphi(\theta^\nu_\mu) D^{\alpha-\beta}_\varphi(s^\mu) F_\nu \\ &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \sum_{\nu=1}^{s} \sum_{\mu=1}^{r} D^\beta_{\varphi,\Phi,\Psi}(\theta)^\nu_\mu D^{\alpha-\beta}_\varphi(s^\mu) F_\nu \\ &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \sum_{\nu=1}^{s} (D^\beta_{\varphi,\Phi,\Psi}(\theta) D^{\alpha-\beta}_{\varphi,\Phi}(s))^\nu F_\nu \\ &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta_{\varphi,\Phi,\Psi}(\theta) D^{\alpha-\beta}_{\varphi,\Phi}(s). \end{aligned}$$

□

2.2.13 Convention. It is very common to use the notation D^α for $D^\alpha_{\varphi,\Phi}$ etc. as well. This notation convention is very convenient, but you have to keep in mind that this operator depends on the chosen chart and trivialization.

2.2.14 Theorem (local independence)**.** Let $P : \Gamma(M,E) \to \Gamma(M,F)$ be a linear map, let $\varphi : U \to V$, $\psi : U \to \tilde V$ be any charts and $\Phi, \tilde\Phi : E_U \to U \times \mathbb{C}^r$, $\Psi, \tilde\Psi : F_U \to U \times \mathbb{C}^s$ be local trivializations.

(i) Then

$$\begin{aligned} D &:= \varphi_*\Psi_* \circ P \circ (\varphi_*\Phi_*)^{-1} \in \mathrm{Diff}^k(V, \mathbb{C}^r, \mathbb{C}^s) \\ \Longrightarrow \tilde D &:= \psi_*\tilde\Psi_* \circ P \circ (\psi_*\tilde\Phi_*)^{-1} \in \mathrm{Diff}^k(\tilde V, \mathbb{C}^r, \mathbb{C}^s). \end{aligned}$$

So the local property of beeing a differential operator does not depend on the choice of charts or trivializations, but only on the smooth structures of M, E and F.

(ii) Denote by $F := \psi \circ \varphi^{-1} : V \to \tilde V$ the transition map between the charts, $A := \nabla F \circ F^{-1}$, and by g_E and g_F the transition functions between the local trivializations (see equation (2.10)) and let

$$D = \sum_{|\alpha| \le k} P_\alpha D^\alpha \in \mathrm{Diff}^k(\tilde V, \mathbb{C}^r, \mathbb{C}^s).$$

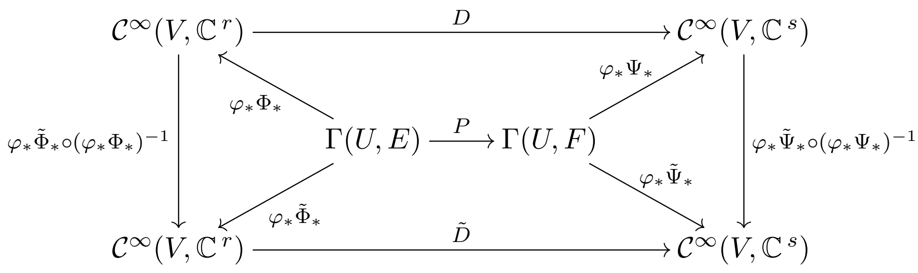
Then the symbol satisfies

$$\forall \tilde x \in \tilde V : \forall \xi \in \mathbb{R}^n : \sigma_{\tilde D}(\tilde x, \xi) = \sum_{|\alpha| = k} (g_F P_\alpha g_E^{-1})(F^{-1}(\tilde x))(A^t(\tilde x)\xi)^\alpha.$$

Proof.
Step 1 (Independence of trivializations): First we fix the chart φ and consider different trivializations. There exist functions (c.f. [4, 5.4])) $g_E \in \mathcal{C}^\infty(V, GL(r, \mathbb{C}))$, $g_F \in \mathcal{C}^\infty(V, GL(s, \mathbb{C}))$ such that

$$\begin{aligned} \forall x \in V : \forall v \in \mathbb{C}^r : (\tilde\Phi \circ \Phi^{-1})(\varphi^{-1}(x), v) = (\varphi^{-1}(x), g_E(x)v) \\ \forall x \in V : \forall w \in \mathbb{C}^s : (\tilde\Psi \circ \Psi^{-1})(\varphi^{-1}(x), w) = (\varphi^{-1}(x), g_F(x)w). \end{aligned} \tag{2.10}$$

We redefine $\tilde D := \varphi_*\tilde\Psi_* \circ P \circ (\varphi_*\tilde\Phi_*)^{-1}$ (valid for this step of the proof) and remark that the following diagram commutes:



We calculate

$$\begin{aligned} \tilde D &= \varphi_*\tilde\Psi_* \circ P \circ (\varphi_*\tilde\Phi_*)^{-1} = \varphi_*\tilde\Psi_* \circ (\varphi_*\Psi_*)^{-1} \circ D \circ \varphi_*\Phi_* \circ (\varphi_*\tilde\Phi_*)^{-1} \\ &= \varphi_* \circ \tilde\Psi_* \circ \Psi_*^{-1} \circ \varphi_*^{-1} \circ D \circ \varphi_* \circ \Phi_* \circ \tilde\Phi_*^{-1} \circ \varphi_*^{-1}. \end{aligned}$$

The map $\varphi_* \circ \tilde{\Psi}_* \circ \Psi_*^{-1} \circ \varphi_*^{-1} : \mathcal{C}^\infty(V, \mathbb{C}^r) \to \mathcal{C}^\infty(V, \mathbb{C}^r)$ can be simplified drastically: For any $f \in \mathcal{C}^\infty(V, \mathbb{C}^r)$, we conclude from 2.2.10

$$\begin{aligned}\varphi_*(\tilde{\Psi}_*(\Psi_*^{-1}(\varphi_*^{-1}(f)))) &= \varphi_*(\tilde{\Psi}_*(\Psi_*^{-1}(f \circ \varphi))) = \varphi_*(\tilde{\Psi}_*(\Psi^{-1} \circ \mathrm{id}_U \times (f \circ \varphi))) \\ &= \varphi_*(\mathrm{pr}_2 \circ \tilde{\Psi} \circ \Psi^{-1} \circ \mathrm{id}_U \times (f \circ \varphi)) = g_F f \circ \varphi \circ \varphi^{-1} = g_F f\end{aligned}$$

and analogously

$$(\varphi_* \circ \Phi_* \circ \tilde{\Phi}_*^{-1} \circ \varphi_*^{-1})(f) = g_E^{-1} f.$$

Since $D \in \mathrm{Diff}^k(M; E, F)$ by hypothesis, there exist $P_\alpha \in \mathcal{C}^\infty(V, \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s))$ such that

$$D = \sum_{|\alpha| \leq k} P_\alpha D^\alpha \in \mathrm{Diff}^k_{\mathbb{C}}(V, r, s).$$

Alltogether, we obtain

$$\begin{aligned}\tilde{D} f &= g_F \Big(\sum_{|\alpha| \leq k} P_\alpha D^\alpha \Big)(g_E^{-1} f) = \sum_{|\alpha| \leq k} g_F P_\alpha D^\alpha (g_E^{-1} f) \\ &\overset{\text{A.1.5}}{=} \sum_{|\alpha| \leq k} g_F P_\alpha \Big(\sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (D^{\alpha - \beta} g_E^{-1}) D^\beta f \Big) \\ &= \sum_{|\alpha| \leq k} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} g_F P_\alpha (D^{\alpha-\beta} g_E^{-1}) D^\beta f,\end{aligned}$$

which shows $\tilde{D} \in \mathrm{Diff}^k(V, \mathbb{C}^r, \mathbb{C}^s)$.
We analyze the highest order terms: These occur precisely, if $|\beta| = k$. But since $\beta \leq \alpha$ this happens if and only if $\beta = \alpha$. So the symbol is given by

$$\sigma_{\tilde{D}}(x, \xi) = \sum_{|\alpha| = k} g_F(x) P_\alpha(x) g_E^{-1}(x) \xi^\alpha.$$

Step 2 (Independence of the chart): Now fix the trivializations Φ, Ψ and consider the two different charts φ, ψ. Analogously we redefine $\tilde{D} := \psi_* \Psi_* \circ P \circ (\psi_* \Phi_*)^{-1}$ (valid for this step of the proof) and calculate

$$\begin{aligned}\tilde{D} &= \psi_* \Psi_* \circ P \circ (\psi_* \Phi_*)^{-1} = \psi_* \Psi_* \circ (\varphi_* \Psi_*)^{-1} \circ D \circ \varphi_* \Phi_* \circ (\psi_* \Phi_*)^{-1} \\ &= \psi_* \circ \Psi_* \circ \Psi_*^{-1} \circ \varphi_*^{-1} \circ D \circ \varphi_* \circ \Phi_* \circ \Phi_*^{-1} \circ \psi_*^{-1} = \psi_* \circ \varphi_*^{-1} \circ D \circ \varphi_* \circ \psi_*^{-1}.\end{aligned}$$

Thus for any $\tilde{f} \in \mathcal{C}^\infty(\tilde{V}, \mathbb{C}^r, \mathbb{C}^s)$, we obtain

$$\tilde{D}(\tilde{f}) = D(\tilde{f} \circ F) \circ F^{-1} = F_*(D)(f),$$

which implies $\tilde{D} \in \mathrm{Diff}^k(\tilde{V}, \mathbb{C}^r, \mathbb{C}^s)$ by Theorem 2.2.3. It was already shown there that the symbol is given by

$$\sigma_{\tilde{D}}(\tilde{x}, \xi) = \sum_{|\alpha| = k} P_\alpha(F^{-1}(\tilde{x}))(A^t(\tilde{x})\xi)^\alpha.$$

Redefining $\tilde{D} := \tilde{\psi}_* \tilde{\Psi}_* \circ P \circ (\psi_* \tilde{\Phi}_*)^{-1}$ as in the statement of the theorem and combining both steps, we obtain both claims. □

2.2.15 Definition (symbol)**.** Let $P \in \mathrm{Diff}^k(M; E, F)$ be a PDO. For any $p \in M$ and any $\xi \in T_p^* M$, define $\sigma_P(p, \xi) \in \mathrm{Hom}(E_p, F_p)$ to be the homomorphism given as follows: Choose a chart $\varphi : U \to V$ near p and local trivializations $\Phi : E_U \to U \times \mathbb{C}^r$, $\Psi : F_U \to U \times \mathbb{C}^s$. Let D be the local coordinate representation of P with respect to this chart and these trivializations and define

$$\forall e \in E_p : \sigma_P(p, \xi)(e) := \sigma_P(\xi)e := \Psi^{-1}(p, \sigma_D(\varphi(p), \varphi_* \xi)(\Phi_2(e))),$$

We call σ_P the *symbol of* P. (For an alternative approach see Theorem 2.2.19.)

2.2.16 Remark. This definition produces two problems: First of all, the homomorphism $\sigma_P(p, \xi)$ is defined in terms of various non canonical choices, so we have to show that it is well-defined, c.f. Lemma 2.2.17 (assume for the moment this has been done). Secondly, we would like to state more precisely, what kind of σ_P is. Recall that the symbol of the local representation D as defined in 2.1.4 is a smooth map $\sigma_P : V \times \mathbb{R}^n \to \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s)$. This is no longer possible. On a manifold σ_P is not a map on $M \times T^*M$, since it is only defined for those $(p, \xi) \in M \times T^*M$ such that $\pi(\xi) = p$. This problem can be easily circumvented by thinking of σ as a map on T^*M, since we can recover the base point of any $\xi \in T^*M$ by $p := \pi(\xi)$. We can also no longer think of σ_P as a map with range $\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^s)$. For any $\xi \in T^*M$ the map $\sigma_P(\xi)$ is an element of $\mathrm{Hom}(E_{\pi(\xi)}, F_{\pi(\xi)})$. To define a suitable image space, let

$$\mathrm{Hom}(E, F) := \bigcup_{p \in M} \mathrm{Hom}(E_p, F_p) = E^* \otimes F$$

and endow this set with the canonical topology and smooth structure from the tensor product bundle, which turns it into a smooth vector bundle over M. Then

$$\sigma : T^*M \to \mathrm{Hom}(E, F).$$

Now we have a suitable domain and range for σ_P, but this map satisfies a bit more. We have not yet encoded the condition $\sigma(\xi) \in \mathrm{Hom}(E_p, F_p)$, $p = \pi(\xi)$. This condition almost looks like the condition for a section. In fact σ is a section, but not of the bundle $\mathrm{Hom}(E, F)$ (since this is a bundle over M). Therefore, we just have to think of E and F as bundles over T^*M which are "constant" on any T_p^*M. This enables us to think of $\mathrm{Hom}(E, F)$ as a bundle over T^*M.
One can make this rigorous: Denote by $\pi : T^*M \to M$ the cotangent bundle and let $\pi_E : E \to M$, $\pi_F : F \to M$ be the vector bundles and consider the pull-back bundles π^*E and π^*F . By construction, we obtain a commutative diagram

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$$\begin{array}{ccc} \pi^*(E) & \longrightarrow & E \\ {\scriptstyle \pi_E^*}\big\downarrow & & \big\downarrow{\scriptstyle \pi_E} \\ T^*M & \xrightarrow{\pi} & M, \end{array}$$

where π_E^* is the projection of the pull-back bundle $\pi^*(E)$ and the fibres satisfy

$$\forall \xi \in T^*M : (\pi^*E)_\xi = \{e \in E \mid \pi_E(e) = \pi(\xi)\} = E_{\pi(\xi)}$$

by definition. Consequently σ_P is a section of the bundle $\mathrm{Hom}(\pi^*(E), \pi^*(F))$ over T^*M. We will use the notation $\sigma_P(\xi)$ and $\sigma_P(p, \xi)$ interchangably.

2.2.17 Lemma. The symbol is a well-defined section

$$\sigma_P \in \Gamma(T^*M, \mathrm{Hom}(\pi^*E, \pi^*F)),$$

i.e.: Let $\tilde{\varphi} : \tilde{U} \to \tilde{V}$ be another chart, $\tilde{\Phi}$, $\tilde{\Psi}$ be other local trivializations for E and F and let $\tilde{\sigma}_P$ be the symbol defined in terms of this chart and these local trivializations. Then using notation from Definition 2.2.15

$$\forall p \in U \cap \tilde{U} : \forall \xi \in T_p^*M : \forall e \in E_p : \sigma_P(p,\xi)(e) = \tilde{\sigma}_P(p,\xi)(e).$$

Proof. By shrinking the coordinate neighbourhoods if necessary, we may assume that $U = \tilde{U}$, and calculate there. As usual, we define $F := \tilde{\varphi} \circ \varphi^{-1} \in \mathcal{C}^\infty(V, \tilde{V})$, $A := \nabla F \circ F^{-1} \in \mathcal{C}^\infty(\tilde{V}, GL(n))$. Denote by $\Xi := (\xi_1, \ldots, \xi_n) \in \mathcal{C}^\infty(T^*U \to \mathbb{R}^n)$ the coordinate vector function of ξ with respect to φ seen as a column vector in $\mathbb{R}^n$ (define $\tilde{\Xi}$ analogously). The transformation law for the cotangent bundle states that

$$\xi = \xi_i d\varphi^i = \tilde{\xi}_i d\tilde{\varphi}^i,$$

where $\Xi = \nabla F^t \circ \varphi \cdot \tilde{\Xi}$. This implies $\Xi = A^t \circ \tilde{\varphi} \cdot \tilde{\Xi}$, which is equivalent to

$$\tilde{\Xi} = (A^t)^{-1} \circ \tilde{\varphi} \cdot \Xi \tag{2.11}$$

Remember the defining equations (2.10) for the transition functions. Define $\tau : U \times \mathbb{C}^r \to U \times \mathbb{C}^r$, $(p, v) \mapsto (p, g_E(\varphi(p))v)$. We can reformulate

$$\tilde{\Phi} \circ \Phi^{-1} = \tau \iff \tilde{\Phi} = \tau \circ \Phi,$$

which implies in particular

$$\forall p \in U : \forall e \in E_p : \tilde{\Phi}(e) = (\tau(\Phi(e))) = \tau(\Phi_1(e), \Phi_2(e)) = (p, g_E(\varphi(p))\Phi_2(e)) \tag{2.12}$$

and analogously for $\tilde{\Psi}$. Now let $\tilde{D}$ be a coordinate representation of P with respect to $\tilde{\varphi}$, $\tilde{\Phi}$, $\tilde{\Psi}$. We calculate for any $p \in U$, $\xi \in T^*U$, $e \in E_U$,

$$\begin{aligned}
\tilde{\sigma}_P(\xi)(e) &= \tilde{\Psi}^{-1}(p, \sigma_{\tilde{D}}(\tilde{\varphi}(p), \tilde{\Xi}(p))(\tilde{\Phi}_2(e))) \\
&\overset{2.2.14(ii)}{=} \tilde{\Psi}^{-1}\Big(p, \sum_{|\alpha|=k} (g_F P_\alpha g_E^{-1})(F^{-1}(\tilde{\varphi}(p)))(\tilde{\Phi}_2(e)))(A^t(\tilde{\varphi}(p))\tilde{\Xi}(p))^\alpha\Big) \\
&\overset{(2.11)}{=} \tilde{\Psi}^{-1}\Big(p, \sum_{|\alpha|=k} (g_F P_\alpha g_E^{-1})(\varphi(p))(\tilde{\Phi}_2(e))(A^t(\tilde{\varphi}(p))(A^t)^{-1}(\tilde{\varphi}(p))\Xi(p)^\alpha\Big) \\
&\overset{(2.12)}{=} \Psi^{-1}\Big(p, g_F^{-1}(\varphi(p)) \sum_{|\alpha|=k} (g_F P_\alpha g_E^{-1})(\varphi(p)) g_E(\varphi(p))(\Phi_2(e))\Xi(p)^\alpha\Big) \\
&= \Psi^{-1}(p, \sigma_D(\varphi(p), \Xi(p))(\Phi_2(e))) = \sigma_P(p, \xi)(v).
\end{aligned}$$

□

2.2.18 Remark. This theorem is precisely the reason why the symbol is defined on the contangent bundle rather than on the tangent bundle.

There is an alternative approach to the symbol, which is more coordinate invariant and therefore sometimes useful. For reasons of completeness, we establish this symbol from scratch.

2.2.19 Theorem (Alternative approach to the symbol).

(i) Let $U \subset \mathbb{R}^n$ be open, $p \in U$, $f \in \mathcal{C}^\infty(U, \mathbb{R})$, such that $f(p) = 0$, $df|_p = \xi$. Let $k \in \mathbb{N}$, $k \geq 1$. Then for any multi-index $\alpha \in \mathbb{N}^n$, $1 \leq |\alpha| \leq k$,

$$\partial^\alpha(f^k)(p) = \begin{cases} k!\xi^\alpha, & |\alpha| = k, \\ 0, & |\alpha| < k. \end{cases}$$

(ii) Let $\tilde{f} \in \mathcal{C}^\infty(U, \mathbb{R})$ satisfy $f(p) = 0$ and $df|_p = \xi$ as well. In addition let $e \in \mathbb{C}^r$, $s, \tilde{s} \in \mathcal{C}^\infty(U, \mathbb{C}^r)$ such that $s(p) = \tilde{s}(p) = e$. Then for any $P \in \mathrm{Diff}^k(V, \mathbb{C}^r, \mathbb{C}^s)$

$$P(f^k s)(p) = P(\tilde{f}^k \tilde{s})(p).$$

(iii) Let $P \in \mathrm{Diff}^k(U, \mathbb{C}^r, \mathbb{C}^s)$, $e \in \mathbb{C}^r$ and $s \in \mathcal{C}^\infty(U, \mathbb{C}^r)$ such that $s(p) = e$. Then

$$\sigma_P(p, \xi)(e) = \frac{i^k}{k!} P(f^k s)(p).$$

(iv) Let $E \to M$, $F \to M$ be smooth complex vector bundles and $P \in \mathrm{Diff}^k(M; E, F)$. Let $p \in M$, $\xi \in T_pM$, $e \in E_p$ and let $f \in \mathcal{C}^\infty(M, \mathbb{R})$ such that $df|_p = \xi$ and $s \in \Gamma(M, E)$ such that $s(p) = e$. Then

$$\sigma_P(p, \xi)(e) = \frac{i^k}{k!} P(f^k s)(p). \tag{2.13}$$

We could have used this as a definition for σ_P as well.

Proof.

(i) We prove the statement by induction over k. In case $k = 1$ the only multi-index α satisfying $|\alpha| = 0 < 1$ is $\alpha = 0$. Consequently

$$\partial^\alpha(f^k)(p) = \partial^0(f)(p) = f(p) = 0.$$

In case $|\alpha| = 1$, there exists $1 \leq j \leq n$ such that $\alpha = e_j$. Consequently

$$\partial^\alpha(f^k)(p) = \partial_j(f)(p) = \xi_j.$$

For the induction step $k \to k+1$ consider a multi-index α with $|\alpha| \leq k+1$. Then we may split $\alpha = \beta + e_j$, where $|\beta| \leq k$ and $1 \leq j \leq n$. We calculate

$$\begin{aligned} \partial^\alpha(f^{k+1})(p) &= \partial^\beta(\partial_j(f^{k+1}))(p) = (k+1)\partial^\beta((f^k \partial_j f))(p) \\ &\overset{\text{A.1.3}}{=} (k+1) \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \partial^\gamma(f^k)(p)(\partial^{\beta-\gamma}\partial_j f)(p) \end{aligned} \tag{2.14}$$

and first analyse this term for $|\alpha| < k+1$. In that case $|\beta| < k$ and since all $\gamma \leq \beta$, this implies $|\gamma| < k$ as well. Therefore (2.14) reveals $\partial^\alpha(f^{k+1})(p) = 0$ by induction hypothesis. In case $|\alpha| = k+1$, the only relevant multi-index $\gamma \leq \beta$ in the remaining sum is is $\beta = \gamma$. Thus we may continue (2.14) by

$$= (k+1)\binom{\beta}{\beta}\partial^\beta(f^k)(p)(\partial_j f)(p) = (k+1)k!\xi^\beta \xi_j = (k+1)!\xi^\alpha,$$

where we used the induction hypothesis.

(ii) Obviously

$$\tilde{f}^k \tilde{s} - f^k s = (\tilde{f}^k - f^k)s + \tilde{f}^k(\tilde{s} - s)$$

and therefore we may calculate

$$\begin{aligned} P((\tilde{f}^k - f^k)s)(p) &= \sum_{|\alpha| \leq k} P_\alpha(p) D^\alpha((\tilde{f}^k - f^k)s)(p) \\ &= \sum_{|\alpha| \leq k} P_\alpha(p) \sum_{\beta \leq \alpha} (D^\beta(\tilde{f}^k)(p) - D^\beta(f^k)(p)) D^{\alpha-\beta}(s)(p) \overset{\text{(i)}}{=} 0 \end{aligned}$$

and

$$\begin{aligned} D(\tilde{f}^k(\tilde{s} - s))(p) &= \sum_{|\alpha| \leq k} P_\alpha(p) D^\alpha(\tilde{f}^k(\tilde{s} - s))(p) \\ &= \sum_{|\alpha| \leq k} P_\alpha(p) \sum_{\beta \leq \alpha} D^\beta(\tilde{f}^k)(p) D^{\alpha-\beta}(\tilde{s} - s))(p) \overset{\text{(i)}}{=} 0. \end{aligned}$$

(iii) We calculate

$$\begin{aligned} \frac{i^k}{k!} P(f^k s)(p) &= \frac{i^k}{k!} \sum_{|\alpha| \leq k} P_\alpha(p) D^\alpha(f^k s)(p) \\ &= \frac{i^k}{k!} \sum_{|\alpha| \leq k} P_\alpha(p) \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta(f^k)(p) D^{\alpha-\beta} s(p) \\ &\overset{\text{(i)}}{=} \frac{i^k}{k!} \sum_{|\alpha| = k} P_\alpha(p)(-i)^k k! \xi^\alpha e = \sigma_P(p, \xi)(e). \end{aligned}$$

(iv) Choose a chart $\varphi : U \to V$ near p and local trivializations $\Phi : E_U \to U \times \mathbb{C}^r$, $\Psi : F_U \to U \times \mathbb{C}^s$. Let D be the local coordinate representation of P with respect to this chart and these trivializations. We calculate for any $p \in U$, $\xi \in T^*U$, $v \in E_U$

$$\begin{aligned} \frac{k!}{i^k} \sigma_P(p, \xi)(v) &= \frac{k!}{i^k} \Psi^{-1}(p, \sigma_D(\varphi(p), \xi_i e^i)(\Phi_2(v))) \\ &\overset{\text{(iii)}}{=} \Psi^{-1}\Big(p, D((f \circ \varphi^{-1})^k(\Phi_2(s) \circ \varphi^{-1}))(\varphi(p))\Big) \\ &= \Psi^{-1}\Big(p, (D \circ f^k \Phi_2(s))(p)\Big) \\ &= (\Psi^{-1} \circ \mathrm{id} \times D \circ \Phi_2 \circ f^k s)(p) \\ &\overset{2.2.10}{=} ((\varphi_* \Psi_*)^{-1} \circ D \circ \varphi_* \Phi_*)(f^k s)(p) = P(f^k s)(p). \end{aligned}$$

□

2.2.20 Theorem (Symbol via exponential function). Let $P \in \mathrm{Diff}^k(M; E, F)$, $p \in M$, $\xi \in T_p^* M$, $e \in E_p$, $g \in \mathcal{C}^\infty(M, \mathbb{R})$ such that $dg|_p = \xi$ and $s \in \Gamma(M, E)$ such that $s(p) = e$. Then

$$\sigma_P(p, \xi)e = \lim_{t \to \infty} t^{-k} \Big(e^{-itg} P(e^{itg} s)\Big)(p).$$

Proof.

Step 1 (Preparation): We prove the following claim by induction over k: In case $M = U \subset \mathbb{R}^n$,

$$\forall |\alpha| \leq k : \lim_{t\to\infty} t^{-k}\Big(e^{-itg}D^\alpha(e^{itg})\Big)(p) = \begin{cases} \xi^\alpha, & |\alpha| = k, \\ 0, & |\alpha| < k. \end{cases} \tag{2.15}$$

Step 1.1 (Induction start $k = 0$): No multi-index α satisfies $|\alpha| < 0$ and the only multi-index α such that $|\alpha| = 0$ is $\alpha = 0$. Therefore

$$\lim_{t\to\infty} t^{-k}\Big(e^{itg}D^\alpha(e^{itg})\Big)(p) = \lim_{t\to\infty} e^{-itg(p)}e^{itg(p)} = 1 = \xi^0 = \xi^\alpha.$$

Step 1.2 (Induction step $k \to k+1$): Assume $|\alpha| \leq k+1$. Then we may decompose α into $\alpha = \beta + e_j$, where $|\beta| \leq k$ and $1 \leq j \leq n$. We calculate

$$\begin{aligned} &\lim_{t\to\infty} t^{-(k+1)}\Big(e^{-itg}D^\alpha(e^{itg})\Big)(p) = \lim_{t\to\infty} t^{-(k+1)}e^{-itg(p)}\Big(D^\beta D_j(e^{itg})\Big)(p) \\ &= \lim_{t\to\infty} t^{-k-1}e^{-itg(p)}D^\beta\Big(e^{itg}tiD_j(g)\Big)(p) = \lim_{t\to\infty} t^{-k}e^{-itg(p)}D^\beta\Big(e^{itg}iD_j(g)\Big)(p) \\ &= \lim_{t\to\infty} t^{-k}e^{-itg(p)}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}D^\gamma(e^{itg})(p)iD^{\beta-\gamma+e_j}(g)(p) \\ &= \sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\underbrace{\lim_{t\to\infty} t^{-k}e^{-itg(p)}D^\gamma(e^{itg})(p)}_{=:(*)}iD^{\beta-\gamma+e_j}(g)(p) \end{aligned} \tag{2.16}$$

By construction $|\beta| \leq k$. Consequently, the expression $(*)$ certainly equals zero for all $\gamma < \beta$ by induction hypothesis. In case $|\beta| < k$, the entire equation (2.16) equals zero, which we wanted to show. In case $|\beta| = k$ ($\Leftrightarrow |\alpha| = k+1$), we may continue (2.16) by

$$= \xi^\beta iD^{e_j}(g)(p) = \xi^\beta\partial_j(g)(p) = \xi^\beta\xi_j = \xi^\alpha.$$

Step 2 (in case $M = U \subset \mathbb{R}^n$): We calculate

$$\begin{aligned} &\lim_{t\to\infty} t^{-k}\Big(e^{-itg}P(e^{itg}s)\Big)(p) = \lim_{t\to\infty} t^{-k}e^{-itg(p)}\Big(\sum_{|\alpha|\leq k}P_\alpha D^\alpha(e^{itg}s)\Big)(p) \\ &= \lim_{t\to\infty} t^{-k}e^{-itg(p)}\Big(\sum_{|\alpha|\leq k}P_\alpha(p)\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}D^\beta(e^{itg})(p)D^{\alpha-\beta}(s)(p)\Big) \\ &= \sum_{|\alpha|\leq k}P_\alpha(p)\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\Big(\lim_{t\to\infty} t^{-k}e^{-itg}D^\beta(e^{itg})\Big)(p)D^{\alpha-\beta}(s)(p)\Big) \\ &\overset{(2.15)}{=} \sum_{|\alpha|=k}P_\alpha(p)(e)\xi^\alpha = \sigma_P(p,\xi)e. \end{aligned}$$

Step 3 (general case): Let D be a local coordinate representation of P with respect to a chart $\varphi_U \to V$ and local trivializations Φ, Ψ. The function $\tilde{g} := g \circ \varphi^{-1} \in \mathcal{C}^\infty(V, \mathbb{R})$ satisfies

$$d\tilde{g}|_{\varphi(p)}d(g\circ\varphi^{-1})(\varphi(p)) = \varphi_*dg|_p = \varphi_*\xi$$

and the section $\tilde{s} := \varphi_*\Phi_*(s) \in \mathcal{C}^\infty(V, \mathbb{C}^r)$ satisfies $\tilde{s}(\varphi(p)) = \Phi_2(e)$. Therefore by what we have proven so far:

$$\sigma_D(\varphi(p), \varphi_*\xi)(\Phi_2(e)) = \lim_{t\to\infty} t^{-k}\Big(e^{-it\tilde{g}}D(e^{it\tilde{g}\tilde{s}})\Big)(\varphi(p)) \tag{2.17}$$

By definition of the symbol

$$\begin{aligned}\sigma_P(p,\xi)v &= \Psi^{-1}(p,\sigma_D(\varphi(p),\varphi_*\xi)(\Phi_2(v))) \overset{(2.17)}{=} \Psi^{-1}\Big(p, \lim_{t\to\infty} t^{-k}(e^{-it\tilde{g}}D(e^{it\tilde{g}}\tilde{s}))(\varphi(p))\Big) \\ &= \lim_{t\to\infty} \Psi^{-1}\Big(p, t^{-k}(e^{-it\tilde{g}}D(e^{it\tilde{g}}\tilde{s}))(\varphi(p))\Big) \\ &= \lim_{t\to\infty} t^{-k}e^{-itg(p)}((\varphi_*\Psi_*)^{-1}\circ D\circ\varphi_*\Phi_*)(e^{itg}s))(p) \\ &= \lim_{t\to\infty} t^{-k}\Big(e^{-itg}P(e^{itg}s)\Big)(p).\end{aligned}$$

□

2.2.21 Lemma (calculating symbols)**.** Let $P\in\mathrm{Diff}^k(M;E,F)$ be a PDO and let

$$P=\sum_{|\alpha|\le k} P_\alpha D^\alpha \in \mathrm{Diff}^k(U;E,F)$$

be a local representation of P as in (2.6). Then the symbol of P has a local representation

$$\sigma_P(\xi)=\sum_{|\alpha|=k} P_\alpha \xi_\varphi^\alpha, \tag{2.18}$$

where

$$\xi_\varphi^\alpha := ((\varphi_*(\xi))^\alpha.$$

Proof. Let $\varphi:U\to V$ be a chart and Φ, Ψ be local trivializations of E and F. It follows directly from the definitions that for any $p\in U$, $e\in E_p$, $\xi\in T_p^*U$

$$\sigma_{D^\alpha_{\varphi,\Phi}}(\xi)(e) \overset{2.2.15}{=} \Psi^{-1}(p,\sigma_{D^\alpha}(\varphi(p),\varphi_*\xi)(\Phi_2(e))) \overset{2.1.4}{=} \Psi^{-1}(p,(\varphi_*\xi)^\alpha(\Phi_2(e))) = \xi^\alpha_\varphi e \tag{2.19}$$

Now let $f\in\mathscr{C}^\infty(U)$ such that $df|_p=\xi$ and $s\in\Gamma_c(U,E)$ such that $s(p)=e$. By 2.2.19, we obtain

$$\begin{aligned}\sigma_P(\xi)(e) &\overset{(2.13)}{=} \frac{i^k}{k!}P(f^ks)(p) = \frac{i^k}{k!}\sum_{|\alpha|\le k} P_\alpha D^\alpha_{\varphi,\Phi}(f^ks)(p) \overset{2.2.19(i)}{=} \sum_{|\alpha|=k} P_\alpha \frac{i^k}{k!} D^\alpha_{\varphi,\Phi}(f^ks)(p) \\ &\overset{(2.13)}{=} \sum_{|\alpha|=k} P_\alpha \sigma_{D^\alpha_{\varphi,\Phi}}(\xi)(e) \overset{(2.19)}{=} \sum_{|\alpha|=k} P_\alpha \xi^\alpha_\varphi e.\end{aligned}$$

□

2.3. Properties of the PDO-Algebra

The set $\mathrm{Diff}(M;E,F)$ has much more hidden structure than just beeing a set. One of them is rather obvious.

2.3.1. Vector Space Structure

2.3.1 Definition (Linear combinations of PDOs)**.** Let $P,Q\in\mathrm{Diff}(M;E,F)$, $\lambda\in\mathbb{C}$, $f\in\mathcal{C}^\infty(M,\mathbb{C})$. We define $P+Q:\Gamma(M,E)\to\Gamma(M,F)$ by

$$(P+Q)(s) := P(s)+Q(s),$$

$\lambda P : \Gamma(M,E) \to \Gamma(M,E)$ by

$$(\lambda P)(s) := \lambda P(s),$$

and $fP : \Gamma(M,E) \to \Gamma(M,E)$ by

$$(fP)(s) := fP(s).$$

2.3.2 Theorem (Vector space structure)**.** Let $P \in \mathrm{Diff}^k(M;E,F)$, $Q \in \mathrm{Diff}^l(M;E,F)$, $f \in \mathcal{C}^\infty(M,\mathbb{C})$. Then

$$P + fQ \in \mathrm{Diff}^{\max(k,l)}(M;E,F)$$

and

$$\sigma_{P+fQ} = \sigma_P + f\sigma_Q.$$

Thus $\mathrm{Diff}(M;E,F)$ is a complex vector space and a module over $\mathcal{C}^\infty(M,\mathbb{C})$.

Proof. By setting $P_\alpha := 0$, if $|\alpha| > k$, and $Q_\alpha := 0$, if $|\alpha| > l$, the local representations of P, Q satisfy

$$\sum_{|\alpha|\leq k} P_\alpha D^\alpha + f \sum_{|\alpha|\leq k} Q_\alpha D^\alpha = \sum_{|\alpha|\leq \max(k,l)} (P_\alpha + fQ_\alpha) D^\alpha.$$

Now the statement follows from the local definition of the symbol. □

2.3.2. Sheaf Axioms

2.3.3 Definition (restriction)**.** Let $P \in \mathrm{Diff}^k(M;E,F)$ considered as a linear map

$$P : \Gamma_c(M,E) \to \Gamma_c(M,F).$$

Let $U \overset{\circ}{\subseteq} M$. Define

$$\begin{aligned} P|_U : \Gamma_c(U,E) &\to \Gamma_c(U,F) \\ s &\mapsto P(s), \end{aligned}$$

where $s \in \Gamma_c(U,E)$ is extended by zero to an element of $\Gamma_c(M,E)$. We say

$$P|_U \in \mathrm{Diff}^k(U;E,F)$$

is the *restriction of P to U*.

2.3.4 Theorem. $\mathrm{Diff}^k(\_;E,F)$ is a sheaf of $\mathbb{C}$-vector spaces on M, i.e. for any open cover $\{U_j\}_{j\in J}$ of M that is countable and locally finite, we obtain

(i) first sheaf axiom: Any $P \in \mathrm{Diff}(M;E,F)$ satisfies

$$\forall j \in J : P|_{U_j} = 0 \Longrightarrow P = 0.$$

(ii) second sheaf axiom: For any system $P_j \in \mathrm{Diff}(U_j;E,F)$

$$\forall j,k \in J : P_j|_{U_j\cap U_k} = P_k|_{U_j\cap U_k} \Longrightarrow \exists P \in \mathrm{Diff}(M;E,F) : \forall j \in J : P|_{U_j} = P_j.$$

Proof. Let $\{\psi_j\}_{j\in J}$ be a partition of unity subordinate to the cover $\{U_j\}_{j\in J}$, $p \in M$.

(i) Since the cover is locally finite there exists a finite subset $\tilde{J} \subset J$ such that $s \in \Gamma_c(M;E)$

$$P(s)|_p = P\Big(\sum_{j\in J} \psi_j s\Big)|_p = \sum_{j\in\tilde{J}} P|_{U_j}(\psi_j s)|_p = 0$$

(ii) Define

$$P(s) := \sum_{j \in J} P_j(\psi_j s),$$

which is well-defined since the sum is locally finite. Let $t \in \Gamma_c(U_j)$ and $K := \operatorname{supp} t$. Since K is compact, there exists a finite subset $\tilde{J} \subset J$ such that

$$K \subset \bigcup_{k \in \tilde{J}} U_k.$$

We calculate

$$\begin{aligned} P|_{U_j}(t) &= \sum_{k \in \tilde{J}} P_k(\psi_k t) = \sum_{k \in \tilde{J}} P_k|_{U_j \cap U_k}(\psi_k t) = \sum_{k \in \tilde{J}} P_j|_{U_j \cap U_k}(\psi_k t) \\ &= P_j\Big(\sum_{k \in \tilde{J}} \psi_k s\Big) = P_j(t). \end{aligned}$$

□

2.3.3. Compositions

It is very natural to ask, if the composition of two PDOs is again a PDO. The answer is always "yes" and locally, we even have explicit formulae for the symbol of the composition.

2.3.5 Theorem (Local compositions)**.** Let $U \subset \mathbb{R}^n$ be open and

$$P = \sum_{|\alpha| \le k} P_\alpha D^\alpha \in \operatorname{Diff}^k(U, \mathbb{C}^r, \mathbb{C}^s) \qquad \text{and} \qquad Q = \sum_{|\beta| \le l} Q_\beta D^\beta \in \operatorname{Diff}^l(U, \mathbb{C}^s, \mathbb{C}^t)$$

be two PDO with symbols

$$\Sigma_P(x,\xi) = p(x,\xi) = \sum_{|\alpha| \le k} P_\alpha(x)\xi^\alpha \qquad \text{and} \qquad \Sigma_Q(x,\xi) = q(x,\xi) = \sum_{|\beta| \le l} Q_\beta(x)\xi^\beta$$

Then the composition $Q \circ P \in \operatorname{Diff}^{k+l}(U, \mathbb{C}^r, \mathbb{C}^t)$ is a PDO with symbol

$$\Sigma_{Q \circ P}(x,\xi) = \sum_{|\gamma| \le l} \frac{(-i)^\gamma}{\gamma!} (\partial_\xi^\gamma \Sigma_Q)(x,\xi)(\partial_x^\gamma \Sigma_P)(x,\xi) = \sum_{|\gamma| \le l} \frac{i^\gamma}{\gamma!} (D_\xi^\gamma \Sigma_Q)(x,\xi)(D_x^\gamma \Sigma_P)(x,\xi)$$

and principal symbol

$$\sigma_{Q \circ P}(x,\xi) = \sum_{|\beta| = l} \sum_{|\alpha| = k} Q_\beta(x) P_\alpha(x) \xi^{\alpha+\beta} = \sigma_Q(x, \sigma_P(x,\xi)).$$

Proof. For any $f \in \mathcal{C}^\infty(U, \mathbb{C}^r)$

$$\begin{aligned} (Q \circ P)(f)(x) &= \sum_{|\beta| \le l} Q_\beta(x) D_x^\beta \left(\sum_{|\alpha| \le k} P_\alpha(x) D_x^\alpha f \right)(x) \\ &= \sum_{|\beta| \le l} \sum_{|\alpha| \le k} Q_\beta(x) D_x^\beta (P_\alpha D_x^\alpha f)(x) \\ &\overset{\text{A.1.5}}{=} \sum_{|\beta| \le l} \sum_{|\alpha| \le k} Q_\beta(x) \sum_{\gamma \le \beta} \binom{\beta}{\gamma} (D_x^\gamma P_\alpha)(x)(D_x^{\alpha+\beta-\gamma} f)(x) \\ &= \sum_{|\beta| \le l} \sum_{|\alpha| \le k} \sum_{\gamma \le \beta} \binom{\beta}{\gamma} Q_\beta(x)(D_x^\gamma P_\alpha)(x)(D_x^{\alpha+\beta-\gamma} f)(x). \end{aligned} \tag{2.20}$$

This implies $Q \circ P \in \mathrm{Diff}^{k+l}(U, \mathbb{C}^r, \mathbb{C}^t)$. Furthermoore (2.20) implies that the full symbol is given by

$$\begin{aligned}
\Sigma_{Q\circ P}(x,\xi) &= \sum_{|\beta|\leq l}\sum_{|\alpha|\leq k}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}Q_\beta(x)(D_x^\gamma P_\alpha)(x)\xi^{\alpha+\beta-\gamma} \\
&= \sum_{|\beta|\leq l}\sum_{|\gamma|\leq\beta}\binom{\beta}{\gamma}Q_\beta(x)\xi^{\beta-\gamma}D_x^\gamma\Big(\sum_{|\alpha|\leq k}P_\alpha(x)\xi^\alpha\Big) \\
&= \sum_{|\beta|\leq l}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\xi^{\beta-\gamma}Q_\beta(x)D_x^\gamma p(x,\xi) \overset{(1)}{=} \sum_{|\beta|\leq l}\sum_{\gamma\leq\beta}\frac{1}{\gamma!}(\partial_\xi^\gamma\xi^\beta)Q_\beta(x)D_x^\gamma p(x,\xi) \\
&\overset{(2)}{=} \sum_{|\beta|\leq l}\sum_{|\gamma|\leq l}\frac{1}{\gamma!}(\partial_\xi^\gamma\xi^\beta)Q_\beta(x)D_x^\gamma p(x,\xi) = \sum_{|\gamma|\leq l}\Big(\frac{1}{\gamma!}\partial_\xi^\gamma\Big(\sum_{|\beta|\leq l}Q_\beta(x)\xi^\beta\Big)D_x^\gamma p(x,\xi)\Big) \\
&= \sum_{|\gamma|\leq k}\frac{1}{\gamma!}(\partial_\xi^\gamma q)(x,\xi)D_x^\gamma p(x,\xi) = \sum_{|\gamma|\leq k}\frac{(-i)^\gamma}{\gamma!}(\partial_\xi^\gamma q)(x,\xi)\partial_x^\gamma p(x,\xi).
\end{aligned}$$

Remember from Lemma A.1.6, that for any two multi-indices β, γ, we have

$$\partial_\xi^\gamma\xi^\beta = \begin{cases}\gamma!\binom{\beta}{\gamma}\xi^{\beta-\gamma}, & \gamma\leq\beta, \\ 0, & \text{otherwise.}\end{cases}$$

This is the justification for (1) and also for (2) since we only added zero summands! Analysing the highest order terms in (2.20), we see that

$$\begin{aligned}
\sigma_{Q\circ P}(x,\xi) &= \sum_{|\beta|=l}\sum_{|\alpha|=k}Q_\beta(x)P_\alpha(x)\xi^{\alpha+\beta} \\
&= \sum_{|\beta|=l}Q_\beta(x)\xi^\beta\sum_{|\alpha|=k}P_\alpha(x)\xi^\alpha = \sigma_Q(x,\sigma_P(x,\xi)).
\end{aligned}$$

□

2.3.6 Theorem (Global composition)**.** Let $P \in \mathrm{Diff}^k(M;E,F)$ and $Q \in \mathrm{Diff}^l(M;F,G)$. Then $Q \circ P \in \mathrm{Diff}^{k+l}(M;E,G)$ and

$$\forall\xi\in T^*M : \sigma_{Q\circ P}(\xi) = \sigma_Q(\xi)\circ\sigma_P(\xi).$$

Proof. Clearly $Q\circ P : \Gamma(M,E)\to\Gamma(M,G)$ is a linear map. To check the PDO property choose an open set $U\subset M$ such that there exists a chart $\varphi : U\to V$ and local trivializations Φ, Ψ, Θ for E, F, G over U. Let $\tilde{P}$, $\tilde{Q}$ be local representations of P, Q. Then the equation

$$Q\circ P = (\varphi_*\Theta_*)^{-1}\circ\tilde{Q}\circ\varphi_*\Psi_*\circ(\varphi_*\Psi_*)^{-1}\circ\tilde{P}\circ\varphi_*\Psi_* = (\varphi_*\Theta_*)^{-1}\circ\tilde{Q}\circ\tilde{P}\circ\varphi_*\Psi_*$$

holds on U. This proves that $Q\circ P$ is a PDO with local representation $\tilde{Q}\circ\tilde{P}$. By definition the symbol satisfies for any $p\in M$, $\xi\in T_p^*M$, $e\in E_p$:

$$\begin{aligned}
\sigma_{Q\circ P}(\xi)(e) &= \Theta_2^{-1}\Big(p, \sigma_{\tilde{Q}\circ\tilde{P}}(\varphi(p),\varphi_*\xi)(\Phi_2(e))\Big) \\
&\overset{2.3.5}{=} \Theta_2^{-1}\Big(p, \sigma_{\tilde{Q}}(\varphi(p),\sigma_{\tilde{P}}(\varphi(p),\varphi_*\xi)(\Phi_2(e)))\Big) \\
&= \Theta_2^{-1}\Big(p, \sigma_{\tilde{Q}}(\varphi(p),\varphi_*\xi)(\Psi_2(\Psi^{-1}(p,\sigma_{\tilde{P}}(p,\varphi_*\xi)(\Phi_2(e)))))\Big) \\
&= \Theta_2^{-1}\Big(p, \sigma_{\tilde{Q}}(\varphi(p),\varphi_*\xi)(\Psi_2(\sigma_P(\xi)(e)))\Big) = \sigma_Q(\xi)\circ\sigma_P(\xi)(e).
\end{aligned}$$

□

2.3.4. Adjoints

In this subsection let (M, g) be a smooth oriented Riemannian m-manifold and let (E, h^E), (F, h^F) be hermitian vector bundles over M.

2.3.7 Convention. A PDO $P \in \operatorname{Diff}^k(M; E, F)$ can be thought of as a linear map

$$P : \Gamma(M, E) \to \Gamma(M, F), \qquad\qquad P : \Gamma_c(M, E) \to \Gamma_c(M, F).$$

In this section, we will always choose the later one. Nevertheless one may apply such an operator P to a section s that is not compactly supported.

2.3.8 Definition (induced L^2-space)**.** For any two sections $s, t \in \Gamma_c(M, E)$, define

$$\langle s, t\rangle_{L^2(h^E)} := \int_M h^E(s, t) d_g V \in \mathbb{C}\,,$$

the *L^2-scalar product on $\Gamma_c(M, E)$ induced by h^E*. This induces a norm via

$$\|s\|^2_{L^2(h^E)} := \langle s, s\rangle_{L^2(h^E)}.$$

We will sometimes drop the index and just write $\langle \_, \_\rangle$, $\|\_\|$.

2.3.9 Lemma. The L^2-scalar product is bilinear even over $\mathcal{C}^\infty(M, \mathbb{R})$ respectively sesquilinear over $\mathcal{C}^\infty(M, \mathbb{C})$.

2.3.10 Definition (formally adjoint)**.** Let $P \in \operatorname{Diff}^k(M; E, F)$ be a PDO. A linear map $Q : \Gamma_c(M, F) \to \Gamma_c(M, E)$ is *formally adjoint to P*, if

$$\forall s \in \Gamma_c(M; E) : \forall t \in \Gamma_c(M; F) : \langle P(s), t\rangle_{L^2(h^F)} = \langle s, Q(t)\rangle_{L^2(h^E)}.$$

2.3.11 Remark. We will later define the hilbert space $L^2(E)$ to be the completion of all sections $s \in \Gamma_c(M, E)$ such that $\|s\|_{L^2} < \infty$. The problem is that this space contains sections that are not differentiable and therefore P does not (yet) operate on this space. This is the reason why we speak of *formally adjoint* and why we can't just take the Hilbert space adjoint of P. Our ultimate goal is to show that there exists a unique formal adoint $Q = P^* \in \operatorname{Diff}^k(M; F, E)$.

The uniqueness is the much easier part.

2.3.12 Lemma (Uniqueness of formal adjoints)**.** Any formal adjoint to $P \in \operatorname{Diff}^k(M; E, F)$ is unique.

Proof. Assume Q, $\tilde{Q}$ are adjoint to P. Then for any $s \in \Gamma_c(M; E, F)$, $t \in \Gamma_c(M; F)$

$$\langle P(s), t\rangle = \langle s, Q(t)\rangle = \langle s, \tilde{Q}(t)\rangle$$

This implies

$$\langle s, (Q - \tilde{Q})(t)\rangle = 0,$$

thus $Q = \tilde{Q}$. □

2.3.13 Example (standard L^2 scalar product)**.** In case $M = U \subset \mathbb{R}^n$ is endowed with the Euclidean metric, $\mathbb{C}^r$ is given the the canonical hermitian form

$$\langle v, w\rangle := \sum_{i=1}^{r} v_i \bar{w}_i.$$

The induced L^2-scalar product on the trivial bundle $M \times \mathbb{C}^r$ is given by

$$\forall f, g \in \Gamma_c(M, M \times \mathbb{C}^r) = \mathscr{C}_c^\infty(U, \mathbb{C}^r) : \langle f, g\rangle_{L^2} := \langle f, g\rangle_{L^2(U)} := \int_U \langle f(x), g(x)\rangle dx.$$

2.3.14 Lemma (Euclidean adjoint operator)**.** In the situation of 2.3.13 above, let $P \in \mathrm{Diff}^k(U, \mathbb{C}^{r' \times r})$ be a PDO with full symbol

$$\Sigma_P(x, \xi) = \sum_{|\alpha| \leq k} P_\alpha(x)\xi^\alpha.$$

Then there exists a unique PDO $P^* \in \mathrm{Diff}^k(U, \mathbb{C}^{r \times r'})$, which is formally adjoint to P and which satisfies

$$\forall g \in \mathscr{C}^\infty(U, \mathbb{C}^{r'}) : P^*(g) = \sum_{|\alpha| \leq k} D^\alpha(P_\alpha^* g) \tag{2.21}$$

In particular, D^α is formally self-adjoint. The symbol is given by

$$\Sigma_{P^*}(x, \xi) = \sum_{|\beta| \leq k} \frac{i^\beta}{\beta!} D_x^\beta D_\xi^\beta \Sigma_P^*(x, \xi).$$

In particular

$$\sigma_{P^*}(x, \xi) = \sigma_P^*(x, \xi).$$

Proof. Uniqueness follows from Lemma 2.3.12.

Step 1 (Existence):

Step 1.1 ($P = D_j$): Let $1 \leq j \leq n$ and $P = D_j \in \mathrm{Diff}^1(U, \mathbb{C}^r, \mathbb{C}^r)$. We claim

$$\forall f \in \mathcal{C}_c^\infty(U, \mathbb{C}^r) : \forall g \in \mathcal{C}_c^\infty(U, \mathbb{C}^r) : \langle D_j(f), g\rangle_{L^2} = \langle f, D_j(g)\rangle_{L^2}, \tag{2.22}$$

hence the analogous statement is true for the operator $D^\alpha \in \mathrm{Diff}^k(U, \mathbb{C}^r, \mathbb{C}^r)$. To prove this consider any $f \in \mathcal{C}_c^\infty(U, \mathbb{C}^r)$, $g \in \mathcal{C}_c^\infty(U, \mathbb{C}^r)$ and consider

$$\begin{aligned}\langle D_j(f), g\rangle_{L_2} &= \int_U \langle D_j(f), g\rangle dx = \sum_{\nu=1}^{n} \frac{1}{i} \int_U \partial_j(f_\nu)\bar{g}_\nu dx = \sum_{\nu=1}^{n} \frac{-1}{i} \int_U f_\nu \partial_j(\bar{g}_\nu) dx \\ &= \sum_{\nu=1}^{n} \int_U f_\nu \overline{D_j g_\nu} dx = \langle f, D_j(g)\rangle_{L^2}\end{aligned}$$

Step 1.2 (general case): Now let $P \in \mathrm{Diff}^k(U, \mathbb{C}^r, \mathbb{C}^{r'})$ be arbitrary. For any $f \in \mathcal{C}_c^\infty(U, \mathbb{C}^r)$ and any $g \in \mathcal{C}_c^\infty(U, \mathbb{C}^{r'})$ we calculate

$$\begin{aligned}\langle P(f), g\rangle_{L^2} &= \int_{\mathbb{R}^n} \sum_{|\alpha| \leq k} \langle P_\alpha D^\alpha(f), g\rangle dx = \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} \langle D^\alpha(f), P_\alpha^* g\rangle dx \\ &\overset{(2.22)}{=} \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} \langle f, D^\alpha(P_\alpha^* g)\rangle dx = \int_{\mathbb{R}^n} \langle f, \underbrace{\sum_{|\alpha| \leq k} D^\alpha(P_\alpha^* g)}_{=:P^*(g)}\rangle dx = \langle f, P^*(g)\rangle_{L^2}\end{aligned}$$

Now, the Leibniz rule implies

$$P^*(g) = \sum_{|\alpha|\leq k} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta(P^*_\alpha) D^{\alpha-\beta}(g), \tag{2.23}$$

thus $P^* \in \mathrm{Diff}^k(U, \mathbb{C}^{r\times r'})$.

Step 2 (full symbol): This allows us to calculate the full symbol by Consequently

$$\begin{aligned}\Sigma_{P^*}(x,\xi) &\overset{(2.23)}{=} \sum_{|\alpha|\leq k} \sum_{\beta\leq a} D^\beta_x(P^*_\alpha)(x) \binom{\alpha}{\beta} \xi^{\alpha-\beta} \overset{(1)}{=} \sum_{|\alpha|\leq k} \sum_{\beta\leq\alpha} D^\beta_x(P^*_\alpha)(x) \frac{1}{\beta!} \partial^\beta_\xi \xi^\alpha \\ &\overset{(2)}{=} \sum_{|\alpha|\leq k} \sum_{|\beta|\leq k} \frac{1}{\beta!} D^\beta_x(P^*_\alpha)(x) \partial^\beta_\xi \xi^\alpha = \sum_{|\beta|\leq k} \frac{1}{\beta!} D^\beta_x \partial^\beta_\xi \Big(\sum_{|\alpha|\leq k} P^*_\alpha \xi^\alpha \Big) \\ &= \sum_{|\beta|\leq k} \frac{i^\beta}{\beta!} D^\beta_x D^\beta_\xi \Sigma^*_P(x,\xi).\end{aligned}$$

(1),(2): Remember from Lemma A.1.6 that

$$\partial^\beta_\xi \xi^\alpha = \begin{cases} \beta! \binom{\alpha}{\beta} \xi^{\alpha-\beta}, & \beta \leq \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

This justifies (1) and it also justifies (2), since we only added zero summands.

Step 3 (principal symbol): From (2.23) we conclude directly

$$\sigma_{P^*}(x,\xi) = \sum_{|\alpha|\leq k} P^*_\alpha \xi^\alpha = \sigma^*_P(x,\xi).$$

□

2.3.15 Theorem (global adjoints)**.** For any PDO $P \in \mathrm{Diff}^k(M; E, F)$ there exists a unique $P^* \in \mathrm{Diff}^k(M; F, E)$, which is formally adjoint to P. If P has a local representation

$$P = \sum_{|\alpha|\leq k} P_\alpha D^\alpha$$

as in 2.2.12(ii), then

$$\forall t \in \Gamma_c(M, F) : P^* = \frac{1}{\sqrt{g}} \sum_{|\alpha|\leq k} D^\alpha(\sqrt{g} P^*_\alpha t), \tag{2.24}$$

where $\sqrt{g} := \sqrt{\det(g_{ij})} \neq 0$ is the Riemannian volume function. The principal symbols satisfy

$$\forall \xi \in T^*_p M : \sigma_{P^*}(\xi) = \sigma_P(\xi)^*.$$

Proof. Uniqueness is clear. We will establish a local version of this result first. So assume $U \overset{\circ}{\subseteq} M$ such that $\varphi : U \to V$ is a chart and $E_1, \ldots, E_r$, $F_1, \ldots, F_s$ are local orthonormal frames of E and F. Let Φ, Ψ be the associated local trivializations as in 2.2.8.

Step 1 ($D^\alpha_{\varphi,\Phi}$): The local trivializations map the ONB $E_i \in E_p$ to the ONB $e_i \in \mathbb{C}^r$, therefore for any $s, t \in \Gamma_c(U, E)$

$$\begin{aligned} h_E(D^\alpha_{\varphi,\Phi}(s), t) &= \sum_{\mu,\nu=1}^{r} D^\alpha_\varphi(s^\mu) t^\nu h_E(E_\mu, E_\nu) = \sum_{\mu,\nu=1}^{r} D^\alpha_\varphi(s^\mu) t^\nu h_E(e_\mu, e_\nu) \\ &= \sum_{\mu,\nu=1}^{r} \langle D^\alpha_\varphi(s^\mu) e_\mu, t^\nu e_\nu \rangle_{\mathbb{C}^r} = \left\langle \sum_{\mu=1}^{r} D^\alpha_\varphi(s^\mu) e_\mu, \sum_{\nu=1}^{r} t^\nu e_\nu \right\rangle_{\mathbb{C}^r} \end{aligned} \tag{2.25}$$

Define and calculate

$$\begin{aligned} \langle D^\alpha_{\varphi,\Phi}(s), t \rangle_{L^2(h^E)} &= \int_U h_E(D^\alpha_{\varphi,\Phi}(s), t) d_g V = \int_V h_E(D^\alpha_{\varphi,\Phi}(s), t) \circ \varphi^{-1} \sqrt{g} \circ \varphi^{-1} dx \\ &\overset{(2.25)}{=} \sum_{\mu,\nu=1}^{r} \int_V \langle D^\alpha_\varphi(s^\mu), t^\nu e_\nu \rangle_{\mathbb{C}^r} \circ \varphi^{-1} \sqrt{g} \circ \varphi^{-1} dx \\ &= \sum_{\mu,\nu=1}^{r} \int_V \langle D^\alpha(s^\mu \circ \varphi^{-1} e_\mu), (t^\nu \sqrt{g} e_\nu) \circ \varphi^{-1} \rangle_{\mathbb{C}^r} dx \\ &\overset{2.3.14}{=} \sum_{\mu,\nu=1}^{r} \int_V \langle s^\mu \circ \varphi^{-1} e_\mu, D^\alpha((t^\nu \sqrt{g} e_\nu) \circ \varphi^{-1}) \rangle_{\mathbb{C}^r} dx \\ &= \sum_{\mu,\nu=1}^{r} \int_U \langle s^\mu e_\mu, \frac{1}{\sqrt{g}} D^\alpha_\varphi((t^\nu \sqrt{g} e_\nu)) \rangle_{\mathbb{C}^r} d_g V \\ &\overset{(2.25)}{=} \int_U h_e(s, \frac{1}{\sqrt{g}} D^\alpha_{\varphi,\Phi}((t\sqrt{g}))) d_g V \\ &= \langle s, \frac{1}{\sqrt{g}} D^\alpha_{\varphi,\Phi}((t\sqrt{g})) \rangle_{L^2(h^E)} \end{aligned} \tag{2.26}$$

Step 2 (local adjoint):

$$\begin{aligned} \langle P(s), t \rangle_{L^2(h^F)} &\overset{(2.7)}{=} \sum_{|\alpha| \le k} \int_U h_F(P_\alpha D^\alpha_{\varphi,\Phi}(s), t) d_g V = \sum_{|\alpha| \le k} \int_U h_E(D^\alpha_{\varphi,\Phi}(s), P^*_\alpha(t)) d_g V \\ &\overset{(2.26)}{=} \sum_{|\alpha| \le k} \int_U h_E(s, D^\alpha_{\varphi,\Phi}(P^*_\alpha(t))) d_g V = \sum_{|\alpha| \le k} \langle s, \frac{1}{\sqrt{g}} D^\alpha_{\varphi,\Phi}(P^*_\alpha t \sqrt{g}) \rangle_{L^2(h^E)}, \end{aligned}$$

wich proves (2.24). Since for any $t \in \Gamma_c(U, F)$

$$P^*(t) = \frac{1}{\sqrt{g}} \sum_{|\alpha| \le k} D^\alpha_{\varphi,\Phi}(\sqrt{g} P^*_\alpha t) \overset{(2.9)}{=} \sum_{|\alpha| \le k} \sum_{\beta \le \alpha} \binom{\beta}{\alpha} \frac{1}{\sqrt{g}} D^\beta_{\varphi,\Phi,\Psi}(\sqrt{g} P^*_\alpha) D^{\alpha-\beta}_{\varphi,\Psi}(t),$$

we obtain on the one hand that $P^* \in \mathrm{Diff}^k(U; F, E)$ is the adjoint of $P \in \mathrm{Diff}^k(U; E, F)$. By analyzing the highest order terms, we conclude on the other hand that for any $\xi \in T^*U$

$$\sigma_{P^*}(\xi) \overset{2.2.21}{=} \sum_{|\alpha| = k} \binom{0}{\alpha} \frac{1}{\sqrt{g}} \sqrt{g} P^*_\alpha \xi^\alpha_\varphi = \sum_{|\alpha| = k} P^*_\alpha \xi^\alpha_\varphi \overset{2.2.21}{=} \sigma_P(\xi)^*.$$

Step 3 (global result): Now take any open cover $\{U_j\}_{j \in J}$ of M that is countable, locally finite and such that any U_j satisfies the hypothesis of the previous step. By what we have proven so far, for any $j \in J$ there exists an operator $P_j^* \in \operatorname{Diff}^k(U_j; F, E)$ that is formally adjoint to $P|_{U_j}$. Since formal adjoints are unique by 2.3.12, we obtain for any $k \in J$

$$P_j^*|_{U_j \cap U_k} = (P|_{U_j \cap U_k})^* = P_k^*|_{U_j \cap U_k}. \tag{2.27}$$

Consequently, by 2.3.4, there exists a unique $P^* \in \operatorname{Diff}^k(M; E, F)$ such that $P^*|_{U_j} = P_j^*$. Let $\{\psi_j\}_{j \in J}$ be a partition of unity subordinate to the cover $\{U_j\}_{j \in J}$ and let $\chi_j \in \mathscr{C}_c^\infty(U_j, \mathbb{R})$ be a smooth bump function that equals 1 in a neighbourhood of $\operatorname{supp} \psi_j$. We claim that

$$P^*(s) = \sum_{j \in J} P_j^*(\psi_j s)$$

is indeed the global adjoint of P. To see this let $s \in \Gamma_c(M, E)$, $t \in \Gamma_c(M, F)$, notice that

$$\psi_j P(s) = \begin{cases} 0, & \text{on } \operatorname{supp} \psi_j, \\ \psi_j P(1 \cdot s), & \text{oustide } \operatorname{supp} \psi_j \end{cases} = \psi_j P(\chi_j s), \tag{2.28}$$

$$\chi_j P_j^*(\psi_j t) = P_j^*(\psi_j t) \tag{2.29}$$

and calculate

$$\begin{aligned} \langle P(s), t \rangle_{L^2} &= \Big\langle \sum_{j \in J} \psi_j P(s), t \Big\rangle_{L^2} \overset{(2.28)}{=} \sum_{j \in J} \langle \psi_j P(\chi_j s), t \rangle_{L^2} = \sum_{j \in J} \langle P|_{U_i}(\chi_j s), \psi_j t \rangle_{L^2} \\ &= \sum_{j \in J} \langle \chi_j s, P_j^*(\psi_j t) \rangle_{L^2} = \sum_{j \in J} \langle s, \chi_j P_j^*(\psi_j t) \rangle_{L^2} \overset{(2.29)}{=} \sum_{j \in J} \langle s, P_j^*(\psi_j t) \rangle_{L^2} \\ &\overset{(2.27)}{=} \langle s, P^*(t) \rangle_{L^2}. \end{aligned}$$

□

2.3.16 Corollary. Let $P \in \operatorname{Diff}^k(M; E, F)$ and $P^* \in \operatorname{Diff}^k(M; F, E)$ be formally adjoint to P.

(i) $P^{**} = P$.

(ii) If either $s \in \Gamma_c(M; E, F)$ and $t \in \Gamma(M; F, E)$ or $s \in \Gamma(M; E, F)$ and $t \in \Gamma_c(M; F, E)$

$$\langle P(s), t \rangle_{L^2(h^F)} = \langle s, P^*(t) \rangle_{L^2(h^E)}. \tag{2.30}$$

Proof.

(i) By 2.3.12 formal adjoints are unique. Consequently, it suffices to check that for any $s \in \Gamma_c(M, E)$, $t \in \Gamma_c(M, F)$

$$\langle P^*(t), s \rangle_{L^2} = \overline{\langle s, P^*(t) \rangle}_{L^2} = \overline{\langle P(s), t \rangle}_{L^2} = \langle t, P(s) \rangle_{L^2}.$$

(ii) By definition (2.30) holds, if both sections are compactly supported. In the first case let $K := \operatorname{supp} s \Subset M$ and let $\chi \in \mathscr{C}_c^\infty(M, \mathbb{R})$ be a smooth cutoff function, i.e. $\chi \equiv 1$ in an open neighbourhood $U \Supset K$. Clearly $t\chi \in \Gamma_c(M, F)$. Using the facts that

$$\operatorname{supp} P(s) \subset K, \qquad \operatorname{supp} P^*(\chi t) \subset U,$$

we obtain:

$$\begin{aligned}\langle P(s),t\rangle_{L^2} &= \int_M h_F(P(s),t)d_gV = \int_K \chi h_F(P(s),t)d_gV = \int_M h_F(P(s),\chi t)d_gV \\ &= \langle s, P^*(\chi t)\rangle_{L^2} = \int_M h_E(s,P^*(\chi t))d_gV = \int_U h_E(s,P^*(t))d_gV \\ &= \int_M h_E(s,P^*(t))d_gV = \langle s,P^*(t)\rangle_{L^2}.\end{aligned}$$

The other claim follows from (i). □

2.4. Ellipticity

2.4.1 Definition (elliptic PDO)**.** An operator $P \in \mathrm{Diff}^k(M;E,F)$ is called *elliptic*, if its symbol is invertible outside the zero section, i.e.

$$\forall p \in M : \forall 0 \neq \xi \in T_p^*M : \sigma_P(\xi) \in \mathrm{Iso}(E_p, F_p).$$

2.4.2 Lemma. If there exists an elliptic operator $P \in \mathrm{Diff}^k(M;E,F)$, then

$$\mathrm{rg}\, E = \mathrm{rg}\, F.$$

Proof. This follows directly from the definitions. □

2.4.1. The Hodge Laplacian

We would like to discuss a famous example of an elliptic operator, namely the Hodge Laplacian. We assume some basic familiarity with the de Rham complex and the Hodge operator in this section. From now on M is a smooth oriented Riemannian manifold without boundary.

2.4.3 Definition (complexified de Rham complex)**.** For any $0 \leq k \leq m$ the bundle

$$\Lambda^k_{\mathbb{C}} T^*M := \Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C}$$

is the *complexified exterior algebra of order k*. Its sections are denoted by

$$\Omega^k_{\mathbb{C}}(M) := \Gamma_c(\Lambda^k_{\mathbb{C}} T^*M).$$

Notice that any complex valued differential form $\omega \in \Omega^k_{\mathbb{C}}(M)$ can be decomposed into

$$\forall p \in M : \omega(p) = \omega_1(p) \otimes 1 + \omega_2(p) \otimes i =: \omega_1(p) + i\omega_2(p).$$

In particular, we have a complex exterior differential $d : {\Omega_{\mathbb{C}}}^k(M) \to {\Omega_{\mathbb{C}}}^{k+1}(M)$ by defining

$$d(\omega) := d(\omega_1) + id(\omega_2).$$

2.4.4 Definition (hermitian metric)**.** We assume that the Riemannian metric g on M is canonically extended to a fibre metric on $\Lambda^k T^*M$. Now we extend it further by complexification to a metric $g = g^{\mathbb{C}}$, defined by

$$g^{\mathbb{C}}(\omega_1 + i\omega_2, \eta_1 + i\eta_2) := g(\omega_1,\eta_1) + g(\omega_2,\eta_2) + i(g(\eta_2,\omega_1) - g(\eta_1,\omega_2))$$

We assume that $\Lambda^k_{\mathbb{C}} T^*M$ is endowed with this metric.

2.4.5 Definition (hermitian form on de Rham complex)**.** For any two differential forms $\omega, \eta \in \Omega^k_{\mathbb{C}}(M)$ define

$$\langle \omega, \eta \rangle_{L^2} := \int_M \omega \wedge \overline{*\eta},$$

where $*$ is the Hodge operator. This defines a hermitian scalar product on $\Omega^k_{\mathbb{C}}(M)$.

2.4.6 Theorem (duality of int and ext)**.** For any $\xi \in \Omega^*(M)$ define the *exterior multiplication*

$$\begin{aligned} \operatorname{ext}_\xi : \Omega^k(M) &\to \Omega^{k+1}(M) \\ \omega &\mapsto \xi \wedge \omega \end{aligned}$$

and for any $X \in T(M)$ define the *interior multiplication*

$$\begin{aligned} \operatorname{int}_X : \Omega^{k+1}(M) &\to \Omega^k(M) \\ \omega &\mapsto \omega(X, \_) \end{aligned}$$

(i) For any $X \in T(M)$, $\xi \in T^*(M)$

$$\operatorname{int}_X^2 = 0, \qquad \operatorname{ext}_\xi^2 = 0.$$

(ii) On any Riemannian manifold (M, g)

$$\operatorname{ext}_\xi^* = \operatorname{int}_{\xi^\flat}, \tag{2.31}$$

where the adjoint is taken with respect to the canonical fibre metric in $\Lambda^k T^*M$.

(iii) Furthermoore

$$\operatorname{ext}_\xi \circ \operatorname{int}_{\xi^\flat} + \operatorname{int}_{\xi^\flat} \circ \operatorname{ext}_\xi = \|\xi\|^2 \operatorname{id}.$$

Proof.

(i) This is clear.

(ii) We prove this locally and choose a local ONF $E_1, \ldots, E_m$ of TU. Hence $E^1, \ldots, E^m$ is a local ONF of T^*U and

$$\{E^{i_1} \wedge \ldots \wedge E^{i_k} \mid 1 \leq i_1 < \ldots < i_k \leq m\}$$

is a local ONF for $\Lambda^k T^*M$. Now choose any $1 \leq \mu \leq m$, $i_1 < \ldots < i_k$, $j_1 < \ldots < j_{k+1}$. We calculate on the one hand

$$\begin{aligned} \langle \operatorname{ext}_{E^\mu}(E^{i_1} \wedge \ldots \wedge E^{i_k}), &E^{j_1} \wedge \ldots \wedge E^{j_{k+1}} \rangle \\ &= \langle E^\mu \wedge E^{i_1} \wedge \ldots \wedge E^{i_k}, E^{j_1} \wedge \ldots \wedge E^{j_{k+1}} \rangle \\ &= \delta^{\mu, i_1, \ldots, i_k}_{j_1, \ldots, j_{k+1}} \end{aligned} \tag{2.32}$$

To calculate $\operatorname{int}_{E^\mu}$, notice the following useful formula

$$\begin{aligned} \operatorname{int}_{E_\mu}(E^{j_1} \wedge \ldots \wedge E^{j_{k+1}}) &= \sum_{\nu=1}^{k+1} (-1)^{\nu-1} E^{j_\nu}(E_\mu) E^{j_1} \wedge \ldots \widehat{E^{j_\nu}} \ldots \wedge E^{j_{k+1}} \\ &= \begin{cases} (-1)^{\nu-1} E^{j_1} \wedge \ldots \widehat{E^\mu} \ldots \wedge E^{j_{k+1}}, & \exists \nu : \mu = j_\nu, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{2.33}$$

Therefore, we distinguish two cases: If there exists $1 \leq \nu \leq m$ such that $\mu = j_\nu$, we calculate

$$\begin{aligned}
&\langle E^{i_1} \wedge \dots E^{i_k}, \mathrm{int}_{E^\mu}(E^{j_1} \wedge \dots \wedge E^{j_{k+1}})\rangle \\
&\overset{(2.33)}{=} \langle E^{i_1} \wedge \dots E^{i_k}, (-1)^{\nu-1} E^{j_1} \wedge \dots \widehat{E^{j_\nu}} \wedge \dots \wedge E^{j_{k+1}})\rangle \\
&= (-1)^{\nu-1} \delta^{i_1,\dots,i_k}_{j_1,\dots,\widehat{j_\nu},\dots,j_{k+1}} = (-1)^{\nu-1} \delta^{\mu,i_1,\dots,i_k}_{j_\nu,j_1,\dots,\widehat{j_\nu},\dots,j_{k+1}} = \delta^{\mu,i_1,\dots,i_k}_{j_1,\dots,j_\nu,\dots,j_{k+1}} \\
&\overset{(2.32)}{=} \langle \mathrm{ext}_{E^\mu}(E^{i_1} \wedge \dots \wedge E^{i_k}), E^{j_1} \wedge \dots \wedge E^{j_{k+1}}\rangle.
\end{aligned}$$

In case $\mu \notin \{j_1, \dots, j_{k+1}\}$, we obtain

$$\begin{aligned}
\langle \mathrm{ext}_{E^\mu}(E^{i_1} \wedge \dots \wedge E^{i_k}), E^{j_1} \wedge \dots \wedge E^{j_{k+1}}\rangle &\overset{(2.32)}{=} 0 \\
&\overset{(2.33)}{=} \langle E^{i_1} \wedge \dots E^{i_k}, \mathrm{int}_{E^\mu}(E^{j_1} \wedge \dots \wedge E^{j_{k+1}})\rangle.
\end{aligned}$$

Since $(E^\mu)^\flat = E_\mu$ and both sides of (2.31) are linear in all arguments, the claim follows.

(iii) Both sides are linear in the argument, so let $E^{i_1} \wedge \dots \wedge E^{i_k}$, $\xi \in TU$, be arbitrary. We calculate for any $1 \leq \nu, \mu \leq m$

$$\begin{aligned}
\mathrm{int}_{E_\mu}(\,\mathrm{ext}_{E^\nu}(E^{i_1} \wedge \dots \wedge E^{i_k}) &= \begin{cases} \mathrm{int}_{E_\mu}(E^\nu \wedge E^{i_1} \wedge \dots \wedge E^{i_k}), & \nu \notin \{i_1, \dots, i_k\} \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} E^{i_1} \wedge \dots \wedge E^{i_k}, & \nu \notin \{i_1, \dots, i_k\}, \nu = \mu \\ (-1)^r E^\nu \wedge E^{i_1} \wedge \dots \widehat{E^{i_r}} \dots \wedge E^{i_k}, & \nu \notin \{i_1, \dots, i_k\}, \mu = i_r \\ 0, & \text{otherwise} \end{cases}
\end{aligned} \tag{2.34}$$

$$\begin{aligned}
\mathrm{ext}_{E^\mu}(\,\mathrm{int}_{E_\nu}(E^{i_1} \wedge \dots \wedge E^{i_k})) &= \begin{cases} \mathrm{ext}_{E^\mu}((-1)^{r-1} E^{i_1} \wedge \dots \widehat{E^{i_r}} \dots \wedge E^{i_k}), & i_r = \nu, \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} E^{i_1} \wedge \dots \wedge E^{i_k}, & i_r = \nu = \mu, \\ (-1)^{r-1} E^\mu \wedge E^{i_1} \wedge \dots \widehat{E^{i_r}} \dots \wedge E^{i_k}, & i_r = \nu, \mu \notin \{i_1, \dots, i_k\} \\ 0, & \text{otherwise} \end{cases}
\end{aligned} \tag{2.35}$$

Now we distinguish several cases, in which we calculate using (2.34) and (2.35).

Case 1 ($\nu = \mu$): Now exactly two subcases may occur.

Case 1.1 ($\nu \in \{i_1, \dots, i_k\}$): We calculate

$$\begin{aligned}
&(\mathrm{int}_{E_\mu} \circ \mathrm{ext}_{E^\nu} + \mathrm{ext}_{E^\mu} \circ \mathrm{int}_{E^\nu})(E^{i_1} \wedge \dots \wedge E^{i_k}) \\
&\quad = (\mathrm{ext}_{E^\mu} \circ \mathrm{int}_{E^\nu})(E^{i_1} \wedge \dots \wedge E^{i_k}) \\
&\quad = E^{i_1} \wedge \dots \wedge E^{i_k}
\end{aligned}$$

Case 1.2 ($\nu \notin \{i_1, \dots, i_k\}$):

$$\begin{aligned}
&(\mathrm{int}_{E_\mu} \circ \mathrm{ext}_{E^\nu} + \mathrm{ext}_{E^\mu} \circ \mathrm{int}_{E^\nu})(E^{i_1} \wedge \dots \wedge E^{i_k}) \\
&\quad = (\mathrm{int}_{E_\mu} \circ \mathrm{ext}_{E^\nu})(E^{i_1} \wedge \dots \wedge E^{i_k}) \\
&\quad = E^{i_1} \wedge \dots \wedge E^{i_k}
\end{aligned}$$

In both subcases we obtain

$$(\mathrm{int}_{E_\mu} \circ \mathrm{ext}_{E^\mu} + \mathrm{ext}_{E^\mu} \circ \mathrm{int}_{E^\mu})(E^{i_1} \wedge \ldots \wedge E^{i_k}) = E^{i_1} \wedge \ldots \wedge E^{i_k} \tag{2.36}$$

CASE 2 ($\nu \neq \mu$): Several subcases occur.
CASE 2.1 ($\nu \in \{i_1, \ldots, i_k\}$, $\mu \in \{i_1, \ldots, i_k\}$):

$$\begin{aligned}(\mathrm{int}_{E_\mu} \circ \mathrm{ext}_{E^\nu} + \mathrm{ext}_{E^\mu} \circ \mathrm{int}_{E^\nu})&(E^{i_1} \wedge \ldots \wedge E^{i_k})\\ &= (\mathrm{ext}_{E^\mu} \circ \mathrm{int}_{E^\nu})(E^{i_1} \wedge \ldots \wedge E^{i_k})\\ &= 0\end{aligned}$$

CASE 2.2 ($\nu \in \{i_1, \ldots, i_k\}$, $\mu \notin \{i_1, \ldots, i_k\}$):

$$\begin{aligned}(\mathrm{int}_{E_\mu} \circ \mathrm{ext}_{E^\nu} + \mathrm{ext}_{E^\mu} \circ \mathrm{int}_{E^\nu})&(E^{i_1} \wedge \ldots \wedge E^{i_k})\\ &= (\mathrm{ext}_{E^\mu} \circ \mathrm{int}_{E^\nu})(E^{i_1} \wedge \ldots \wedge E^{i_k})\\ &= (-1)^{r-1} E^\mu \wedge E^{i_1} \wedge \ldots \widehat{E^{i_r}} \ldots \wedge E^{i_k}\end{aligned}$$

We will write $r = r(\nu)$ in order to stress that r depends on ν.
CASE 2.3 ($\nu \notin \{i_1, \ldots, i_k\}$, $\mu \in \{i_1, \ldots, i_k\}$):

$$\begin{aligned}(\mathrm{int}_{E_\mu} \circ \mathrm{ext}_{E^\nu} + \mathrm{ext}_{E^\mu} \circ \mathrm{int}_{E^\nu})&(E^{i_1} \wedge \ldots \wedge E^{i_k})\\ &= (\mathrm{int}_{E_\mu} \circ \mathrm{ext}_{E^\nu})(E^{i_1} \wedge \ldots \wedge E^{i_k})\\ &= (-1)^{r} E^\nu \wedge E^{i_1} \wedge \ldots \widehat{E^{i_r}} \ldots \wedge E^{i_k}\end{aligned}$$

We will write $r = r(\mu)$ in order to stress that r depends on μ.
CASE 2.4 ($\nu \notin \{i_1, \ldots, i_k\}$, $\mu \notin \{i_1, \ldots, i_k\}$):

$$\begin{aligned}(\mathrm{int}_{E_\mu} \circ \mathrm{ext}_{E^\nu} + \mathrm{ext}_{E^\mu} \circ \mathrm{int}_{E^\nu})&(E^{i_1} \wedge \ldots \wedge E^{i_k})\\ &= (\mathrm{int}_{E_\mu} \circ \mathrm{ext}_{E^\nu})(E^{i_1} \wedge \ldots \wedge E^{i_k})\\ &= 0\end{aligned}$$

Combining all these subcases, we obtain

$$\begin{aligned}&\sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{m} \xi_\mu \xi_\nu (\mathrm{int}_{E_\mu} \circ \mathrm{ext}_{E^\nu} + \mathrm{ext}_{E^\mu} \circ \mathrm{int}_{E^\nu})(E^{i_1} \wedge \ldots \wedge E^{i_k})\\
&= \sum_{\substack{\nu\in\{i_1,\ldots,i_k\}\\ \mu\notin\{i_1,\ldots,i_k\}}} \xi_\mu \xi_\nu (-1)^{r(\nu)-1} E^\mu \wedge E^{i_1} \wedge \ldots \widehat{E^{i_{r(\nu)}}} \wedge \ldots E^{i_k}\\
&+ \sum_{\substack{\nu\notin\{i_1,\ldots,i_k\}\\ \mu\in\{i_1,\ldots,i_k\}}} \xi_\mu \xi_\nu (-1)^{r(\mu)} E^\nu \wedge E^{i_1} \wedge \ldots \widehat{E^{i_{r(\mu)}}} \wedge \ldots E^{i_k}\\
&= \sum_{\substack{\nu\in\{i_1,\ldots,i_k\}\\ \mu\notin\{i_1,\ldots,i_k\}}} \xi_\mu \xi_\nu (-1)^{r(\nu)-1} E^\mu \wedge E^{i_1} \wedge \ldots \widehat{E^{i_{r(\nu)}}} \wedge \ldots E^{i_k}\\
&+ \sum_{\substack{\nu\in\{i_1,\ldots,i_k\}\\ \mu\notin\{i_1,\ldots,i_k\}}} \xi_\nu \xi_\nu (-1)^{r(\nu)} E^\mu \wedge E^{i_1} \wedge \ldots \widehat{E^{i_{r(\nu)}}} \wedge \ldots E^{i_k}\\
&= 0\end{aligned} \tag{2.37}$$

Combining both cases, we obtain

$$\begin{aligned}
(\operatorname{ext}_\xi \circ \operatorname{int}_{\xi^\flat} + \operatorname{int}_{\xi^\flat} \circ \operatorname{ext}_\xi)(E^{i_1} \wedge \ldots \wedge E^{i_k}) \\
&= \sum_{\mu,\nu=1}^m \xi_\nu \xi_\mu (\operatorname{int}_{E_\mu} \circ \operatorname{ext}_{E^\nu} + \operatorname{ext}_{E^\mu} \circ \operatorname{int}_{E^\nu})(E^{i_1} \wedge \ldots \wedge E^{i_k}) \\
&\overset{(2.37)}{=} \sum_{\mu=1}^m \xi_\mu^2 (\operatorname{int}_{E_\mu} \circ \operatorname{ext}_{E^\mu} + \operatorname{ext}_{E^\mu} \circ \operatorname{int}_{E^\mu})(E^{i_1} \wedge \ldots \wedge E^{i_k}) \\
&\overset{(2.36)}{=} \sum_{\mu=1}^m \xi_\mu^2 E^{i_1} \wedge \ldots \wedge E^{i_k} = \|\xi\|^2 E^{i_1} \wedge \ldots \wedge E^{i_k}.
\end{aligned}$$

□

2.4.7 Theorem (de Rahm complex)**.** Let $(\Omega^k_{\mathbb{C}}(M), d)$ be the complexified de Rahm complex over a Riemannian manifold (M, g).

(i) The exterior derivative d satisfies $d \in \operatorname{Diff}^1(M; \Omega^k_{\mathbb{C}} M, \Omega^{k+1}_{\mathbb{C}} M)$ and

$$\forall \omega \in \Omega^k_{\mathbb{C}}(M) : \sigma_d(\xi)(\omega|_p) = i \operatorname{ext}_\xi(\omega)|_p.$$

Clearly d is not elliptic. Locally d is given on functions by

$$\forall f \in \mathcal{C}^\infty(U, \mathbb{C}) : d(f) = \sum_{j=1}^m \partial\varphi_j(f) d\varphi^j.$$

(ii) Let $d^* = (-1)^{m(k+1)+1} * d* : \Omega^{k+1}_{\mathbb{C}}(M) \to \Omega^k_{\mathbb{C}}(M)$ be the codifferential. Then $d^* \in \operatorname{Diff}^1(M; \Omega^{k+1}_{\mathbb{C}} M, \Omega^k_{\mathbb{C}} M)$ is the adjoint of d and its symbol is given by

$$\sigma_{d^*}(\xi)(\omega|_p) = -i \operatorname{int}_\xi(\omega|_p).$$

Clearly d^* is not elliptic. Locally d^* is given on functions by

$$\forall \omega = \sum_{j=1}^m \omega_j d\varphi^j \in \Omega^1_{\mathbb{C}} : (U) : d^*(\omega) = \frac{1}{\sqrt{g}} \sum_{j,k=1}^m \partial\varphi_j(\sqrt{g} g^{kj} \omega_k)$$

(iii) The *Dirac operator*

$$D := d + d^* \in \operatorname{Diff}^1(M; \Lambda^k_{\mathbb{C}} M, \Lambda^k_{\mathbb{C}} M)$$

is elliptic and has the symbol

$$\sigma_D(\xi) = i(\operatorname{ext}_\xi - \operatorname{int}_\xi).$$

(iv) The *Hodge Laplacian*

$$\Delta := D^2 = d \circ d^* + d^* \circ d \in \operatorname{Diff}^2(M; \Lambda^k_{\mathbb{C}} M, \Lambda^k_{\mathbb{C}} M)$$

is elliptic and has the symbol

$$\sigma_\Delta(\xi) = -\|\xi\|^2 \operatorname{id}$$

Locally Δ is given on funtions by

$$\Delta(f) = (d^* \circ d)(f) = \frac{1}{\sqrt{g}} \sum_{j,k=1}^m \partial\varphi_j(\sqrt{g} g^{kj} \partial\varphi_k(f)).$$

Proof.

(i) In any chart φ of M $d\omega = \sum_I \sum_{i=1}^m \partial\varphi_i(\omega_I)\omega_I d\varphi I$, where the sum is taken over all increasing multi-indices I. Thus d is a differential operator. Using 2.2.19 we choose a function $f \in \mathcal{C}^\infty(M)$ such that $f(p) = 0$ and $df|_p = \xi$ and calculate

$$\sigma_d(\xi)(\omega_p) = id(f\omega)|_p = i(df \wedge \omega)|_p + i(f \wedge d\omega)(p) = i\xi \wedge \omega(p) = i\,\mathrm{ext}_\xi(\omega)|_p.$$

(ii) By d^* is adjoint to d. Therefore 2.3.15 implies ref

$$d^* \in \mathrm{Diff}^1(M; \Omega^{k+1}_{\mathbb{C}} M, \Omega^{k+1}_{\mathbb{C}} M).$$

Its easyier to calculate the principal symbol directly instead of using 2.3.15. But first of all, we establish a Leibniz rule for d^*. For any $f \in \mathscr{C}^\infty(M, \mathbb{R})$, $\eta \in \Omega^k_{\mathbb{C}}(M)$, $\omega \in \Omega^{k+1}_{\mathbb{C}}(M)$:

$$\begin{aligned}\langle d^*(f\omega), \eta\rangle_{L^2} &= \langle f\omega, d\eta\rangle_{L^2} = \int_M f\omega \wedge \overline{*d\eta} = \int_M \omega \wedge \overline{*\bar{f}d\eta} \\ &= \langle \omega, \bar{f}d\eta\rangle_{L^2} = \langle \omega, fd\eta\rangle_{L^2} = \langle \omega, d(f\eta) - df \wedge \eta\rangle_{L^2} \\ &= \langle d^*(\omega), f\eta\rangle - \langle \omega, \mathrm{ext}_{df}(\eta)\rangle_{L^2} \overset{2.4.6}{=} \langle fd^*\omega, \eta\rangle - \langle \mathrm{int}_{df}\,\omega, \eta\rangle_{L^2} \\ &= \langle fd^*(\omega) - \mathrm{int}_{df}(\omega), \eta\rangle_{L^2}\end{aligned}$$

Since η was arbitrary, we obtain

$$d^*(f\omega) = fd^*(\omega) - \mathrm{int}_{df}(\omega) \tag{2.38}$$

In case $f(p) = 0$, $df|_p = \xi$, we obtain

$$\sigma_{d^*}(\xi) \overset{2.13}{=} id^*(f\omega)(p) \overset{2.38}{=} i(f(p)d^*(\omega)|_p - \mathrm{int}_{df(p)}(\omega|_p)) = -i\,\mathrm{int}_\xi(\omega|_p).$$

To obtain the local coordinate representation for functions, notice that for any local chart $\varphi : U \to V$ and any $f \in \mathscr{C}^\infty(U)$

$$df = \sum_{j=1}^m \partial\varphi_j(f)d\varphi^j = \sum_{j=1}^m d_j\partial\varphi_j(f),$$

where $d_j : \mathscr{C}^\infty(U) = \Lambda^0 T^*_{\mathbb{C}} U \to \Lambda^1 T^*_{\mathbb{C}} U$ is the homomorphism $f \mapsto fd\varphi^j$. We have to calculate its adjoint d_j^*. We choose the coordinate frame 1 on for $\Lambda^0 T^*_{\mathbb{C}} U$ and the coordinate frame $\{d\varphi^j\}$ for $\Lambda^1 T^*_{\mathbb{C}} U$. Then the coordinate matrix of d_j with respect to these frames is $e_j \in \mathbb{C}^{m\times 1}$. The coordinate matrix of the fibre metric in $\Lambda^0 T^*_C U$ is $(1) \in \mathbb{C}^{1\times 1}$ and the coordinate matrix of the fibre metric in $\Lambda^1 T^*_{\mathbb{C}} U$ is by definition the same as the coordinate matrix of the fibre metric in $\Lambda^1 T^* U$, which is given by

$$\begin{aligned}\langle d\varphi^j, d\varphi^i\rangle = g((d\varphi^j)^\sharp, (d\varphi^i)^\sharp) &= \sum_{\nu,\mu=1}^m g(g^{\nu i}\partial\varphi_\nu, g^{\mu j}\partial\varphi_\mu = \sum_{\nu,\mu=1}^m g^{\nu i}g^{\mu j}g_{\nu\mu} \\ &= \sum_{\mu=1}^m (G^{-1}G)_{i\mu}g^{\mu j} = \sum_{\mu=1}^m \delta_{i\mu}g^{\mu j} = g^{ij}.\end{aligned}$$

Using A.3.1 the coordinate matrix of d_j^* is given by

$$1\bar{e}_j^t(G^{-1}) = (g^{j1}, \dots, g^{jm}).$$

Therefore

$$d^*(\omega) \overset{(2.24)}{=} \frac{1}{\sqrt{g}} \sum_{j,k=1}^m \partial\varphi_j(\sqrt{g}g^{kj}\omega_k)$$

(iii) It follows from (i) and (ii) that

$$\sigma_D(\xi) = i(\mathrm{ext}_\xi - \mathrm{int}_\xi).$$

It remains to show that D is elliptic. By definition we have to show that for any $\xi \neq 0$, $\sigma_D(\xi)$ is an isomorphism. It will be more convenient to proceed with the next part first.

(iv) We obtain

$$\begin{aligned}\Delta = D^2 = (d + d^*)^2 &= d^2 + d \circ d^* + d^* \circ d + (d^*)^2 \\ &= d \circ d^* + d^* \circ d \in \mathrm{Diff}^2(M; \Lambda^k_{\mathbb{C}} M, \Lambda^k_{\mathbb{C}} M)\end{aligned}$$

by 2.3.6. This theorem also implies for any $\xi \in T_pM$

$$\begin{aligned}\sigma_\Delta(\xi) = (i(\mathrm{ext}_\xi - \mathrm{int}_\xi))^2 &= -(\mathrm{ext}_\xi^2 - \mathrm{ext}_\xi \circ \mathrm{int}_\xi - \mathrm{int}_\xi \circ \mathrm{ext}_\xi + \mathrm{int}_\xi^2) \\ &= \mathrm{int}_\xi \circ \mathrm{ext}_\xi + \mathrm{ext}_\xi \circ \mathrm{int}_\xi \overset{2.4.6(iii)}{=} \|\xi\|^2 \mathrm{id}\,.\end{aligned}$$

This implies that $\sigma_\Delta(\xi)$ is an isomorphism for any $\xi \neq 0$. In turn this implies also that σ_D is elliptic: If it were not ellitplic, there would be a $\xi \neq 0$ such that σ_D was no isomorphism. But then $\sigma_\Delta(\xi) = \sigma_D(\xi) \circ \sigma_D(\xi)$ would not be an isomorphism as well.

□

2.4.8 Definition (elliptic complex)**.** Let M be a smooth compact manifold. For any $j \in J \subset \mathbb{Z}$ let $\pi_j : E_j \to M$ be a smooth vector bundle over M and

$$P_j \in \mathrm{Diff}(M; E_j, E_{j+1})$$

be a PDO. We say $(E_j, P_j)_{j \in J}$ is a *complex over* M, if for any $j \in \mathbb{Z}$, $P_{j+1} \circ P_j = 0$. The complex is *elliptic*, if for any $j \in J$, $D_j := P_j + P_j^*$ is an elliptic operator.

2.4.9 Corollary. The de Rham complex is an elliptic complex.

Proof. This is just a reformulation of 2.4.7. □

2.4.10 Lemma. Let $(E_j, P_j)_{j \in J}$ be a complex and let $\sigma_j := \sigma_{P_j}$.

(i) For any $\xi \in T^*M$

$$\mathrm{im}\, \sigma_j(\xi) \subset \ker \sigma_{j+1}(\xi).$$

(ii) Denote by $\pi : T^*M \to M$ the cotangent bundle. In view of 2.2.17, we may also consider the bundles $\pi^*(E_j)$ over T^*M and think of σ_j as a section $\sigma_j \in \Gamma(T^*M; \pi^*(E_j), \pi^*(E_{j+1})$.

(iii)

2.5. Locality

2.5.1 Definition (local)**.** A linear map $P : \Gamma(M, E) \to \Gamma(M, F)$ is *local*, if

$$\forall s \in \Gamma(M, E) : \mathrm{supp}\, P(s) \subset \mathrm{supp}(s).$$

2.5.2 Theorem (Peetre)**.** A linear map $P : \Gamma(M, E) \to \Gamma(M, F)$ is a PDO if and only if it is local.

3. Basics from Functional Analysis

Topological Vector Spaces and Basic Properties: Separation Axioms, Basis for Topology, Morphisms, Characarization of Continuity, Completeness, Frechet Spaces, Locally Convex Spaces, Minkowski-Functionals, Half-Norms => Vector Space Normed, Semi-Normed, Banach, Hilbert Spaces: Definitions Notation Konvergenz, Konvergenz und Folgenkonvergenz, Abzählbarkeitsaxiome Dicht Definierte Operatoren Fourier-Transformation und Faltung

3.1. Topological Vector Spaces

3.1.1 Definition (Topological vector space)**.** A *topological vector space* X is a real or complex vector space endowed with a topology such that every point is closed and that addition $X \times X \to X$ and scalar multiplication $\mathbb{K} \times X \to X$ are continuous (w.r.t. the product topology). We define TVS to be the category whose objects are topological vector spaces and whose morphisms are continuous linear maps between them.

3.1.2 Definition (operations on sets)**.** Let X be any vector space, $A, B \subset X$, $x \in X$, $\lambda \in \mathbb{K}$. We define

$$\begin{aligned} x + A &:= \{x + a \mid a \in A\} \\ x - A &:= \{x - a \mid a \in A\} \\ A + A &:= \{a - b \mid a \in A, b \in B\} \\ \lambda A &:= \{\lambda a \mid a \in A\} \end{aligned}$$

3.1.3 Definition (special subsets)**.** Let X be a vector space.

(i) $Y \subset X$ is a *subspace* if it is itself a vector space with the restricted vector space operations.

(ii) $C \subset X$ is *convex* if

$$\forall t \in [0, 1] : tC + (1 - t)C \subset C.$$

This is equivalent of stating that for any two points $x, y \in C$ the entire line $tx+(1-t)y$ is contained in C. Notice that it is superflous to check this condition for $t \in \{0, 1\}$.

(iii) $B \subset X$ is *balanced* if

$$\forall \lambda \in \mathbb{K} : |\lambda| \leq 1 \Rightarrow \lambda B \subset B.$$

3.1.4 Lemma. Topological vector spaces are Hausdorff.

3.1.5 Definition. For any topological space X and any $x \in X$ we denote by

$$\begin{aligned} \mathfrak{U}(x) &:= \{U \subset X \mid U \text{ is a neighbourhood of } x\} \\ \mathfrak{O}(x) &:= \{U \subset X \mid U \text{ is an open neighbourhood of } x\} \end{aligned}$$

A subset $\mathfrak{B}(x) \subset \mathfrak{U}(x)$ is a *local base* at x if

$$\forall U \in \mathfrak{U}(x) : \exists B \in \mathfrak{B}(x) : B \subset U.$$

3.1.6 Lemma. The topology of a TVS X is completely determined by its topology at 0. More precisely:

$$\forall x \in X : \mathfrak{U}(x) = x + \mathfrak{U}(0).$$

Therefore the word "neighbourhood", "local base" etc. always refer to the point 0 and we write $\mathfrak{U} := \mathfrak{U}(0)$.

3.1.7 Definition. Let X be a TVS and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X.

(i) We say (x_n) is a *Cauchy sequence* if

$$\forall U \in \mathfrak{U} : \exists N \in \mathbb{N} : \forall n, m \geq N : x_n - x_m \in U$$

(ii) We say (x_n) *converges in* X if

$$\exists x \in X : \forall U \in \mathfrak{U}(x) : \exists N \in \mathbb{N} : \forall n \geq N : x_n \in U.$$

We denote this by

$$x_n \xrightarrow[X]{n\to\infty} x \ .$$

(iii) A subset $E \subset X$ is *bounded* if

$$\forall U \in \mathfrak{U} : \exists s > 0 : \forall t > s : E \subset tU.$$

A sequence (x_n) is bounded if $\{x_n | n \in \mathbb{N}\} \subset X$ is bounded.

3.1.8 Remark. The definition of boundedness may eventually seem odd. If $(X, \|\_\|)$ is a normed space, we temporariliy define a set $E \subset X$ to be $\|\_\|$-*bounded*, if

$$\exists R > 0 : E \subset B_R(0).$$

This is the usuall definition of boundedness. Now $\|\_\|$ induced a topology $\mathcal{O}$ on X. Let's say E is $\mathcal{O}$-bounded, if the definition 3.1.7,(iii) holds. Then a set E is $\|\_\|$-bounded if and only if it is $\mathcal{O}$-bounded:
"$\Rightarrow$": Assume $E \subset B_R(0)$. Let $U \in \mathfrak{U}$ be arbitrary. By definition there exists $r > 0$ such that $B_r(0) \subset U$. Define $s := R/r$. Then for any $t > s$

$$\forall x \in E : \|x\| < R = sr < tr \Longrightarrow x \in tB_r(0) \subset tU.$$

"$\Leftarrow$": Conversely consider $B_1(0) \in \mathfrak{U}$. There exists $s > 0$ such that for any $t > s$

$$E \subset tB_1(0) = B_t(0).$$

3.1.9 Corollary. Let X be a TVS.

(i) X has a balanced local base.

(ii) If X is locally convex, then X has a balanced convex local base.

Proof. [5, 1.14] □

3.1.10 Definition (Operator)**.** Let X, Y be topological vector spaces. A continuous linear map $T : X \to Y$ is an *operator*. An operator $X \to \mathbb{K}$ is a *functional.*
A linear map $T : X \to Y$ is *bounded*, if for any bounded set $E \subset X$, the set $T(E) \subset Y$ is bounded.

3.1.11 Theorem (Characterizations of Operators)**.** Let X, Y be TVS and $T : X \to Y$ be a linear map. Among the following properties

(i) T is continuous.

(ii) T is bounded.

(iii) If $x_n \xrightarrow[X]{} 0$ then $\{T(x_n) | n \in \mathbb{N}\}$ is bounded.

(iv) If $x_n \xrightarrow[X]{} 0$, then $T(x_n) \xrightarrow[Y]{} 0$.

the implications

$$(i) \Rightarrow (ii) \Rightarrow (iii)$$

are always true. If X is metrizable, the implications

$$(iii) \Rightarrow (iv) \Rightarrow (i)$$

are also true. Hence in that case all properties are equivalent.

3.1.12 Definition (invariant metric)**.** A metric d on a vector space X is *(translation-)invariant*, if

$$\forall x, y, z \in X : d(x + z, y + z) = d(x, y).$$

3.1.13 Definition (Types of TVS)**.** Let $(X, \mathcal{O})$ be a TVS.

(i) X is *locally convex* if there exists a local base $\mathfrak{B}$ whose members are all convex.

(ii) X is *metrizable* if $\mathcal{O}$ is induced by some metric d.

(iii) X is an *F-space* if it is complete and metrizable by an invariant metric.

(iv) X is a *Fréchet space* if it is a locally convex F-space.

(v) X is *normable* if $\mathcal{O}$ is induced by some norm.

(vi) X has the *Heine-Borel property* if every closed and bounded subset is compact.

3.1.14 Definition (seminorm)**.** Let V be a vector space. A function $p : V \to \mathbb{R}$ is a *seminorm* if

(i) *Subadditivity*: $\forall x, y \in V : p(x + y) \leq p(x) + p(y)$.

(ii) *Semi-homogenity*: $\forall x \in V : \forall \lambda \in \mathbb{K} : p(\lambda x) = |\lambda| p(x)$.

A seminorm is a *norm* provided

$$\forall x \in V : p(x) = 0 \Rightarrow x = 0$$

and usually is denoted by $p = \| \ \|$. The tuple $(V, \|\_\|)$ is a *normed space.* The category Nrm consists of all normed spaces and continuous maps between them.

A family $\mathcal{P}$ of seminorms is *separating*, if

$$\forall x \in V : x \neq 0 \Rightarrow \exists p \in \mathcal{P} : p(x) \neq 0.$$

3.1.15 Definition (absorbing, Minkowski functional)**.** A set $A \subset X$ is *absorbing*, if

$$\bigcup_{t \in]0,\infty[} tA = X.$$

In that case we call $\mu_A : X \to [0, \infty]$,

$$x \mapsto \inf t > 0 \mid x \in tA,$$

the associated *Minkowski functional.*

3.1.16 Theorem (properties of seminorms)**.** Let X be a vector space and let p be a seminorm on X.

(i) $p(0) = 0$.

(ii) $\forall x, y \in X : |p(y) - p(x)| \leq p(y - x)$.

(iii) $\forall x \in X : p(x) \geq 0$.

(iv) $\{x \in X \mid p(x) = 0\} \subset X$ is a vector space.

(v) The set $B := \{x \in X \mid p(x) < 1\}$ is convex, balanced, absorbing and $\mu_B = p$.

Proof. [5, 1.34] □

3.1.17 Lemma (operations on seminorms)**.** Let X be a $\mathbb{K}$ vector space, let $p, q : X \to \mathbb{R}$ be semi-norms and $\lambda \in \mathbb{R}_{\geq 0}$. Then

(i) $p + q$,

(ii) cp,

(iii) $\max(p, q)$

are seminorms on X as well.
If X is a topological vector space and p and q are continuous, so are $p+q$, cp and $\max(p, q)$.

3.1.18 Theorem. Let $A \subset X$ be a convex, absorbing set in a vector space X.

(i) $\forall x, y \in X : \mu_A(x + y) \leq \mu_A(x) + \mu_A(y)$.

(ii) $\forall x \in X : \forall t \geq 0 : \mu_A(tx) = t\mu_A(x)$.

(iii) If A is balanced, then μ_A is a seminorm.

(iv) If $B := \{x \in X \mid \mu_A(x) < 1\}$ and $C := \{x \in X \mid \mu_A(x) \leq 1\}$, then $A \subset B \subset C$ and $\mu_A = \mu_B = \mu_C$.

Proof. [5, 1.35] □

3.1.19 Theorem (seminorms induced by local base)**.** Let X be a locally convex TVS. By 3.1.9 X has a convex balanced local base $\mathfrak{B}$. For any $V \in \mathfrak{B}$ let μ_B be the associated Minkowski function.

(i) $\forall V \in \mathfrak{B} : \{x \in X \mid \mu_V(x) < 1\} = V$.

(ii) $\{\mu_V \mid V \in \mathfrak{B}\}$ is a separating family of continuous seminorms on X.

Proof. [5, 1.36] □

3.1.20 Theorem (Topological vector spaces induced by Seminorms)**.** Let $\mathfrak{P}$ be a family of seminorms on a vector space V. For any $p \in \mathfrak{P}$ and every positive $n \in \mathbb{N}$ define

$$B(p, n) := \{x \in V \mid p(x) < \frac{1}{n}\}.$$

Then the collection $\mathfrak{B}$ of finite intersections of those $B(n, p)$ is a convex balanced local base for a topology $\mathcal{O} := \mathcal{O}_{\mathfrak{P}}$ on V, which turns V into a locally convex space such that

(i) Every $p \in \mathfrak{P}$ is continuous.

(ii) A set $E \subset X$ is bounded if and only if every $p \in \mathfrak{P}$ is bounded on M.

(iii) A sequence (x_j) in X converges to x with respect to the induced topology if and only if

$$\forall p \in \mathfrak{P} : \; x_j \xrightarrow[p]{j \to \infty} x \; .$$

The analogous statement holds for Cauchy sequences.

If additionally $\mathfrak{P} = \{p_i\}_{i\in\mathbb{N}}$ is countable, then $\mathcal{O}_{\mathfrak{P}}$ is metrizable. If (c_i) is any positive real sequence, such that $c_i \to 0$, the function $d : X \times X \to \mathbb{R}$

$$d(x,y) := \max_{i\in\mathbb{N}} c_i \frac{p_i(x-y)}{1+p_i(x-y)}$$

is a translation invariant metric such that $\mathcal{O}_d = \mathcal{O}_{\mathfrak{P}}$.

Proof.
Step 1 (Construction of $\mathcal{O}$): We just declare

$$\mathcal{O} := \{O \subset X \mid \forall x \in O : \exists B \in \mathfrak{B} : x + B \subset O\}.$$

Step 1.1 (Topology Axioms): Clearly $\emptyset, X \in \mathcal{O}$. It is also clear that $\mathcal{O}$ is closed under arbitrary unions. It is closed under finite intersections by construction of $\mathfrak{B}$ (this is the reason why we defined $\mathfrak{B}$ to be the set of finite intersections of the $B(p,n)$). Thus $\mathcal{O}$ is a topology. By construction $\mathcal{O}$ is translation invariant.
Step 1.2 (Closed points): We show that $\{0\} \in X$ is closed: Let $0 \neq x \in X$ be arbitrary. Since $\mathfrak{P}$ is separating, there exists $p \in \mathfrak{P}$ such that $p(x) > 0$. Thus there exists $n \in \mathbb{N}$, such that $p(x) > \frac{1}{n}$. Therefore $x \notin B(p,n)$, thus $0 \notin x + B(p,n)$. Consequently $X \setminus \{0\}$ is open and therefore $\{0\}$ is closed.
Step 1.3 (Continuity of Addition): Denote by $A : X \times X \to X$ the addition. It suffices to show that A is continuous at $(0,0) \in X \times X$. Let $U \in \mathfrak{U}(0)$ be any neighbourhood. Then there exist $n_1, \ldots, n_m \in \mathbb{N}$, $p_1, \ldots, p_m \in \mathfrak{P}$, such that

$$U \supset B(p_1,n_1) \cap \ldots \cap B(p_m,n_m) \tag{3.1}$$

Define

$$V := B(p_1,2n_1) \cap \ldots \cap B(p_m,2n_m) \tag{3.2}$$

and observe

$$\forall (x,y) \in V \times V : \forall 1 \leq \nu \leq m : p_\nu(x+y) \leq p_\nu(x) + p_\nu(y) = \frac{1}{2n_\nu} + \frac{1}{2n_\nu} = \frac{1}{n_\nu}.$$

Therefore $V + V \subset U$, i.e. $(0,0) \in V \times V \subset A^{-1}(U)$.
Step 1.4 (Continuity of Scalar Multiplication): Let $(\alpha, x) \in \mathbb{K} \times X$ and U,V as in (3.1) and (3.2) above. There exists $s > 0$ such that $x \in sV$. Define $t := s/(a + |\alpha|s)$. Denoting the scalar multiplication by $SM : \mathbb{K} \times X \to X$ we claim that $(\alpha, x) \in B_{1/s}(\alpha) \times (x + tV) \subset SM^{-1}(U)$. Therefore let $(\beta, y) \in B_{1/s} \times (x + tV)$ be arbitrary. We calculate

$$\begin{aligned} |\beta|t &= \frac{|\beta|s}{1+|\alpha|s} < \frac{|\alpha|s + \frac{1}{s}s}{1+|\alpha|s} = 1 \\ \Rightarrow \beta y - \alpha x &= \beta(y-x) + (\beta-\alpha)x \in |\beta|tV + |\beta-\alpha|sV \subset V + V \subset U, \end{aligned}$$

since V is balanced.
Step 2: Now we proof the additional properties.
Step 2.1 (Continuity of the Semi-Norms): This follows directly from the definitions.
Step 2.2 (Bounded Sets): Let $E \subset X$ and U, V as in (3.1) and (3.2) above.
"$\Rightarrow$": Let $p \in \mathfrak{P}$ be arbitrary. Sicne $B(p,1)$ is a neighboorhood of 0 by construction there exists $k \in \mathbb{N}$ such that $E \subset kB(p,1)$. Therefore any $x \in E$ satisfies $p(x) < k$. Therefore p

is bounded on E.
"$\Leftarrow$": The definition of V and the hypothesis implies

$$\forall 1 \leq \nu \leq m : \exists M_\nu \in \mathbb{R}_{>0} : \forall x \in E : p_\nu(x) < M_\nu.$$

Take any $n > \max_{1\leq\nu\leq m}(M_\nu n_\nu)$. This implies

$$\forall x \in E : \forall 1 \leq \nu \leq m : p_\nu(x) < M_\nu < n\frac{1}{n_\nu} \Rightarrow x \in nU,$$

thus $E \subset nU$ and E is bounded.
STEP 2.3 (Sequential Convergence): If f_j converges to 0 with respect to the induced topology, item (i) implies converges with respect to all the seminorms. Conversely, assume a sequence converges with respect to all the seminorms. Let $U \in \mathfrak{U}(0)$ be arbitrary. By definition there exists $B \in \mathfrak{B}$, such that $B \subset U$. By definition there exist $p_1, \ldots, p_k \in \mathfrak{P}$, $n_1, \ldots, n_k \in \mathbb{N}$ such that $B = B(p_1, n_1) \cap \ldots \cap B(p_k, n_k)$. By hypothesis

$$\forall 1 \leq \nu \leq k : \exists N_\nu : \forall j \geq N_\nu : p_\nu(f_j) < \frac{1}{n_\nu}.$$

Consequently for any $j \geq \max_{1\leq\nu\leq k} N_\nu : f_j \in B \subset U$.
STEP 3 (Metrizability): We now assume that $\mathfrak{P}$ is countable.
STEP 3.1 (Metric Axioms): Since $p_i \geq 0$ the sequence

$$\frac{p_i(x-y)}{1+p_i(x-y)} \in [0,1]$$

is bounded and non-negative. Since c_i is positive and converges to zero, the sequence $d(x,y)$ is positive and converges to zero from above. Therefore d is well-defined. It is clear that d translation-invariant and symmetric. Since $\mathfrak{P}$ is separating

$$d(x,y) = 0 \Leftrightarrow x = y.$$

To see the triangle inequality, notice that the function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \frac{x}{1+x}$, satisfies

$$f'(x) = \frac{1+x-x}{(1+x)^2} > 0$$

and therefore f is monotonously increasing. Consequently since every p_i is subadditive

$$\frac{p_i(x-z)}{1+p_i(x-z)} = f(p_i((x-y)-(z-y))) \leq f(p_i(x-y)+p_i(z-y))$$
$$= \frac{p_i(x-y)}{1+p_i(z-y)+p_i(x-y)} + \frac{p_i(z-y)}{1+p_i(z-y)+p_i(x-y)} \leq \frac{p_i(x-y)}{1+p_i(x-y)} + \frac{p_i(z-y)}{1+p_i(z-y)}.$$

Therefore d satisfies the triangle inequality.
STEP 3.2 ($\mathcal{O}_\mathfrak{P} = \mathcal{O}_d$): First we claim that the balls

$$B_r := \{x \in X \mid d(0,x) < r\}, 0 < r < \infty,$$

are a convex balanced local base for $\mathcal{O}_\mathfrak{P}$. This basically follows from the identity

$$\forall\, 0 < r < \infty : \; B_r = \bigcap_{i\in\mathbb{N}: c_i > r} \Big\{x \in X \mid p_i(x) < \frac{r}{c_i - r}\Big\}.$$

First notice that since c_i converges to zero, the intersection is finite. Since all the p_i are $\mathcal{O}_{\mathfrak{P}}$-continuous, the right hand side is an $\mathcal{O}_{\mathfrak{P}}$-open set. The identitiy follows from the fact, that if $c_i > r$,

$$\frac{c_i p_i(x)}{1+p_i(x)} < r \Leftrightarrow c_i p_i(x) < r + p_i(x) r \Leftrightarrow (c_i - r) p_i(x) < r \Leftrightarrow p_i(x) < \frac{r}{c_i - r}.$$

This proves $B_r \in \mathcal{O}_{\mathfrak{P}}$. The B_r are also convex and balanced (!ToDoRef).
Now assume that $W \in \mathcal{O}_{\mathfrak{P}}$ is an open neighbourhood of $0 \in X$. By definition there exist $p_1, \ldots, p_k \in \mathfrak{P}$, $\delta_1, \ldots, \delta_k \in]0,1[$ such that

$$W \supset \bigcap_{i=1}^{k} B(p_i, \delta_i) =: B.$$

Choose $0 < r < \infty$, such that $2r < \min\{c_1\delta_1, \ldots, c_k\delta_k\}$. This implies

$$\forall x \in B_r : \forall 1 \leq i \leq k : \frac{c_i p_i(x)}{1+p_i(x)} < r < \frac{c_i \delta_i}{2} \Rightarrow 2p_i(x) < \delta_i(1+p_i(x)) \Rightarrow p_i(x)(2-\delta_i) < \delta_i$$

$$\Rightarrow p_i(x) < \frac{\delta_i}{2-\delta_i} < 1 \Rightarrow x \in B(p_i, \delta_i).$$

Therefore $B_r \subset B \subset W$. Consequently the B_r are a local base as claimed, W is an $\mathcal{O}_d$-neighbourhood of 0 and alltogether $\mathcal{O}_d = \mathcal{O}_{\mathfrak{P}}$.

□

3.1.21 Lemma (continuous seminorms)**.** Let X be topologized as in 3.1.20 above with a family $\mathfrak{P} = \{p_i\}_{i \in I}$ of seminorms. Let $q : X \to \mathbb{R}$ be an arbitrary seminorm. The following are equivalent:

(i) q is continuous on X.

(ii) There exists a finite subset $J \subset I$ and a constant $C > 0$ such that

$$\forall x \in X : q(x) \leq C \max_{j \in J} p_j(x). \tag{3.3}$$

(iii) There exists a finite subset $J \subset I$ and a constant $C > 0$ such that

$$\forall x \in X : q(x) \leq C \sum_{j \in J} p_j(x).$$

Proof. "(i)⇒(ii)": Let q be continuous. By definition

$$0 \in q^{-1}(I_1(0)) \overset{\circ}{\subseteq} X,$$

where $I_1(0) =]-1,1[\subset \mathbb{R}$ is open. Consequently there exists an open neighbourhood U of 0 such that $V \subset q^{-1}(I_q(0))$. By definition 3.1.20 of the topology on X there exists a finite $J \subset I$ and $\varepsilon_j > 0$ such that

$$0 \in V := \bigcap_{j \in J} B_j \subset U,$$

where $B_j := B^{p_j}_{\varepsilon_j}(0)$. Define

$$\varepsilon := \frac{1}{2} \min_{j \in J} \varepsilon_j > 0.$$

Now let $x \in X$ be arbitrary. Clearly $x = 0$ satisfies inequality (3.3), so let $x \neq 0$. Define

$$x' := \frac{\varepsilon x}{\max_{j \in J} p_j(x)}.$$

This implies

$$\forall i \in J : p_i(x') = \frac{p_i(x)}{\max_{j \in J} p_j(x)} \frac{1}{2} \min_{j \in J} \varepsilon_j \leq \frac{\varepsilon_i}{2} < \varepsilon_i.$$

Consequently $x' \in B_i$ for any $i \in J$. By definition this implies

$$1 \geq q(x') = \frac{\varepsilon}{\max_{j \in J} p_j(x)} q(x),$$

thus

$$q(x) \leq \varepsilon^{-1} \max_{j \in J} p_j(x) = \underbrace{\frac{2}{\min_{j \in J} \varepsilon_j}}_{=:C} \max_{j \in J} p_j(x).$$

"(ii)⇒(i)": We show that q is continuous at $0 \in X$: Let $\varepsilon > 0$ be arbitrary. The set

$$V := \bigcap_{j \in J} B^{p_j}_{\frac{\varepsilon}{2C}}(0)$$

is open by construction and it satifies

$$\forall x \in V : q(x) \overset{(3.3)}{\leq} C \max_{j \in J} p_j(x) \leq C \frac{\varepsilon}{2C} < \varepsilon,$$

thus

$$V \subset q^{-1}(I_\varepsilon(0)).$$

"(ii)⇔(iii)": This follows from

$$\max_{j \in J} p_j(x) \leq \sum_{j \in J} p_j(x) \leq |J| \max_{j \in J} p_j(x).$$

□

3.1.22 Theorem (Characterization of continuous maps)**.** Assume X, Y are locally convex spaces, let $\{q_j\}_{j \in J}$ be a family of seminorms on Y that generate the topology on Y as in 3.1.20 and let $\{p_i\}_{i \in I}$ be the analogous family for X. Let $T : X \to Y$ be linear. The following are equivalent:

(i) T is continuous.

(ii) For any continuous seminorm q on Y there exists a finite subset $\tilde{I} \subset I$ and a $C > 0$ such that
$$\forall x \in X : q(T(x)) \leq C \max_{i \in \tilde{I}} p_i(x).$$

(iii) For any continuous seminorm q on Y there exists a continuous seminorm p on X such that
$$\forall x \in X : q(T(x)) \leq p(x).$$

(iv) For any $j \in J$ there exists a continuous seminorm p on X such that
$$\forall x \in X : q_j(T(x)) \leq p(x).$$

(v) For any $j \in J$ there exists a finite subset $\tilde{I} \subset I$ and a $C > 0$ such that

$$\forall x \in X : q_j(T(x)) \leq C \max_{i \in \tilde{I}} p_i(x).$$

Proof.
"(i)⇒(ii)": Let q be a continuous seminorm on Y. By 3.1.21 there exists a finite subset $\tilde{J} \subset J$ and $C' > 0$ such that

$$\forall y \in Y : q(y) \leq C' \max_{j \in \tilde{J}} q_j(y). \tag{3.4}$$

By hypothesis T is linear and continuous. Thus for any $j \in \tilde{J}$, the map $q_j \circ T : X \to \mathbb{R}$ is a continuous seminorm on X. Again by 3.1.21 this implies that there exists a constant $C_j > 0$ and a finite $\tilde{I}_j \subset I$ such that

$$\forall x \in X : q_j(T(x)) \leq C_j \max_{i \in \tilde{I}_j} p_i(x). \tag{3.5}$$

Define $\tilde{I} := \tilde{I}_1 \cup \ldots \cup \tilde{I}_{|J|}$. Combining both, we obtain

$$\forall x \in X : q(T(x)) \overset{(3.4)}{\leq} C' \max_{j \in \tilde{J}} q_j(T(x)) \overset{(3.5)}{\leq} \underbrace{C' \max_{j \in J} C_j}_{=:C} \max_{i \in \tilde{I}} p_i(x).$$

"(ii)⇒(iii)": By 3.1.17

$$p(x) := C \max_{i \in \tilde{I}} p_i(x)$$

is a continuous seminorm on X.
"(iii)⇒ (iv)": By construction q_j is a continuous seminorm on Y.
"(iv)⇒(v)": Follows from 3.1.21.
"(v)⇒(i)": Let $\tilde{J} \subset J$ be finite, $B_j := B^{q_j}_{\varepsilon_j}(0)$ and

$$V = \bigcap_{j \in \tilde{J}} B_j$$

be an element of the local base for Y. By hypothesis, for any $j \in \tilde{J}$ there exists a finite $\tilde{I}_j \subset I$ and $C_j > 0$ such that

$$\forall x \in X : q_j(T(x)) \leq C_j \max_{i \in \tilde{I}_j} p_i(x).$$

Define

$$\tilde{I} := \bigcup_{j \in J} \tilde{I}_j, \qquad C := \max_{j \in J} C_j, \qquad \varepsilon := \frac{1}{2} \min_{j \in J} \varepsilon_j, \qquad U := \bigcap_{i \in I} B^{p_i}_{\varepsilon}(0) \overset{\circ}{\subseteq} X$$

For any $x \in U$ we calculate

$$q_j(T(x)) \leq C_j \max_{i \in \tilde{I}_j} p_i(x) \leq C \max_{i \in \tilde{I}} p_i(x) < \varepsilon_j.$$

Thus $U \subset T^{-1}(V)$. □

3.1.23 Definition (equivalence seminorms)**.** Let X be a $\mathbb{K}$ vector space and let $P := \{p_i\}_{i \in I}$, $Q := \{q_j\}_{j \in J}$ be two families of seminorms on X. Both induce a topology τ_P, τ_Q on X according to 3.1.20. We say P *is equivalent to* Q, if $\tau_P = \tau_Q$.

3.1.24 Lemma. In the situation of 3.1.23 above: P is equivalent to Q if and only if $\mathrm{id} : (X, \tau_P) \to (X, \tau_Q)$ is a homeomorphism.

3.1.25 Remark. That statement is of course totally trivial. Its strength comes from the fact that one may check the continuity of id and id^{-1} using the various characterizations given in 3.1.22.

3.1.26 Definition (weak-*-topology)**.** Let X be a TVS and X' be its topological dual space. For any $x \in X$, let

$$\begin{aligned} p_x : X' &\to \mathbb{K} \\ x' &\mapsto |x'(x)| \end{aligned}$$

be the seminorm on X' induced by X. The topology on X' generated by the family $\{p_x \mid x \in X\}$ via 3.1.20 is the *weak-*-topology on* X'.

3.1.27 Theorem (Topologization of the Dual)**.** Let X be an F-space and assume X' has the weak-*-topology.

(i) For any sequence (x'_j) in X'

$$x'_j \xrightarrow[X']{} x' \iff \forall x \in X : \; x'_j(x) \xrightarrow[\mathbb{K}]{} x'(x) \; .$$

(ii) X' is complete.

Proof.

(i) It suffices to check this for $x' = 0$. By 3.1.20 the sequence (x'_j) converges in X' if and only if it converges with respect to all the seminorms p_x, $x \in X$. By construction

$$|x_j(x)| = p_x(x'_j).$$

(ii) Let (x'_j) be a Cauchy sequence in X'. By 3.1.20, this implies that x'_j is a Cauchy sequence with respect to all the p_x. So let $x \in X$, $\varepsilon > 0$. There exists $N_0 \in \mathbb{N}$ such that

$$\forall j, k \geq N_0 : |x'_j(x) - x'_k(x)| = p_x(x'_j - x'_k) < \varepsilon.$$

This implies that $x'_k(x)$ is a Cauchy sequence in $\mathbb{K}$. Thus

$$\exists x'(x) \in \mathbb{K} : \; x'_j(x) \xrightarrow[\mathbb{K}]{} x'(x) \; .$$

This defines a linear map $x' : X' \to \mathbb{K}$. By construction

$$x'_j \xrightarrow[X']{} x' \; .$$

The fact that $x' \in X'$ follows from the Banach-Steinhaus theorem, c.f. [5, 2.8].

□

3.1.28 Theorem (Dual Operator)**.** Let X, Y be TVS and $T : X \to Y$ be linear. Let X', Y' be the topological dual spaces endowed with the weak*-topology (i.e. the topology of pointwise convergence). Then $T' : Y' \to X'$ defined by $T'(y')(x) := y'(T(x))$ is a continuous operator $Y' \to X'$.

Proof. Assume
$$y'_j \xrightarrow[Y']{} 0 \,.$$
By definition this is equivalent to
$$\forall y \in Y : \; y'_j(y) \xrightarrow[\mathbb{C}]{} 0 \,.$$
Thus
$$\forall x \in X : \; T'(y'_j)(x) = y'_j(T(x)) \xrightarrow[\mathbb{C}]{} 0$$
and therefore
$$T'(y'_j) \xrightarrow[X']{} 0 \,.$$
□

3.2. Completeness and dense subspaces

3.2.1 Theorem (Extension of Operators)**.** Let $(X, \|\_\|_X)$ be a normed space, such that $D \subset X$ is a dense subspace with the induced norm $\|\_\|_D := \|\_\|_X|_D$. Let $(Y, \|\_\|_Y)$ be a Banach space and $T \in \mathscr{L}((D, \|\_\|_D), (Y, \|\_\|_Y))$ be a continuous linear operator. Then there exists a unique continuous operator $\hat{T} \in \mathscr{L}((X, \|\_\|_X), (Y, \|\_\|_Y))$, such that
$$\hat{T}|_D = T \qquad\qquad \|\hat{T}\|_{\mathscr{L}(X,Y)} = \|T\|_{\mathscr{L}(D,Y)}.$$

3.2.2 Theorem (Continuity of bilinear forms)**.** c.f. Rudin 2.17

3.2.3 Theorem. An operator $T \in \mathscr{L}(X, Y)$ between Banach spaces, that is norm-preserving has closed image.

Proof. Assume
$$Tx_j \xrightarrow[Y]{j\to\infty} y \,.$$
Since T preserves the norm,
$$\|x_j - x_i\|_X = \|Tx_j - Tx_i\|_Y.$$
Since (Tx_j) is a Cauchy sequence in Y, this implies that (x_j) is a Cauchy sequence in X. Since this space is complete,
$$\exists X \in X : \; x_j \xrightarrow[X]{j\to\infty} x \,.$$
Since T is continuous
$$Tx = \lim_{j\to\infty} Tx_j = y.$$
□

3.2.4 Theorem. Let $T \in \mathscr{L}(X, Y)$ be an operator between Banach spaces. For any subset $D \subset X$, we obtain $T(\bar{D}) \subset \overline{T(D)}$. In case T is an isometry, $T(\bar{D}) = \overline{T(D)}$.

Proof. Let $x \in \bar{D}$. Then there exists $(x_j) \in D$, such that

$$x_j \xrightarrow[X]{} x\,.$$

Since T is continuous,

$$Tx_j \xrightarrow[Y]{} Tx\,,$$

thus $Tx \in \overline{T(D)}$. In case T is an isometry, its image is closed by 3.2.3. Therefore

$$\overline{T(D)} = T(D) \subset T(\bar{D}).$$

□

3.3. Complex interpolation method

3.3.1 Theorem (Hadamard-3-Line-Theorem)**.** Let

$$\Omega := \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z < 1\}$$

and let $f : \bar{\Omega} \to \mathbb{C}$ be continuous and bounded and let $f|_\Omega$ be holomorphic. Define

$$M_j := \sup_{t \in \mathbb{R}} |f(\theta + it)|,\ j = 0, 1.$$

Then

$$\forall z \in \Omega : |f(z)| \leq M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z}.$$

Proof. We proceed in two steps.
STEP 1 (Case $M_0 = M_1 = 1$): Assume $M_0 = M_1 = 1$. Define

$$\begin{aligned} f_n : \bar{\Omega} &\to \mathbb{C} \\ z &\mapsto \exp\left(\tfrac{z^2-1}{n}\right) f(z). \end{aligned}$$

Then f_n is continuous on $\bar{\Omega}$ and holomorphic on Ω.
STEP 1.1 ($|f_n| \leq 1$ on $\Omega \setminus [0,1] \times]-R, R[$): Since f is bounded, there exists $C > 0$ such that

$$\forall z \in \bar{\Omega} : |f(z)| \leq C.$$

This implies for any $x \in [0, 1]$ and any $y \in \mathbb{R}$

$$|f_n(x + iy)| \leq C \left|\exp\left(\frac{x^2 + 2iy - y^2 - 1}{n}\right)\right| \leq C \exp\left(\frac{-y^2}{n}\right). \tag{3.6}$$

Choose $R > 0$ such that $C \exp(-R^2/n) \leq 1$. This directly implies

$$\forall z \in \bar{\Omega} : \operatorname{Im}(z) \geq R \Rightarrow |f_n(z)| \leq 1. \tag{3.7}$$

STEP 1.2 ($|f_n| \leq 1$ on $[0, 1] \times [-R, R]$): On the other hand, we estimate for any $y \in \mathbb{R}$:

$$\begin{aligned} |f_n(iy)| &= \left|\exp\left(\frac{-y^2 - 1}{n}\right)\right| |f(iy)| \leq \exp\left(\frac{-y^2 - 1}{n}\right), \\ |f_n(1 + iy)| &= \left|\exp\left(\frac{(1 + iy)^2 - 1}{n}\right)\right| |f(1 + iy)| = \left|\exp\left(\frac{1 + 2iy - y^2 - 1}{n}\right)\right| \leq \left|\exp\left(\frac{-y^2}{n}\right)\right|. \end{aligned}$$

Consequently,

$$\forall z \in \partial\Omega : |f_n(z)| \leq \exp\left(\frac{-\operatorname{Im}(z)^2}{n}\right) \leq 1.$$

Combining this with (3.7), we obtain the same estimate on $\partial([0,1] \times [-R,R])$. Thus by the maximum principle, we obtain

$$\forall z \in [0,1] \times [-R,R] : |f_n(z)| \leq 1.$$

Step 1.3: Alltogether we obtain

$$\forall z \in \bar{\Omega} : |f_n(z)| \leq 1.$$

Since for any $z \in \Omega$

$$f_n(z) \xrightarrow[\mathbb{C}]{n\to\infty} f(z) ,$$

this implies

$$\forall z \in \bar{\Omega} : |f(z)| \leq 1.$$

Step 2 (Reduction to the case $M_0 = M_1 = 1$): Define

$$\begin{array}{rcl} g : \bar{\Omega} & \to & \mathbb{C} \\ z & \mapsto & M_0^{z-1} M_1^{-z} f(z). \end{array}$$

Then g is continuous on $\bar{\Omega}$, holomorphic on Ω. Since f is bounded and $\operatorname{Re}(z) \in [0,1]$, the estimate

$$\forall z \in \bar{\Omega} : |g(z)| \leq M_0^{\operatorname{Re}(z)-1} M_1^{-\operatorname{Re}(z)} |f(z)|$$

shows that g is bounded. Furthermore

$$\begin{aligned} \forall y \in \mathbb{R} \ : |g(iy)| &= |M_0^{iy-1} M_1^{-iy}||f(iy)| \leq M_0^{-1}|f(iy)| \leq 1 \\ \forall y \in \mathbb{R} \ : |g(1+iy)| &= |M_0^{iy} M_1^{-1-iy}||f(1+iy)| \leq M_1^{-1}|f(1+iy)| \leq 1. \end{aligned}$$

Thus the first step implies

$$\forall z \in \bar{\Omega} : |g(z)| \leq 1.$$

Consequently

$$\forall z \in \Omega : |f(z)| = |M_0^{1-z} M_1^{z} g(z)| \leq M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z}.$$

□

3.3.2 Theorem and Definition (Existence of interpolation spaces)**.** Let E and F be Banach spaces and assume there exists a continuous inclusion $E \hookrightarrow F$. Define

$$\Omega := \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z < 1\},$$

$$\begin{aligned} \mathcal{H}(F,E) := \{u \in \mathcal{C}_b^0(\bar{\Omega}, F) \mid & \ u \textit{ is holomorphic on } \Omega \textit{ and} \\ & \forall t \in \mathbb{R} \ : u(1+it) \in E \text{ and } \sup_{t\in\mathbb{R}} \|u(1+it)\|_E < \infty\} \end{aligned}$$

and for any $u \in \mathcal{H}(F,E)$

$$\|u\|_{\mathcal{H}(F,E)} := \sup_{z\in\bar{\Omega}} \|u(z)\|_E + \sup_{t\in\mathbb{R}} \|u(1+it)\|_F = \|u\|_{\mathcal{C}_b^0(\bar{\Omega},E)} + \|u(1+i\_)\|_{\mathcal{C}_b^0(\mathbb{R},F)}.$$

Then $\mathcal{H}(F,E)$ is a Banach space.
For any $0 \leq \theta \leq 1$, define the *interpolation spaces*

$$[F,E]_\theta := \{u(\theta) \mid u \in \mathcal{H}(F,E)\}, \qquad [E,F]_\theta := [F,E]_{1-\theta}.$$

The map

$$\begin{array}{rcl} \varphi_\theta : \mathcal{H}(F,E) & \to & [F,E]_\theta \\ u & \mapsto & u(\theta) \end{array}$$

is surjective with kernel $\ker \varphi_\theta = \{u \in \mathcal{H}(F,E) | u(\theta) = 0\}$. Therefore it descends to an isomorphism

$$\bar{\varphi}_\theta : \bar{\mathcal{H}}(F,E) := \frac{\mathcal{H}(F,E)}{\ker \varphi_\theta} \to [F,E]_\theta.$$

The space $[F,E]_\theta$ is a Banach space itself by declaring φ_θ to be an isometry, i.e.

$$\forall u(\theta) \in [F,E]_\theta : \|u(\theta)\|_{[F,E]_\theta} := \|\bar{\varphi}_\theta^{-1}(u)\|_{\bar{\mathcal{H}}(F,E)}.$$

Furthermore there are isomorphisms $[E,F]_0 \cong E$, $[E,F]_1 \cong 1$.

Proof.
Step 1 ($H(E,F)$ is Banach): Since E and F a vector spaces, the triangle inequality immediately implies that $\mathcal{H}(F,E)$ is a vector space. We have to check that it is complete. Therefore, let (u_j) be a $\mathcal{H}(F,E)$-Cauchy sequence. This implies that (u_j) is a Cauchy sequence in $\mathcal{C}_b^0(\bar{\Omega},F)$ and $(u_j(1+i\_))$ is a Cauchy-sequence in $\mathcal{C}_b^0(\mathbb{R},E)$. Since these space are complete,

$$\exists u \in \mathcal{C}_b^0(\bar{\Omega},F) : \ u_j \xrightarrow[\mathcal{C}_b^0(\bar{\Omega},F)]{} u$$

$$\exists \tilde{u} \in \mathcal{C}_b^0(\mathbb{R},E) : \ u_j(1+i\_) \xrightarrow[\mathscr{C}_b^0(\mathbb{R},E)]{} \tilde{u} \ .$$

By continuity

$$\forall t \in \mathbb{R} : u(1+it) = \tilde{u}(t).$$

Weierstrass' convergence theorem states that the uniform limit of holomorphic function is holomorphic. Therefore $u : \Omega \to F$ is holomorphic, $u \in \mathcal{H}(F,E)$ and

$$u_j \xrightarrow[\mathcal{H}(F,E)]{} u \ .$$

Step 2: The statements concerning φ and $\bar{\varphi}$ follows directly from the defininitions. It is also clear that $[E,F]_\theta$ is a Banach space.
Step 3 ($\theta \in \{0,1\}$): For any $u \in \mathcal{H}(E,F)$, $u(0) \in E$ and $u(1) = u(1+i\cdot 0) \in F$. Consequently

$$\bar{\varphi}_0 : \bar{\mathcal{H}}(F,E) \to [F,E]_0, \qquad \bar{\varphi}_1 : \bar{\mathcal{H}}(F,E) \to [F,E]_1.$$

□

3.3.3 Theorem (Interpolation operators)**.** Assume E, E', F, F' are Banach spaces such that there are continuous inclusions $E \hookrightarrow F$, $E' \hookrightarrow F'$. Let $T : F \to F'$ be a bounded linear operator, such that $T(E) \subset E'$. For any $0 \leq \theta \leq 1$

$$T : [F,E]_\theta \to [F',E']_\theta$$

is a bounded linear operator.

Proof. Let $u(\theta) \in [F,E]_\theta$. Since T is linear and continuous $T \circ u \in \mathcal{H}(F', E')$. Thus

$$\|T(u(\theta))\|_{[F',E']_\theta} = \|\varphi^{-1}((T \circ u)\|_{\mathcal{H}(F',E')} \leq \|T\| \|u\|_{[F,E]_\theta}.$$

□

4. Function Spaces

4.0.4 Definition (compactly contained)**.** Let X be a topological space and $A \subset B \subset X$. Then A is *compactly contained in* B,

$$A \Subset B \ :\Longleftrightarrow \bar{A} \subset B^\circ$$

and $\bar{A}$ is compact.

4.1. Continuously differentiable functions

4.1.1 Definition. Let $U \subset \mathbb{R}^m$ be open. For any $k \in \mathbb{N}$ let $\mathscr{C}^k(U, \mathbb{C}^r)$ be the space of k-times continuously differentiable functions.
Let $U \mathring{\subseteq} \mathbb{R}^m$ be open and bounded (hence $\bar{U}$ is compact). For any $k \in \mathbb{N}$ let

$$\mathscr{C}^k(\bar{U}, \mathbb{C}^r) := \{f \in \mathscr{C}^0(\bar{U}, \mathbb{C}^r) \mid f|_U \in \mathscr{C}^k(U, \mathbb{C}^r)\}$$

endowed with the norm

$$\|f\|_{\mathscr{C}^k(\bar{U})} := \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{\mathscr{C}^0(\bar{U})}.$$

4.1.2 Definition. Let $\pi : E \to M$ be a smooth vector bundle of rank r over a compact manifold M. Denote by $\Gamma^k(M, E)$ the space of $\mathscr{C}^k$ sections in E. The topology in $\Gamma^k(M, E)$ is defined as follows: Let $\{\varphi_i : \bar{U}_i \subset M \to \bar{V}_i \subset \mathbb{R}^m\}_{i \in I}$ be a finite cover of M by compact coordinate neighbourhoods such that there exist trivializations $\Phi_i : E_i := E_{U_i} \to U_i \times \mathbb{C}^r$. For any section $s \in \Gamma^k(M, E)$ define

$$\|s\|_{\mathscr{C}^k(M)} := \sum_{i \in I} \|\varphi_{i*} \Phi_{i*} s\|_{\mathscr{C}^k(\bar{V}_i)}$$

4.1.3 Lemma. The $\mathscr{C}^k$ topology on $\Gamma^k(M, E)$ is independent of the choice of charts and trivializations.

Proof. Assume $\{\tilde{\varphi}_j : \bar{U}_j \to \bar{V}_j\}_{j \in J}$ is another such cover of M. Clearly, for any $j \in J$ and any $i \in I$ such that $U_i \cap U_j \neq \emptyset$

$$\sum_{j \in J} \|\varphi_{j*} \Phi_{j*} s\|_{\mathscr{C}^k(\bar{V}_j)}$$

$$\leq \sum_{i \in I} \|\varphi_{i*} \Phi_{i*} s\|_{\mathscr{C}^k(\bar{V}_i)}$$

□

4.2. The Space of smooth (compactly supported) Functions

4.2.1 Definition ($\mathscr{E}$,$\mathscr{D}$,$\mathscr{D}_K$)**.** Let $U \subset \mathbb{R}^n$ be open. Define

$$\mathscr{E}(U, \mathbb{C}^r) := \mathcal{C}^\infty(U, \mathbb{C}^r) := \{f : U \to \mathbb{C}^r \mid f \text{ is smooth}\}.$$

Remember that if X is any topological space and $f : X \to \mathbb{C}$ is a function

$$\operatorname{supp} f := \overline{\{x \in X \mid f(x) \neq 0\}}.$$

If $K \subset U$ is compact

$$\mathscr{D}_K(U, \mathbb{C}^r) := \{f \in \mathcal{E}(U, \mathbb{C}^r) \mid \operatorname{supp} f \subset K\},$$

and

$$\mathscr{D}(U, \mathbb{C}^r) := \mathcal{C}_c^\infty(U, \mathbb{C}^r) := \{f \in \mathscr{E}(U, \mathbb{C}^r) \mid \operatorname{supp} f \text{ is compact}\}.$$

To simplify notation, we will sometimes just write $\mathscr{E}, \mathscr{D}, \mathscr{D}_K$.

It makes no apparent sense to introduce the new letters $\mathscr{E}$, $\mathscr{D}$ for the well-known spaces $\mathcal{C}^\infty$, $\mathcal{C}_c^\infty$. The reason for this is that both are sets at the moment. We use $\mathscr{E}$, $\mathscr{D}$ in order to stress the fact, that we see them as topological spaces, where the topology is given by the next theorem.

4.2.2 Theorem (Topologization of $\mathscr{E}$,$\mathscr{D}_K$)**.** Let $\emptyset \neq U \subset \mathbb{R}^n$ be open and $K \subset U$ be compact. Let $(K_m)_{m\in\mathbb{N}}$ be a compact exhaustion of U, i.e. all $K_m \subset U$ are compact, $K_m \Subset K_{m+1}$, $U = \bigcup_{m\in\mathbb{N}} K_m$. Then the maps $p_m : \mathscr{E}(U, \mathbb{C}^r) \to \mathbb{R}$

$$p_m(f) := \|f\|_{\mathcal{C}^m(K_m)} := \max_{x\in K_m, |\alpha|\leq m} \|\partial^\alpha f\|(x)$$

assemble to a separating family $\mathfrak{P} := \{p_m\}_{m\in\mathbb{N}}$ of semi-norms and induce a Fréchet-space topology on $\mathscr{E}$, such that $\mathscr{E}$ has the Heine-Borel property and such that $\mathscr{D}_K \subset \mathscr{E}$ is a closed subspace. Therefore $\mathscr{D}_K$ is a Fréchet space as well and its topology is induced by the semi-norms

$$\mathfrak{P}_K := \{\|\_\|_{\mathcal{C}^m(U)} \mid m \in \mathbb{N}\}.$$

Proof.
Step 1 (Topologization): The family $\mathfrak{P}$ is a countable family of separating seminorms. By Theorem 3.1.20 they induce a topology on $\mathscr{E}$, which turns $\mathscr{E}$ into a topological vector space that is locally convex and metrizable by a translation-invariant metric.
Step 2 (Completeness): So the only property $\mathscr{E}$ does not yet posses in order to be a Fréchet space is the completness. Therefore let (f_j) be a Cauchy sequence in $\mathscr{E}$. Thus for any $m \in \mathbb{N}$, the (f_j) are a $\|\_\|_{\mathcal{C}^m(U)}$-Cauchy sequence. Since this is a Banach space, f_j converges uniformly on every compact subset with all its derivatives to some $f \in \mathscr{E}$.
Step 3 (Closedness of $\mathscr{D}_K$): For any $x \in U$ define $\delta_x : \mathscr{E}(U, \mathbb{C}^r) \to \mathbb{C}^r$, $f \mapsto f(x)$. We claim that δ_x is continuous. By 3.1.11 is suffices to show that it is bounded. So let $E \subset \mathscr{E}(U, \mathbb{C}^r)$ be bounded. By 3.1.20 this is the case if and only if all the p_m are bounded on E. Since $x \in U$ and (K_m) is a compact exhaustion, there exists $m \in \mathbb{N}$ such that $x \in K_m$. This implies

$$\forall f \in E : \|\delta_x(f)\| = \|f(x)\| \leq \|f\|_{\mathcal{C}^0(K_m)} \leq \sup_{f\in E} p_m(f) =: R,$$

thus $\delta_x(E) \subset B_R(0)$. So all the δ_x are continuous and

$$\mathscr{D}_K(U, \mathbb{C}^r) = \bigcap_{x\in U\setminus K} \ker \delta_x$$

is closed as an intersection of closed spaces.
Step 4 (Heine-Borel property): Let $E \subset \mathscr{E}(U)$ be closed and bounded...

a sketch of proof is in Rudin, p.35

□

4.2.3 Convention. Without further reference we will always assume the spaces $\mathscr{E}$ and $\mathscr{D}_K$ to be topologized as described in the previous Theorem 4.2.2.

We now proceed to the topologization of $\mathscr{D}$, which is a bit more subtle. We could see $\mathscr{D}$ as an LF-space, i.e. an inductive limit of Frechét spaces (c.f. [6, 13]). This would require an even deeper discussion of the general theory of topological vector spaces. Therefore we follow [5, 6.3-6.6] for a more direct yet less general approach.

4.2.4 Theorem (Topologization of $\mathscr{D}$)**.** Let $\emptyset \neq U \subset \mathbb{R}^n$ be open. For any compact subset $K \subset U$ let τ_K be the Fréchet space topology on $\mathscr{D}_K(U, \mathbb{C}^r)$. For any $m \in \mathbb{N}$ define the norms

$$\|f\|_m := \max_{|\alpha| \leq m, x \in U} \|\partial^\alpha f\|(x)$$

on $\mathscr{D}(U, \mathbb{C}^r)$. Define

$$\begin{aligned}\mathcal{B} := \{W \subset \mathscr{D}(U, \mathbb{C}^r) \,|W \text{ is convex, balanced and for any} \\ K \Subset U : W \cap \mathscr{D}_K(U, \mathbb{C}^r) \in \tau_K\}.\end{aligned}$$

Then $\mathcal{B}$ is intersection stable family of sets all of which contain 0 (since balanced sets always do) and the family

$$\{f + W \mid f \in \mathscr{D}(U, \mathbb{C}^r), W \in \mathcal{B}\}.$$

is a basis for a topology τ such that $(\mathscr{D}(U, \mathbb{C}^r), \tau)$ is a locally convex topological vector space and $\mathcal{B}$ is a local base for τ.

Proof. For simplicity we write $\mathscr{D}_K := \mathscr{D}_K(U, \mathbb{C}^r)$, $\mathscr{D} := \mathscr{D}(U, \mathbb{C}^r)$.
STEP 1 (topology): Let $V_1, V_2 \in \tau$ and $f \in V_1 \cap V_2$. It suffices to show that

$$\exists W \in \mathcal{B} : f + W \subset (V_1 \cap V_2). \tag{4.1}$$

By definition of τ, there exist $f_\nu \in \mathscr{D}$, $W_\nu \in \mathcal{B}$, $\nu = 1, 2$, such that

$$f \in (f_\nu + W_\nu) \subset V_\nu. \tag{4.2}$$

Choose a $K \Subset U$ such that $f, f_1, f_2 \in \mathscr{D}_K$. By construction

$$f - f_\nu \in W_\nu \cap \mathscr{D}_K \overset{\circ}{\subseteq} \mathscr{D}_K. \tag{4.3}$$

Now we claim

$$\exists \delta_\nu > 0 : f - f_\nu \in (1 - \delta_\nu) W_\nu. \tag{4.4}$$

The existence of δ_ν follows by contradiction: If for all $\delta_\nu > 0$, $f - f_\nu \notin (1 - \delta_\nu) W_\nu \cap \mathscr{D}_K$, this implies

$$\tfrac{f - f_\nu}{1 - \delta_\nu} \xrightarrow[\mathscr{D}_K]{\delta \to 0} f - f_\nu \ \notin W_\nu,$$

since the complement of $W_\nu \cap \mathscr{D}_K$ in $\mathscr{D}_K$ is closed. This contradicts (4.3).
By hypothesis W_ν is convex, thus

$$f - f_\nu + \delta_\nu W_\nu \overset{(4.4)}{\subseteq} (1 - \delta_\nu) W_\nu + \delta_\nu W_\nu = W_\nu \tag{4.5}$$

Thus

$$f + \delta_\nu W_\nu \overset{(4.5)}{\subseteq} f_\nu + W_\nu \overset{(4.2)}{\subset} V_\nu.$$

Therefore the set $W := \delta_1 W_1 \cap \delta_2 W_2 \in \mathcal{B}$ satisfies (4.1).

STEP 2 (Hausdorff property): Let $f_1 \neq f_2 \in \mathscr{D}$. Define

$$W := \{f \in \mathscr{D} \mid \|f\|_{\mathcal{C}^0(U)} < \underbrace{\|f_1 - f_2\|\|_{\mathcal{C}^0(U)}}_{\neq 0}\} \in \mathcal{B},$$

since W is clearly convex and balanced; the fact that $W \cap \mathscr{D}_K \in \tau_K$ follows from the definition of τ_K by the family of norms, which includes $\|\_\|_{\mathcal{C}^0(U)}$. Clearly $f_1 \notin f_2 + W$, thus $\{f_1\}$ is closed in $\mathscr{D}$.

STEP 3 (addition): Since all the $W \in \mathcal{B}$ are convex,

$$\forall f_1, f_2 \in \mathscr{D} : \forall W \in \mathcal{B} : (f_1 + \frac{1}{2}W) + (f_2 + \frac{1}{2}W) = f_1 + f_2 + W.$$

This shows that addition is continuous.

STEP 4 (scalar multiplication): First we claim

$$\forall f_0 \in \mathscr{D} : \forall W \in \mathcal{B} : \exists \delta > 0 : \delta f_0 \in \frac{1}{2}W. \tag{4.6}$$

Again this follows by contradiction: Assume there exists $f_0 \in \mathscr{D}_K \subset \mathscr{D}$ such that for all $\delta > 0$, $\delta f_0 \notin \frac{1}{2}W$. This implies

$$2\delta f_0 \xrightarrow[\delta \to 0]{\mathscr{D}_K} 0 \;\notin W,$$

which contradicts the assumption that W is balanced.

So let $\alpha_0 \in \mathbb{C}$, $f_0 \in \mathscr{D}$ and choose $W \in \mathfrak{B}$, and $\delta > 0$ such that (4.6) holds. Define

$$c := \frac{1}{2c(|\alpha_0| + \delta)}.$$

We calculate for any $\alpha \in B_\delta(\alpha_0)$ and any $f \in f_0 + cW$

$$\begin{aligned} \alpha f - \alpha_0 f_0 &= \alpha(f - f_0) + (\alpha - \alpha_0) f_0 \in \alpha c W + \frac{\alpha - \alpha_0}{\delta} \delta f_0 \\ &\subset \frac{1}{2} \underbrace{\frac{\alpha}{c(|\alpha_0| + \delta)}}_{|\_| \leq 1} W + \underbrace{\frac{\alpha - \alpha_0}{\delta}}_{|\_| \leq 1} \frac{1}{2} W \subset \frac{1}{2}W + \frac{1}{2}W = W. \end{aligned}$$

□

4.2.5 Convention. From now on we will always assume that $\mathscr{D} := \mathscr{D}(U, \mathbb{C}^r)$ is endowed with the topology τ defined in 4.2.4.

4.2.6 Theorem (Properties of $\mathscr{D}(U, \mathbb{C}^r)$)**.** The space $(\mathscr{D} := \mathscr{D}(U, \mathbb{C}^r), \tau)$ has the following properties:

(i) A convex balanced subset $V \subset \mathscr{D}$ is open if and only if $V \in \mathcal{B}$, c.f. 4.2.4.

(ii) The topology τ_K conicides with the subspace topology $\tau \cap \mathscr{D}_K$.

(iii) If $E \subset \mathscr{D}$ is bounded, then there exists a $K \Subset U$ such that $E \subset \mathscr{D}_K$ and E is bounded in $\mathscr{D}_K$. Consequently there are numbers $M_N \in \mathbb{R}$ such that

$$\forall f \in E : \forall N \in \mathbb{N} : \|\varphi\|_N \leq M_N.$$

(iv) $\mathscr{D}$ has the Heine-Borel property.

(v) If (f_i) is a Cauchy sequence in $\mathscr{D}$, then there exists a $K \Subset U$ such that $\{f_i\} \subset \mathscr{D}_K$ and for any $N \in \mathbb{N}$ (f_i) is a $\|\_\|_N$-Cauchy sequence.

(vi) Let $(f)_i$ be a sequence in $\mathscr{D}$. Then

$$f_i \xrightarrow[\mathscr{D}]{} f$$

if and only if there exists a compact subset $K \subset U$ such that for all $i \in \mathbb{N}$, $\operatorname{supp} f_i \subset K$ and

$$\forall N \in \mathbb{N} : \; f_i \xrightarrow[\mathcal{C}^N(K)]{} f \; .$$

(vii) The space $\mathscr{D}$ is complete.

Proof.

(i) "$\tau \subset \mathcal{B}$": Let $V \in \tau$ be convex and balanced and $K \Subset U$. For any $f \in \mathscr{D}_K \cap V$, there exists $W \in \mathcal{B}$ such that $f + W \subset V$ by 4.2.4. Thus

$$f + (\mathscr{D}_K \cap W) = \mathscr{D}_K \cap (f + W) \subset \mathscr{D}_K \cap V.$$

By definition $\mathscr{D}_K \cap W \in \tau_K$, hence $\mathscr{D}_K \cap W$ is a neighbourhood of f. Since f was arbitrary, we have shown

$$\forall V \in \tau : \forall K \Subset U : \mathscr{D}_K \cap V \in \tau_K. \tag{4.7}$$

Consequently $\tau \subset \mathcal{B}$.
"$\mathcal{B} \subset \tau$": This follows from the definition.

(ii) "$(\tau \cap \mathscr{D}_K) \subset \tau_K$": This follows directly from (4.7).
"$\tau_K \subset (\mathscr{D}_K \cap \tau)$": Let $E \in \tau_K$. We have to construct a $V \in \tau$ such that $E = \mathscr{D}_K \cap V$. By definition of τ_K via a family of norms, for any $f \in E$ there exists $m \in \mathbb{N}$, $\delta > 0$ such that

$$\{g \in \mathscr{D}_K \mid \|g - f\|_m < \delta\} \subset E.$$

This uses the fact that the semi-balls in τ_K are actually balls since $\|\_\|_m \leq \|\_\|_{m+1}$ (c.f. 4.2.4). Now define

$$W_f := \{g \in \mathscr{D} \mid \|g\|_m < \delta\} \in \mathcal{B}.$$

This implies

$$\mathscr{D}_K \cap (f + W_f) = f + (\mathscr{D}_K \cap W_f) \subset E$$

by definition of W_f and m. Consequently the set

$$V := \bigcup_{f \in E} W_f$$

satisfies

$$\mathscr{D}_K \cap V = \bigcup_{f \in E} \mathscr{D}_K \cap W_f \subset E.$$

Since $E \subset \mathscr{D}_K \cap V$ anyway (note that $0 \in W_f$), this implies the statement.

(iii) Assume that $E \subset \mathscr{D}$ is a set such that $E \subsetneq \mathscr{D}_K$ for any $K \Subset U$. Then there exists a compact exhaustion $K_m \Subset U$, $f_m \in \mathscr{D}_{K_m}$, $x_m \in K_m \setminus K_{m-1}$ such that $f_m \notin \mathscr{D}_{K_{m-1}}$, $f_m(x_m) \neq 0$ and such that the sequence (x_m) has no limit point in U. Define

$$W := \{f \in \mathscr{D} \mid \forall m \in \mathbb{N} : \|f(x_m)\| < \frac{1}{m}\|f_m(x_m)\|\}$$

We claim that $W \in \mathcal{B}$: Notice that

$$\forall K \Subset U : \exists N_0 \in \mathbb{N} : \forall m \geq N_0 : x_m \notin K.$$

Therefore, by choosing

$$0 < \varepsilon < \min_{m \leq N_0} \frac{1}{m}\|f_m(x_m)\|,$$

we obtain that for any $f \in \mathscr{D}_K \cap W$

$$B^0_\varepsilon(f) \subset W,$$

where the ball is formed with respect to the $\|\_\|_0$-norm. Thus $\mathscr{D}_K \cap W \in \tau_K$, hence $W \in \mathcal{B}$. Now this implies that for any $m \in \mathbb{N}$

$$E \subsetneq mW,$$

since $f_m \notin mW$. Consequently E is not τ-bounded in $\mathscr{D}$.
For the second part, recall that by definition $E \subset \mathscr{D}$ is τ-bounded, iff for any τ-neighbourhood W of 0:

$$\exists s > 0 : \forall t > t : E \subset tW.$$

Let $W_K \subset \mathscr{D}_K$ be a τ_K-neighbourhood of 0. By (ii) there exists a τ-neighbourhood $\tilde{W}$ such that $W_K = \tilde{W} \cap \mathscr{D}_K$. Thus for s, t as above

$$E = E \cap \mathscr{D}_K \subset t(\tilde{W} \cap \mathscr{D}_K) = tW_K.$$

Since $\mathscr{D}_K$ carries a Frechét space topology, the rest follows from 3.1.20.

(iv) Assume $E \subset \mathscr{D}$ is closed and τ-bounded. By (iii) there exists $K \Subset U$ such that $E \subset \mathscr{D}_K$ and E is τ_K-bounded. By (ii) E is also τ_K-closed. Since $\mathscr{D}_K$ is Heine-Borel by 4.2.4 E is τ_K-compact. Again by (ii), E is τ-compact.

(v) By the Cauchy sequence (f_i) is τ-bounded. By (iii) there exists $K \Subset U$ such that $\{f_i\} \subset \mathscr{D}_K$ and by (ii), (f_i) is also a τ_K-Cauchy sequence. The rest of the claim follows from 3.1.20 defining the topology τ_K. ref, Rudin 1.29 c)

(vi) Since any convergent sequence is Cauchy, the existence of K follows from (v). By (ii) (f_i) is a τ_K-Cauchy sequence. Since $\mathscr{D}_K$ is complete by 4.2.4 (f_i) converges in $\mathscr{D}_K$, which implies that it converges in all the $\mathscr{C}^N$-norms by 3.1.20. Since $\mathscr{D}_K$ carries the subspace topology and since $\mathscr{D}$ is Hausdorff, the two limits agree.

(vii) Any Cauchy sequence in $\mathscr{D}$ is a Cauchy sequence in some $\mathscr{D}_K$ by (v). Since $\mathscr{D}_K$ is complete, it has a τ_K-limit in $\mathscr{D}_K$ and again this equals the τ-limit of the sequence.

□

4.2.7 Theorem (charactarization of linear operators)**.** Let Y be a locally convex TVS and $T : \mathscr{D} := \mathscr{D}(U, \mathbb{C}^r) \to Y$ be linear. Then the following are equivalent:

(i) T is continuous.

(ii) T is bounded.

(iii) For any sequence $(f_i) \in \mathscr{D}$

$$f_i \xrightarrow[\mathscr{D}]{} 0 \implies T(f_i) \xrightarrow[Y]{} 0 .$$

(iv) For any $K \Subset U$, $T|\mathscr{D}_K : \mathscr{D}_K \to Y$ is continuous.

Proof.
"(i)⇒(ii): This follows from 3.1.11.
"(ii)⇒(iii): Let T be bounded and let (f_i) be a sequence such that $f_i \xrightarrow[\mathscr{D}]{} 0$. By 4.2.6,(vi) there exists $K \Subset U$ such that

$$f_i \xrightarrow[\mathscr{D}_K]{} 0 .$$

Clearly $T|\mathscr{D}_K$ is also bounded. Therefore 3.1.11 applied to $T|\mathscr{D}_K$ implies the claim.
"(iii)⇒(iv)": Since $\mathscr{D}_K$ is metrizable $T|\mathscr{D}_K$ is continuous if and only if it is sequentially continuous. Thus if $f_i \xrightarrow[\mathscr{D}_K]{} 0$, thus by 4.2.6,(ii) $f_i \xrightarrow[\mathscr{D}]{} 0$. By hypothesis, this implies $T(f_i) \xrightarrow[Y]{} 0$. Thus $T|\mathscr{D}_K$ is continuous.
"(iv)⇒(i)": It suffices to check that the reversed images of a local base in Y under T are open in $\mathscr{D}$. So let $W \subset Y$ be a convex balanced neighbourhood of 0. This implies $V := T^{-1}(W)$ is convex and balanced in $\mathscr{D}$, since T is linear. Now for any $K \Subset U$

$$V \cap \mathscr{D}_K = T^{-1}(W) \cap \mathscr{D}_K = (T|\mathscr{D}_K)^{-1}(W) \in \tau_K$$

by hypothesis. By 4.2.6,(i) this implies $V \in \tau$. □

4.2.8 Corollary. For every $\alpha \in \mathbb{N}^n$, $|\alpha| = k$, the operator $D^\alpha \in \mathrm{Diff}^k(U, \mathbb{C}^r, \mathbb{C}^s)$ is a continuous map $D^\alpha : \mathscr{D}(U) \to \mathscr{D}(U)$.

Proof. By 4.2.7 it suffices to check that D^α is continuous from $\mathscr{D}_K(U) \to \mathscr{D}_K(U)$. But for any $m \in \mathbb{N}$

$$\forall f \in \mathscr{D}_K(U, \mathbb{C}^r) : \|D^\alpha f\|_m \leq \|f\|_{k+m},$$

thus $D^\alpha : \mathscr{D}_K \to \mathscr{D}_K$ is continuous by definition 4.2.4 of the topology on $\mathscr{D}_K$. □

4.2.9 Theorem (Smooth Urysohn)**.** Let $K \subset U \subset \mathbb{R}^n$, K compact, U open. Then there exists $\Phi \in \mathscr{D}(U)$, such that

$$\Phi|_K = 1.$$

4.2.10 Theorem (Sum Decomposition)**.** Let $U_1, \ldots, U_k \subset \mathbb{R}^n$ be open and let $U := \bigcup_{j=1}^k U_j$. For any $\varphi \in \mathscr{D}(U)$ there exist $\varphi_j \in \mathscr{D}(U_j)$, such that

$$\varphi = \sum_{j=1}^k \varphi_j.$$

4.3. L_p-Spaces

We assume the reader to be familiar with the notion of L_p-spaces. We will nevertheless introduce some notation and briefly discuss the vector-valued case. For a very elaborate discussion of this topic, the reader may consult [1].

4.3.1 Definition (L_p-space)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(Y, \|\_\|)$ be a $\mathbb{K}$-Banach space. Define

$$L^0(X,Y) := \{f : X \to Y \mid f \text{ is measurable}\},$$

and for any $1 \leq p < \infty$

$$L^p(X,Y) := \{f \in L^0(X,Y) \mid \|f\|_p^p := \|f\|_{L^p(X,Y)}^p := \int_X \|f(y)\|_Y^p d\mu < \infty\}.$$

In case $p = \infty$, we define

$$L^\infty(X,Y) := \{f \in L^0(X,Y) \mid \|f\|_{L^p(X,Y)} := \text{ess sup}_{x\in X} \|f(x)\|_Y < \infty.\}$$

In both cases $L^p(X,Y)$ denotes the space of all those functions modulo equality a.e. In case X is a topological space, we define

$$L^p_{\text{loc}}(X,Y) := \{f \in L^0(X,Y) \mid \forall K \Subset X : f \in L^p(K,Y)\}.$$

Notice that equivalent norms on Y produce equivalent associated L_p-norms. In particular, if Y is finite dimensional, the topology generated on $L_p(X,Y)$ does not depend on the choice of the norm on Y. In particular, if $Y = \mathbb{C}^r$, a natural choice would be $\|\_\|_Y := \|\_\|_p$, where $\|\_\|_p$ is the p-norm on $\mathbb{C}^r$. On the other hand, by choosing the maximum norm on $\mathbb{C}^r$ one may treat the integration of function $f : \mathbb{R}^n \to \mathbb{C}^r$ almost as if one had r functions $f_i : \mathbb{R}^n \to \mathbb{C}$, which is often convenient.

4.4. Convolution, The Schwartz-Space and Fourier Transform

4.4.1. Convolution

4.4.1 Theorem and Definition (Convolution and Young's Inequality)**.** Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. The integral

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

exists for almost every $x \in \mathbb{R}^n$. Therefore it defines an L^p-class $f * g \in L^p(\mathbb{R}^n)$ called the *convolution of f and g.*
Furthermoore the convolution satisfies *Young's Inequality*

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

The same holds if $f \in L^p(\mathbb{R}^n)$, $g \in L^1(\mathbb{R}^n)$.

Proof. The proof works slightly different for the various cases of p, but the structure of argumentation will always rely on the two arguments: A function is integrable if and only if its absolute value is integrable and $\int_X h(x)dx < \infty$ implies $|h(x)| < \infty$ for almost every $x \in X$.
Step 1: Let $1 \leq p < \infty$ and define $h_p : \mathbb{R}^n \to \bar{\mathbb{C}}$

$$h_p(x) := \int_{\mathbb{R}^n} |f(x-y)||g(y)|^p dy$$

$$\|h_p\|_1 = \int_{\mathbb{R}^n} |h_p(x)|dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y)||g(y)|^p dydx = \int_{\mathbb{R}^n} |g(y)|^p \int_{\mathbb{R}^n} |f(x-y)|dxdy$$
$$= \|f\|_1 \int_{\mathbb{R}^n} |g(y)|^p dy = \|f\|_1 \|g\|_p^p.$$

Therefore $h_p(x) < \infty$ for almost every $x \in \mathbb{R}^n$. Since

$$\int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} f(x-y)g(y)dy \Big| dx \leq \int_{\mathbb{R}^n} h_1(x)dx < \infty,$$

both statement are already proven for $p = 1$.
Step 2 ($1 < p < \infty$): Let q be the Hölder conjugate index of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Then Hölder's inequality implies

$$\int_{\mathbb{R}^n} |f(x-y)||g(y)|dy = \int_{\mathbb{R}^n} |f(x-y)|^{\frac{1}{q}} (|f(x-y)|^{\frac{1}{p}} |g(y)|) dy$$
$$\leq \Big(\int_{\mathbb{R}^n} |f(x-y)|dy \Big)^{\frac{1}{q}} \Big(\int_{\mathbb{R}^n} |f(x-y)||g(y)|^p dy \Big)^{\frac{1}{p}} = \|f\|_1^{\frac{1}{q}} \|h_p\|_1^{\frac{1}{p}} < \infty.$$

This proves the existence claim and Young's inequality is proven by

$$\|f * g\|_p^p = \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} f(x-y)g(y)dy \Big|^p dx \leq \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |f(x-y)||g(y)|dy \Big)^p dx$$
$$\leq \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |f(x-y)|^{\frac{1}{q}} \cdot (|f(x-y)|^{\frac{1}{p}} |g(y)|) dy \Big)^p dx$$
$$\leq \int_{\mathbb{R}^n} \Big(\Big(\int_{\mathbb{R}^n} |f(x-y)|dy \Big)^{\frac{1}{q}} \Big(\int_{\mathbb{R}^n} |f(x-y)||g(y)|^p) dy \Big)^{\frac{1}{p}} \Big)^p dx$$
$$\leq \|f\|_1^{\frac{p}{q}} \|h_p\|_1 \leq \|f\|_1^{\frac{p}{q}} \|f\|_1 \|g\|_p^p,$$

thus $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.
Step 3 ($p = \infty$): In that case we may simply argue that

$$\int_{\mathbb{R}^n} |f(x-y)||g(y)|dy \leq \|g\|_\infty \int_{\mathbb{R}^n} |f(x-y)|dy = \|g\|_\infty \|f\|_1.$$

□

4.4.2 Theorem (Properties of Convolutions)**.** Let $f, g, h \in L^1(\mathbb{R}^n)$.

(i) Bilinearity: $\forall \lambda, \mu \in \mathbb{C} : (\lambda f) * (\mu g) = \lambda\mu \cdot f * g$.

(ii) Integral Identity: $\int_{\mathbb{R}^n} (f * g)(x)dx = \int_{\mathbb{R}^n} f(x)dx \int_{\mathbb{R}^n} g(y)dy$.

(iii) Associativity: $(f * g) * h = f * (g * h)$.

(iv) Commutativity: $f * g = g * f$.

(v) Support: $\operatorname{supp}(f * g) \subset \operatorname{supp} f + \operatorname{supp} g$.

4.4.3 Theorem (Differentiation Theorem)**.** Let $f \in L^1(\mathbb{R}^n)$, $g \in \mathcal{C}_b^k(\mathbb{R}^n)$. Then $f * g \in \mathcal{C}^k(\mathbb{R}^n)$ and for any $\alpha \in \mathbb{N}^n$, $|\alpha| \leq k$,

$$\partial^\alpha (f * g) = f * (\partial^\alpha g).$$

4.4.4 Definition (Dirac sequence)**.** A sequence of functions $\delta_k \in L^1(\mathbb{R}^n)$ is a *Dirac-sequence* if

(i) $\forall k \in \mathbb{N} : \delta_k \geq 0$.

(ii) $\forall k \in \mathbb{N} : \int_{\mathbb{R}^n} \delta_k(x)dx =: c \in \mathbb{C}$.

(iii) For any ball $B_r(0)$: $\lim_{k\to\infty} \int_{\mathbb{R}^n \setminus B_r(0)} \delta_k(x)dx = 0$.

In case $c = 1$ the sequence is *normalized*.

4.4.5 Theorem and Definition (Existence of Dirac sequences)**.** Define $\psi : \mathbb{R} \to \mathbb{R}_{\geq 0}$ by

$$t \mapsto \begin{cases} \exp(-\frac{1}{t}) & , t > 0 \\ 0 & , t \leq 0 \end{cases}$$

and for any $\varepsilon > 0$ define $\eta, \eta_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ by

$$c^{-1} := \int_{\mathbb{R}^n} \psi(1 - |x|^2)dx, \qquad \eta(x) := c\psi(1 - |x|^2), \qquad \eta_\varepsilon(x) := \frac{1}{\varepsilon^n}\eta\Big(\frac{x}{\varepsilon}\Big).$$

Then η_ε is the *standard mollifier* and satisfies

(i) $\psi \in \mathcal{C}^\infty(\mathbb{R})$,

(ii) $\eta_\varepsilon \in \mathcal{C}_c^\infty(\mathbb{R}^n)$,

(iii) $\operatorname{supp} \eta_\varepsilon \subset B_\varepsilon(0)$,

(iv) $0 \leq \psi \leq 1$, $0 \leq \eta \leq c$, $0 \leq \eta_\varepsilon \leq \frac{c}{\varepsilon^n}$,

(v) $\int_{\mathbb{R}^n} \eta_\varepsilon(x)dx = 1$.

In particular $\delta_k := \eta_{\frac{1}{k}}$ is a normalized Dirac sequence, the *standard Dirac sequence.*

Proof.

Step 1 ($\psi \in \mathcal{C}^\infty(\mathbb{R})$): This is the decicive point! We will show by induction over n that there are polynomials $p_{2n} \in \mathbb{R}[X]$ satisfying $\deg(p_{2n}) \leq 2n$, such that the n-th derivative of ψ satisfies

$$\psi^{(n)}(t) = p_{2n}(t^{-1})\psi(t).$$

In case $n = 0$ this is clear. For the induction step $n \to n+1$ consider any $t > 0$ and calculate

$$\begin{aligned} \psi^{(n+1)}(t) &= (\psi^{(n)})'(t) = (p_{2n}(t^{-1})e^{-t^{-1}})' = -t^{-2}p_{2n}'(t^{-1})e^{-t^{-1}} - p_{2n}(t^{-1})t^{-2}e^{-t^{-1}} \\ &= \underbrace{(p_{2n}(t^{-1})t^{-2} - t^{-2}p_{2n}'(t^{-1}))}_{=:p_{2(n+1)}(t^{-1})} e^{-t^{-1}}. \end{aligned}$$

Clearly $\deg p_{2(n+1)} \leq 2(n+1)$.

In case $t < 0$ we obtain $\psi^{(n)}(t) = 0$ by definition. Since the exp growth faster than any polynomial

$$\lim_{t\searrow 0} \psi^{(n)}(t) = \lim_{t\searrow 0} p_{2n}(t^{-1})\exp(-t^{-1}) = 0 = \lim_{t\nearrow 0} \psi^{(n)}(t).$$

This implies that every $\psi^{(n)}$ exists and is continuous. Therefore ψ is smooth.

Step 2 (c well-defined): If $|x| > 1$, then $\psi(1 - |x|^2) = 0$. Therefore

$$0 < \int_{\mathbb{R}^n} \psi(1 - |x|^2)dx = \int_{B_1(0)} \psi(1 - |x|^2) < \infty.$$

Step 3 ($\operatorname{supp} \eta_\varepsilon \subset B_\varepsilon(0)$): First we analyse the support of η. Since $c \neq 0$, we obtain

$$\psi(1 - |x|^2) = 0 \Leftrightarrow 1 - |x|^2 \leq 0 \Leftrightarrow |x|^2 \geq 1 \Leftrightarrow x \notin B_1(0).$$

Therefore $\operatorname{supp} \eta = \bar{B}_1(0)$. Similar

$$0 = \eta_\varepsilon(x) = \varepsilon^n \psi(1 - |x/\varepsilon|^2) \Leftrightarrow \frac{x}{\varepsilon} \notin B_1(0) \Leftrightarrow x \notin B_\varepsilon(0).$$

Step 4 (Range): By definition if $t < 0$, $\psi(t) = 0$. Since $\psi'(t) > 0$ on $\mathbb{R}_+$

$$0 = \lim_{t \searrow 0} \psi(t) \leq \psi \leq \lim_{t \nearrow \infty} \psi(t) = 1.$$

This implies the estimates.
Step 5 ($\int \eta_\varepsilon = 1$): By the transformation theorem

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x) dx = \int_{\mathbb{R}^n} \eta(x/\varepsilon) \frac{1}{\varepsilon^n} dx = \int_{\mathbb{R}^n} \eta(x) dx = c \int_{\mathbb{R}^n} \psi(1 - |x|^2) dx = 1.$$

The other statements follow directly from what we have proven so far. □

4.4.6 Theorem (Approximation)**.** Let δ_k be a Dirac sequence. For any $f \in L^1(\mathbb{R}^n)$

$$f * \delta_k \xrightarrow[L^1]{k \to \infty} cf \ ,$$

where $c = \int_{\mathbb{R}^n} \delta_k(x) dx$.

4.4.7 Theorem. For any open set $U \subset \mathbb{R}^n$ the inculsion $\mathcal{C}_c^\infty(U) \to L^p(U)$ is continuous with dense image.

4.4.2. Schwartz Space

4.4.8 Definition (rapidly decreasing)**.** A function $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^r)$ is *rapidly decreasing*, if

$$\forall \alpha \in \mathbb{N}^n : \sup_{x \in \mathbb{R}^n} \|x^\alpha f(x)\| < \infty.$$

Various other characterizations are used throughout the literature.

4.4.9 Lemma (Characterization of rapidly decreasing functions)**.** Let $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^r)$ be arbitrary. The following are equivalent.

(i) f is rapidly decreasing.
(ii) $\forall \alpha \in \mathbb{N}^n : \lim_{|x| \to \infty} x^\alpha f(x) = 0$.
(iii) For all polynomials $p : \mathbb{R}^n \to \mathbb{C}$: $\lim_{|x| \to \infty} p(x) f(x) = 0$.
(iv) $\forall m \in \mathbb{N} : \lim_{|x| \to \infty} |x|^m f(x) = 0$.
(v) $\forall m \in \mathbb{N} : \sup_{x \in \mathbb{R}^n} (1 + |x|^m) \|f(x)\| < \infty$.
(vi) $\forall m \in \mathbb{N} : \sup_{x \in \mathbb{R}^n} (1 + |x|)^m \|f(x)\| < \infty$.

In the last three conditions one may replace $\mathbb{N}$ by an arbitrary unbounded subset of $\mathbb{N}$.

Proof.
"(i)⇒(ii)": Let $\alpha \in \mathbb{N}$ and let (x_j) be a sequence satisfying $\lim_{j \to \infty} |x_j| = \infty$. Then there exists at least one $1 \leq i \leq n$ such that the i-th component satisfies $\lim_{j \to \infty} |x_j^i| = \infty$ as well (in particular this implies $x_j^i \neq 0$ for large j). Apply (i) to $\beta := \alpha + e_i$ and obtain

$$\exists C > 0 : \forall x \in \mathbb{R}^n : \|x^\beta f(x)\| \leq C.$$

For large j this implies $\|x^\alpha f(x)\| \leq C/|x_j^i|$ and therefore

$$0 \leq \lim_{j\to\infty} \|x_j^\alpha f(x_j)\| \leq C \lim_{j\to\infty} \frac{1}{|x_j^i|} = 0.$$

"(ii)⇒(iii)": Any polynomial p has a representation $p = \sum_{k=0}^m \sum_{|\alpha|\leq k} c_\alpha x^\alpha$ for some constants $c_\alpha \in \mathbb{C}$. Therefore (ii) implies

$$\lim_{|x|\to\infty} p(x)f(x) = \sum_{k=0}^m \sum_{|\alpha|\leq k} c_\alpha \lim_{|x|\to\infty} x^\alpha f(x) = 0.$$

"(iii)⇒(iv)": Since the limit approaches infinity and since

$$\forall |x| > 1 : |x|^{m-1} \leq |x|^m \leq |x|^{m+1},$$

we may restrict our attention to even m. In that case

$$p(x) := \Big(\sum_{k=1}^n x_k^2\Big)^{m/2} = |x|^m$$

is a polynomial and therefore (iii) implies

$$\lim_{m\to\infty} |x|^m f(x) = \lim_{m\to\infty} p(x)f(x) = 0.$$

"(iv)⇒(v)": Applying (iv) to 0 and m, we obtain

$$0 = \lim_{|x|\to\infty} |x|^0 |f(x)| = \lim_{|x|\to\infty} |f(x)| \qquad \text{and} \qquad 0 = \lim_{|x|\to\infty} |x|^m |f(x)|.$$

Since f is smooth, this implies (v).
"(v)⇒(vi)": By the binomial theorem and (v)

$$(1+|x|)^m |f(x)|) = \sum_{k=0}^m \binom{m}{k} |x|^k |f(x)| < \infty.$$

"(vi)⇒(i)": Follows from

$$|x^\alpha| \overset{\text{A.2.1}}{\leq} |x|^{|\alpha|} \leq (1+|x|)^{|\alpha|}.$$

□

4.4.10 Definition (Schwartz space)**.** A function $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^r)$ is a *Schwartz-function*, if all its derivatives (including f itself) are rapidly decreasing. The collection $\mathscr{S}$ of these functions is the *Schwartz space*. Somewhat more explicitly

$$\mathscr{S} := \{f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^r) \mid \forall \alpha, \beta \in \mathbb{N}^n : p_{\alpha,\beta}^{\mathscr{S}}(f) := \sup_{x\in\mathbb{R}^n} \|x^\alpha D^\beta(f)(x)\| < \infty\}.$$

4.4.11 Theorem (Topologization of the Schwartz Space)**.** The set

$$\mathfrak{P} := \{p_{\alpha,\beta}^{\mathscr{S}} \mid \alpha, \beta \in \mathbb{N}^n\}$$

is a separating family of seminorms, which induce a Fréchet space topology on $\mathscr{S}$.

Proof. Denote $p_{\alpha,\beta} := p_{\alpha,\beta}^{\mathscr{S}}$.

Step 1: It follows directly from the definition that $p^{\mathscr{S}}_{\alpha,\beta}$ is a semi-norm and thus $\mathscr{S}$ is a vector space. If $f \neq 0$, then $p^{\mathscr{S}}_{0,0}(f) \neq 0$ and therefore $\mathfrak{P}$ is separating.
Consequently we may apply Theorem 3.1.20 to the family $\mathfrak{P}$ and obtain that $\mathscr{S}$ is a locally convex topological vector space. Since $\mathfrak{P}$ is obviously countable, the topology is metrizable by a translation invariant metric.

Step 2 (Completeneess): Let (f_j) be a Cauchy sequence in $\mathscr{S}$. By 3.1.20 this is equivalent to (f_j) being a $p_{\alpha,\beta}$-Cauchy sequence for all $\alpha, \beta \in \mathbb{N}^n$. On the one hand, this implies

$$\forall \alpha, \beta \in \mathbb{N}^n : \exists C_{\alpha,\beta} > 0 : \forall j \in \mathbb{N} : p_{\alpha,\beta}(f_j) \leq C_{\alpha,\beta}, \tag{4.8}$$

since Cauchy sequences are bounded. On the other hand this means that for every α, β the sequence $(x^\alpha D^\beta(f_j))$ is a $\mathcal{C}^0_b := \mathcal{C}^0_b(\mathbb{R}^n, \mathbb{C}^r)$-Cauchy sequence. Since the later is a Banach space,

$$\forall \alpha, \beta \in \mathbb{N}^n : \exists g_{\alpha,\beta} \in \mathcal{C}^0_b : \; x^\alpha D^\beta(f_j) \xrightarrow[\mathcal{C}^0_b]{j\to\infty} g_{\alpha,\beta} \, . \tag{4.9}$$

In particular this holds for $\alpha := 0$ and all β. By a standard theorem from calculus,this implies $f := g_{0,0} \in \mathcal{C}^\infty$ and $g_{0,\beta} = D^\beta(f)$. Since uniform convergence implies pointwise convergence, we deduce from (4.9) reference

$$\forall \alpha, \beta \in \mathbb{N}^n : \forall x \in \mathbb{R}^n : g_{\alpha,\beta}(x) = \lim_{j\to\infty} x^\alpha D^\beta(f_j)(x) = x^\alpha \lim_{j\to\infty} g_{0,\beta}(x) = x^\alpha D^\beta(f)(x).$$

By (4.9) this pointwise convergence is uniform, i.e.

$$x^\alpha D^\beta(f_j) \xrightarrow[\mathcal{C}^0_b]{} x^\alpha D^\beta(f) \, .$$

Thus

$$\forall \alpha, \beta \in \mathbb{N}^n : \; f_j \xrightarrow[p_{\alpha,\beta}]{} f \, ,$$

which by 3.1.20 on the one hand is equivalent to

$$\forall \alpha, \beta \in \mathbb{N}^n : \; f_j \xrightarrow[\mathscr{S}]{} f$$

and on the other hand implies

$$\forall \alpha, \beta \in \mathbb{N}^n : p_{\alpha,\beta}(f) = \lim_{j\to\infty} p_{\alpha,\beta}(f_j) \overset{(4.8)}{\leq} C_{\alpha,\beta}.$$

Consequently $f \in \mathscr{S}$.

□

Sometimes another topologization of the Schwarz space is used and useful.

4.4.12 Theorem (Equivalent seminorms)**.** For any $m \in \mathbb{N}$, $\beta \in \mathbb{N}^n$ define

$$\forall f \in \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{C}^r) : q_{m,\beta}(f) := \sup_{x\in\mathbb{R}^n} (1 + |x|)^m |D^\beta f(x)|.$$

We claim that the families of seminorms

$$\mathcal{P} := \{p_{\alpha,\beta} \mid \alpha, \beta \in \mathbb{N}^n\}, \qquad \mathcal{Q} := \{q_{m,\beta} \mid m \in \mathbb{N}, \beta \in \mathbb{N}^n\}$$

are equivalent on $\mathscr{S}$ in the sense of 3.1.23.

Proof. We already know from 4.4.9 that for any $f \in \mathscr{C}^\infty(U, \mathbb{C}^r)$

$$\forall \alpha, \beta : p_{\alpha,\beta}(f) < \infty \iff \forall m \in \mathbb{N} : \forall \beta \in \mathbb{N}^n : q_{m,\beta}(f) < \infty.$$

But this is not enough. We will prove the equvialence using the strategy explained in 3.1.25. To that end we choose any $f \in \mathscr{C}^\infty(U, \mathbb{C}^r)$ and carry out the following calculations:

STEP 1: For any $\alpha, \beta \in \mathbb{N}^n$

$$\begin{aligned} p_{\alpha,\beta}(f) &= \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta(f)(x)| \overset{\text{A.2.1}}{\leq} \sup_{x \in \mathbb{R}^n} |x|^{|\alpha|} |D^\beta(f)(x)| \\ &\leq \sup_{x \in \mathbb{R}^n} (1 + |x|)^{|\alpha|} |D^\beta(f)(x)| = q_{|\alpha|,\beta}(f). \end{aligned}$$

STEP 2: Let $m \in \mathbb{N}$, $\beta \in \mathbb{N}^n$. By Lemma A.2.2 for any $0 \leq k \leq m$ there are constants $c_\alpha^{(k)} > 0$ such that

$$|x|^k \leq \sum_{|\alpha| \leq 2k} c_\alpha^{(k)} |x^\alpha|$$

Consequently there exist constants $\tilde{c}_\alpha > 0$ such that

$$\sum_{k=0}^m |x|^k \leq \sum_{k=0}^m \sum_{|\alpha| \leq 2k} c_\alpha^{(k)} |x^\alpha| \leq \sum_{|\alpha| \leq 2m} |x^\alpha| \tag{4.10}$$

Thus

$$\begin{aligned} q_{m,\beta}(f) &= \sup_{x \in \mathbb{R}^n} (1 + |x|)^m |D^\beta(f)(x)| \leq \sup_{x \in \mathbb{R}^n} \sum_{k=0}^m |x|^k |D^\beta(f)(x)| \\ &\overset{(4.10)}{\leq} \sum_{|\alpha| \leq 2m} \sup_{x \in \mathbb{R}^n} |c_\alpha x^\alpha| |D^\beta(f)(x)| \leq \underbrace{\max_{|\alpha| \leq 2m} c_\alpha}_{=:C} \sum_{|\alpha| \leq 2m} p_{\alpha,\beta}(f). \end{aligned}$$

□

4.4.13 Theorem (Schwarz space and friends)**.** The Schwartz space $\mathscr{S} := \mathscr{S}(\mathbb{R}^n, \mathbb{C}^r)$ is related to various other important function spaces in the following manner.

(i) For any $k \in \mathbb{N}$

$$\mathcal{C}_c^k \subset \mathcal{C}_c^\infty \subset \mathscr{S} \subset \mathcal{C}_b^k \subset \mathcal{C}_b^\infty.$$

(ii) The inclusion $\mathscr{S} \hookrightarrow \mathcal{C}_b^k$ is continuous.

(iii) The inclusion $\mathscr{D} := \mathscr{D}(\mathbb{R}^n, \mathbb{C}^r) \hookrightarrow \mathscr{S}$ is continuous and

(iv) has dense image.

(v) For any $1 \leq p \leq \infty$, $\mathscr{S} \subset L^p := L^p(\mathbb{R}^n, \mathbb{C}^r)$ and the inclusion $\mathscr{S} \hookrightarrow L^p$ is continuous.

Proof. Let $p_{\alpha,\beta} := p_{\alpha,\beta}^{\mathscr{S}}$.

(i) This follows directly from the definitions of the $p_{\alpha,\beta}$, in particular $p_{0,\beta}$.

(ii) This follows from the fact that for any $f \in \mathscr{S}$

$$\|f\|_{\mathcal{C}^k} = \sum_{|\beta| \leq k} \sup_{x \in \mathbb{R}^n} |D^\beta(f)(x)| \leq C \max_{|\beta| \leq k} p_{0,\beta(f)}.$$

(iii) By 4.2.7 it suffices to check that for any $K \Subset U$ the restriction $\mathscr{D}_K \to \mathscr{S}$ is continuous. For any $f \in \mathscr{D}_K$ we calculate

$$\forall \alpha, \beta \in \mathbb{N}^n : p_{\alpha,\beta}(f) = \sup_{x \in \mathbb{R}^n} \|x^\alpha D^\beta(f)(x)\| = \sup_{x \in K} \|x^\alpha D^\beta(f)(x)\|$$
$$\leq \underbrace{\max_{x \in K} |x|^{|\alpha|}}_{=:C} \|f\|_{\mathcal{C}^{|\beta|}(K)}.$$

Now the claim follows from the definition of the topology on $\mathscr{D}_K$ and **??**.

(iv) To show that $\mathscr{D} \subset \mathscr{S}$ is dense, let $f \in \mathscr{S}$ be arbitrary. Choose a smooth bump function $\rho \in \mathscr{D}$ such that $\rho|_{B_1(0)} \equiv 1$. The existence of such a function is discussed in more detail in . For any $0 < \varepsilon < 1$ define $\rho_\varepsilon(x) := \rho(\varepsilon x)$. This function satisfies $\rho_\varepsilon \in \mathscr{D}$ and $\rho_\varepsilon|_{B_{1/\varepsilon}}(0) \equiv 1$. Clearly, the function $f_\varepsilon := \rho_\varepsilon f \in \mathscr{D}$ satisfies ref

$$f_\varepsilon \xrightarrow[p.w.]{\varepsilon \to 0} f .$$

We calculate for any $\alpha, \beta \in \mathbb{N}^n$, $k := |\beta|$, $x \in \mathbb{R}^n$

$$|x^\alpha D^\beta(f_\varepsilon - f)(x)| = |x^\alpha D^\beta((\rho_\varepsilon - 1)f)(x)|$$
$$\leq \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} |x^\alpha D^\gamma(\rho_\varepsilon - 1)(x) D^{\beta-\gamma}(f)(x)| \tag{4.11}$$

Now we distinguish two cases: If $\gamma \neq 0$,

$$|x^\alpha D^\gamma(\rho_\varepsilon - 1)(x) D^{\beta-\gamma}(f)(x)| = |x^\alpha D^\gamma(\rho)(x) \varepsilon^\gamma D^{\beta-\gamma}(f)(x)|$$
$$\leq \varepsilon^\gamma \|\rho\|_{\mathcal{C}^k} |x^\alpha D^{\beta-\gamma}(f)(x)| \leq \varepsilon \|\rho\|_{\mathcal{C}^k} p_{\alpha,\beta-\gamma}(f) \leq C\varepsilon \xrightarrow{\varepsilon \searrow 0} 0 .$$

In case $\gamma = 0$, we claim that

$$|x^\alpha(\rho_\varepsilon - 1)(x) D^\beta(f)(x)| \leq \sup_{y \in \mathbb{R}^n \setminus B_{1/\varepsilon}(0)} |y^\alpha D^\beta(f)(y)| \xrightarrow{\varepsilon \searrow 0} 0 .$$

To see this last convergence, we argue by contradiction: If this does not hold, there exists a sequence (y_j), $y_j \in \mathbb{R}^n \setminus B_j(0)$, such that $|y_j^\alpha D^\beta(f)(y_j)| \geq \delta$ for some $\delta >$. This is due to the fact that if $\varepsilon \to 0$, $B_{1/\varepsilon}(0)$ becomes larger and larger. This sequence satisfies $|y_j| \to \infty$ and therefore this sequence directly contradicts 4.4.9,(ii).
Both cases together imply that (4.11) tends to zero as well.

(v) For $p = \infty$ this follows from the definitions. So let $1 \leq p < \infty$, $f \in \mathscr{S}$ and

$$m := \lceil (n+1)/p \rceil.$$

Using the fact that $x \mapsto |x|^{-(n+1)} \in L^1(\mathbb{R}^n \setminus B_1(0))$ by A.2.4, we obtain

$$\|f\|_{L_p(\mathbb{R}^n)}^p = \int_{\mathbb{R}^n} |f(x)|^p dx = \int_{B_1(0)} |f(x)|^p dx + \int_{\mathbb{R}^n \setminus B_1(0)} \frac{1}{|x|^{n+1}} \left(|x|^{\frac{n+1}{p}} |f(x)|\right)^p dx$$
$$\leq |B_1(0)| p_{0,0}(f)^p + p_{m,0}(f)^p \|x^{-(n+1)}\|_{L_1(\mathbb{R}^n \setminus B_1(0)} \leq C(p_{0,0}(f)^p + p_{m,0}(f)^p).$$

This shows $f \in L^p(\mathbb{R}^n)$ and the continuity of the inclusion.

□

4.4.14 Lemma. Let $L^p := L^p(\mathbb{R}^n)$. The map

$$\begin{array}{rcl} \langle\_\rangle : L^p & \to & \mathscr{S}' \\ f & \mapsto & (\phi \mapsto \int_{\mathbb{R}^n} f(x)\phi(x)dx) \end{array}$$

is well-defined, linear, injective and continuous.

Proof. For any $f \in L^p$, $\phi \in \mathscr{S}$, we obtain using Hölder's inequality

$$\Big| \int_{\mathbb{R}^n} f(x)\phi(x)dx \Big| \leq \|f\phi\|_{L^1} \leq \|f\|_{L^p)} \|\phi\|_{L^q}, \tag{4.12}$$

where q is Hölder conjugate to p. By 4.4.13(v) this quantity is finite. Thus $\langle f \rangle$ is well-defined. It is clear that $\langle\_\rangle$ and $\langle f \rangle$ are linear. To see that $\langle f \rangle \in \mathscr{S}'$, we have to check continuity: But

$$\phi_j \xrightarrow[\mathscr{S}]{} 0$$

implies

$$\phi_j \xrightarrow[L^q]{} 0 \, ,$$

by 4.4.13(v). Thus

$$\langle f \rangle(\phi_j) \xrightarrow[\mathbb{C}]{} 0$$

by (4.12). Similar if

$$f_j \xrightarrow[L^p]{} 0 \, ,$$

we obtain

$$\forall \phi \in \mathscr{S} : \langle f_j \rangle(\phi) \xrightarrow[L^p]{} 0 \, ,$$

again by (4.12). Consequently $\langle\_\rangle$ is continuous. □

4.4.15 Theorem. The Fréchet Space $\mathscr{S} := \mathscr{S}(\mathbb{R}^n, \mathbb{C}^r)$ is closed under the following operations:

(i) complex conjugation,
(ii) scalar products and products in case $r = 1$,
(iii) differentiation,
(iv) polynomial multiplication,
(v) convolutions.

Proof. Assume that $f, g \in \mathscr{S}$.

(i) Clear.
(ii) In case $r = 1$ the Leibniz rule A.1.3 implies

$$\forall \alpha, \beta \in \mathbb{N}^n : \forall x \in \mathbb{R}^n : x^\alpha \partial^\beta (fg) = \sum_{\gamma \leq \beta} \binom{\gamma}{\beta} x^\alpha \partial^\gamma f x^0 \partial^{\beta - \gamma} g,$$

thus $fg \in \mathscr{S}$. The general case follows from the formula

$$\langle f, g \rangle = \sum_{j=1}^{r} f_j \bar{g}_j .$$

(iii) Follows from the definition of $p^{\mathscr{S}}_{\alpha,\beta}$.

(iv) Follows from 4.4.9.

(v) It suffices to check this for $r = 1$. For any $x \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{N}^n$, we calculate

$$|x^\alpha(\partial^\beta(f * g)(x))| \overset{4.4.3}{=} |x^\alpha(\partial^\beta f * g)(x)| \leq \int_{\mathbb{R}^n} |x^\alpha \partial^\beta(f)(x-y)g(y)|dy$$
$$\leq \sup_{y \in \mathbb{R}^n} |x^\alpha g(y)| \int_{\mathbb{R}^n} |\partial^\beta(f)(x-y)|dy \leq \sup_{y \in \mathbb{R}^n} |y^\alpha g(y)| \int_{\mathbb{R}^n} |\partial^\beta(f)(y)|dy$$
$$= p_{\alpha,0}(g)\|\partial^\beta(f)\|_{L^1} \overset{4.4.13(v)}{<} \infty.$$

□

Although we have already established all this wonderful properties of $\mathscr{S}$, the most famous one is yet missing. As we will see in the next section, the Schwarz space is ideally suited for the *Fourier transform*.

4.4.3. Fourier Transform

4.4.16 Definition (Fourier Transform)**.** Let $f \in L^1(\mathbb{R}^n, \mathbb{C})$. Depending on the context both of the functions

$$\begin{aligned} \hat{f} : \mathbb{R}^n &\to \mathbb{C} \\ \xi &\mapsto \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)dx \end{aligned}$$

$$\begin{aligned} \mathcal{F}(f) : \mathbb{R}^n &\to \mathbb{C} \\ \xi &\mapsto (2\pi)^{-\frac{n}{2}} \hat{f}(\xi) \end{aligned}$$

are the *Fourier transform of* f. For functions $f \in L^1(\mathbb{R}^n, \mathbb{C}^r)$ these operations are defined component wise.

4.4.17 Remark. Notice that the notation conventions concerning the Fourier transform are far from coherent throughout the literature. We have chosen this convention since $\hat{f}$ is very quick to write and usually the constant in $\mathcal{F}(f)$ does not change anything substantial. The constant is relevant for the Inverse Theorem 4.4.26, the fixed point theorem 4.4.23 and the Theorem of Plancherel 4.4.27. The constant in $\mathcal{F}$ is chosen such that it is an isometry (and not only an isometry up to constants).

4.4.18 Lemma (Elementary properties of the Fourier transform)**.** The Fourier transform satisfies the following properties:

(i) For every $f \in L^1 := L^1(\mathbb{R}^n, \mathbb{C}^r)$, $\hat{f}$ exists and is well-defined and $\|\hat{f}(\xi)\|_1 \leq \|f\|_{L^1}$.

(ii) Fourier transform defines an operator $\mathcal{F} \in \mathcal{L}(L^1, \mathcal{C}_b)$.

Proof. For $r = 1$ the simple estimate

$$|\hat{f}(\xi)| \leq \int_{\mathbb{R}^n} |e^{i\langle x,\xi\rangle} f(x)|dx \leq \int_{\mathbb{R}^n} |f(x)|dx = \|f\|_{L^1(\mathbb{R}^n)}$$

shows that the integral $\hat{f}$ always exists. It also shows that $\mathcal{F}$ is a bounded function. Together with the continuity theorem of parameter-dependent integrals , this calculation also implies that $\mathcal{F}(f)$ is continuous. It is clear that $\mathcal{F}$ is linear. To see that $\mathcal{F}$ itself it continuous, assume

ref

$$f_j \xrightarrow[L^1]{} 0\,.$$

This implies

$$\forall \xi \in \mathbb{R}^n : |\hat{f}_j(\xi)| \leq \int_{\mathbb{R}^n} |e^{-i\langle x,\xi\rangle} f_j(x)|dx \leq \|f_j\|_{L^1}.$$

Consequently

$$\mathcal{F}(f_j) \xrightarrow[\mathcal{C}_b^0]{} 0\,.$$

Applying this in all the components yields the statement for general r. □

4.4.19 Theorem (Convolution Theorem)**.** Let $f, g \in L^1(\mathbb{R}^n, \mathbb{C})$. Then

$$\widehat{f * g} = \hat{f} \cdot \hat{g}.$$

Proof. By Theorem 4.4.1 $f * g \in L^1(\mathbb{R}^n)$. By definition and Fubini's theorem

$$\begin{aligned}\widehat{f * g}(\xi) &= \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}(f * g)(x)dx = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} \int_{\mathbb{R}^n} f(x-y)g(y)dydx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle x-y,\xi\rangle} f(x-y)e^{-i\langle y,\xi\rangle} g(y)dydx \\ &= \int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle} g(y) \int_{\mathbb{R}^n} e^{-i\langle x-y,\xi\rangle} f(x-y)dxdy \\ &= \hat{g}(\xi)\hat{f}(\xi).\end{aligned}$$

□

4.4.20 Definition. For any function $f : \mathbb{R}^n \to \mathbb{C}^r$ define the

(i) translation

$$\forall y \in \mathbb{R}^n : \tau_y(f) : \mathbb{R}^n \to \mathbb{C}^r, \quad x \mapsto f(x+y),$$

(ii) rotation

$$\forall y \in \mathbb{R}^n : m_y(f) : \mathbb{R}^n \to \mathbb{C}^r, \quad x \mapsto e^{i\langle x,y\rangle} f(x),$$

(iii) scaling

$$\forall \lambda \in \mathbb{C}^\times : s_\lambda(f) : \mathbb{R}^n \to \mathbb{C}^r, \quad x \mapsto f(\lambda x),$$

(iv) reflection

$$\mathcal{R}(f) : \mathbb{R}^n \to \mathbb{C}^r, \quad x \mapsto \check{f}(x) := f(-x)$$

of f.

4.4.21 Theorem. Let $f \in L^1(\mathbb{R}^n, \mathbb{C}^r)$, $y \in \mathbb{R}^n$, $\lambda \in \mathbb{C}^\times$. Then $\tau_y(f), m_y(f), s_\lambda(f), \mathcal{R}(f) \in L^1(\mathbb{R}^n, \mathbb{C})$ and

(i) $\mathcal{F}(\tau_y(f)) = m_y(\mathcal{F}(f))$.

(ii) $\mathcal{F}(m_y(f)) = \tau_{-y}(\mathcal{F}(f))$.

(iii) $\mathcal{F}(s_\lambda(f)) = |\lambda|^{-n} s_{\frac{1}{\lambda}}\mathcal{F}(f)$.

(iv) $\mathcal{F}(\bar{f}) = \overline{\mathcal{R} \circ \mathcal{F}}(f)$, $\mathcal{F} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{F}$.

The same is true for $\hat{}$ instead of $\mathcal{F}$.

Proof. It suffices to check these statements for r =1. It is clear that $\tau_y(f), m_y(f), s_\lambda(f), \mathcal{R}(f) \in L^1$, since $f \in L^1$ by hypothesis. Since all operations are linear, the prefactor $(2\pi)^{-\frac{n}{2}}$ does not matter. We verify the various formulas using the transformation theorem.

(i) We calculate

$$\begin{aligned}\widehat{\tau_y(f)}(\xi) &= \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\tau_y(f)(x)dx = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x+y)dx \\ &= \int_{\mathbb{R}^n} e^{-i\langle z-y,\xi\rangle} f(z)dz = e^{i\langle y,\xi\rangle}\int_{\mathbb{R}^n} e^{-i\langle z,\xi\rangle} f(z)dz = m_y(\hat{f})(\xi).\end{aligned}$$

(ii) We calculate

$$\widehat{m_y(f)}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} m_y(f)(x)dx = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} e^{i\langle x,y\rangle} f(x)dx$$
$$= \int_{\mathbb{R}^n} e^{-i\langle x,\xi-y\rangle} f(x)dx = \hat{f}(\xi - y) = \tau_{-y}(\hat{f})(\xi).$$

(iii) We calculate

$$\widehat{s_\lambda(f)}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} s_\lambda(f)(x)dx = \int_{\mathbb{R}^n} e^{-i\langle \lambda x,\frac{1}{\lambda}\xi\rangle} f(\lambda x)|\lambda|^n|\lambda|^{-n}dx$$
$$= |\lambda|^{-n}\int_{\mathbb{R}^n} e^{-i\langle z,\frac{1}{\lambda}\xi\rangle} f(z)dz = |\lambda|^{-n}\hat{f}\Big(\frac{\xi}{\lambda}\Big) = |\lambda|^{-n}s_{\frac{1}{\lambda}}(\hat{f})(\xi).$$

(iv) We calculate

$$\widehat{\bar{f}}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\bar{f}(x)dx = \int_{\mathbb{R}^n} \overline{e^{-i\langle x,-\xi\rangle}}\bar{f}(x)dx = \overline{\hat{f}(-\xi)} = \overline{\mathcal{R}(\hat{f})}(\xi).$$

$$\widehat{\mathcal{R}(f)}(\xi) = \int_{\mathbb{R}^n} e^{i\langle -x,\xi\rangle} f(-x)dx = \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} f(x)dx = \int_{\mathbb{R}^n} e^{-i\langle x,-\xi\rangle} f(x)dx = \mathcal{R}(\hat{f})(\xi)$$

□

4.4.22 Theorem (Differentiation Theorem)**.** Let $f \in \mathscr{S} = \mathscr{S}(\mathbb{R}^n, \mathbb{C}^r)$, $\alpha \in \mathbb{N}^n$. Then

$$\mathcal{F}(D^\alpha f)(\xi) = \xi^\alpha \mathcal{F}(f), \qquad D^\alpha\mathcal{F}(f)(\xi) = \mathcal{F}((-x)^\alpha f)(\xi),$$
$$\mathcal{F}(\partial^\alpha f)(\xi) = (i\xi)^\alpha \mathcal{F}(f), \qquad \partial^\alpha\mathcal{F}(f)(\xi) = \mathcal{F}((-ix)^\alpha f)(\xi).$$

Proof. Clearly it suffices to check the last line for $r = 1$. Integrating by parts yields

$$\widehat{\partial_x^\alpha f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\partial_x^\alpha f(x)dx = (-1)^\alpha\int_{\mathbb{R}^n}\partial_x^\alpha(e^{-i\langle x,\xi\rangle})f(x)dx$$
$$= (-1)^\alpha\int_{\mathbb{R}^n}(-i\xi)^\alpha e^{-i\langle x,\xi\rangle}f(x)dx = (i\xi)^\alpha\hat{f}(\xi).$$

The boundary terms vanish since $f \in \mathscr{S}$.
For the second statement we calculate

$$\partial_\xi^\alpha(\hat{f})(\xi) = \partial_\xi^\alpha\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}f(x)dx = \int_{\mathbb{R}^n}\partial_\xi^\alpha(e^{-i\langle x,\xi\rangle})f(x)dx$$
$$= \int_{\mathbb{R}^n}(-ix)^\alpha e^{-i\langle x,\xi\rangle}f(x)dx = ((\widehat{-ix)^\alpha}f)(\xi).$$

□

4.4.23 Theorem (Fixed Point)**.** The function

$$\begin{aligned} f : \mathbb{R}^n &\to \mathbb{R} \\ x &\mapsto e^{-\frac{|x|^2}{2}} \end{aligned}$$

satisfies

$$\mathcal{F}(f) = f.$$

Proof. We will require the equation

$$\int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}, \tag{4.13}$$

which is proven in elementary calculus courses. ref

STEP 1 ($n = 1$): Using partial integration we calculate

$$\begin{aligned}\hat{f}'(\xi) &= \partial_\xi \int_{\mathbb{R}} e^{-ix\xi} f(x)dx = \int_{\mathbb{R}} \partial_\xi (e^{-ix\xi}) f(x)dx = \int_{\mathbb{R}} -ixe^{-ix\xi} e^{-\frac{|x|^2}{2}} dx \\ &= i \int_{\mathbb{R}} e^{-ix\xi} \partial_x (e^{-\frac{|x|^2}{2}}) dx = ie^{-ix\xi} e^{-\frac{|x|^2}{2}} \Big|_{\infty}^{\infty} - i \int_{\mathbb{R}} \partial_x (e^{-ix\xi}) e^{-\frac{|x|^2}{2}} dx \\ &= -\xi \int_{\mathbb{R}} e^{-ix\xi} f(x)dx = -\xi \hat{f}(\xi).\end{aligned}$$

Therefore the Fourier transform satisfies the ODE

$$\mathcal{F}(f)'(\xi) + \xi \mathcal{F}(f)(\xi) = 0$$

as well as the function f. Since

$$\mathcal{F}(f)(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\cdot 0} f(x)dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx \overset{(4.13)}{=} 1 = f(0),$$

we obtain $\mathcal{F}(f) = f$, i.e.

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} e^{-\frac{x^2}{2}} dx = e^{-\frac{\xi^2}{2}}, \tag{4.14}$$

by uniqueness of initial value problems.

STEP 2: For general n this is a consequence of Fubini's Theorem:

$$\begin{aligned}\hat{f}(\xi) &= \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} e^{-\frac{|x|^2}{2}} dx = \int_{\mathbb{R}^n} e^{-i\sum_{j=1}^n x_j\xi_j} e^{-\frac{\sum_{j=1}^n x_j^2}{2}} dx = \int_{\mathbb{R}^n} \prod_{j=1}^n e^{-ix_j\xi_j} e^{-\frac{x_j^2}{2}} dx \\ &= \prod_{j=1}^n \int_{\mathbb{R}} e^{-ix_j\xi_j} e^{-\frac{x_j^2}{2}} dx_j \overset{(4.14)}{=} \prod_{j=1}^n \sqrt{2\pi} e^{-\frac{\xi_j^2}{2}} = (2\pi)^{\frac{n}{2}} f(\xi).\end{aligned}$$

□

4.4.24 Theorem (Adjoint Formula)**.** For any $f, g \in L^1(\mathbb{R}^n, \mathbb{C}^r)$,

$$\langle \mathcal{F}(f), g\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^r)} = \langle f, (\mathcal{R} \circ \mathcal{F})(g)\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^r)}$$

and both sides are finite. In particular for $r = 1$, we obtain the *adjoint formula*

$$\int_{\mathbb{R}^n} \mathcal{F}(f)(\xi) g(\xi) d\xi = \int_{\mathbb{R}^n} f(x) \mathcal{F}(g)(x) dx$$

Proof. First of all, the calculation

$$\begin{aligned}\int_{\mathbb{R}^n} |\langle \mathcal{F}(f)(\xi), \bar{g}(\xi)\rangle| d\xi &\leq \int_{\mathbb{R}^n} \|\mathcal{F}(f)(\xi)\|_2 \|\bar{g}(\xi)\|_2 d\xi \leq \int_{\mathbb{R}^n} \|\mathcal{F}(f)(\xi)\|_1 \|\bar{g}(\xi)\|_1 d\xi \\ &\overset{4.4.18}{\leq} \|f\|_{L^1} \int_{\mathbb{R}^n} \|\bar{g}(\xi)\|_1 d\xi = \|f\|_{L_1} \|g\|_{L^1}\end{aligned}$$

shows that all the integrals exist. Therefore by Fubini's theorem for $r = 1$

$$\begin{aligned} \langle \mathcal{F}(f), \bar{g} \rangle_{L^2(\mathbb{R}^n,\mathbb{C})} &= \int_{\mathbb{R}^n} \mathcal{F}(f)(\xi) g(\xi) d\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i\langle x,\xi \rangle} f(x) dx\, g(\xi) d\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} e^{-i\langle x,\xi \rangle} g(\xi) d\xi dx \\ &= \int_{\mathbb{R}^n} f(x) \mathcal{F}(g)(x) dx = \langle f, \overline{\mathcal{F}(g)} \rangle_{L^2(\mathbb{R}^n,\mathbb{C})} \end{aligned} \tag{4.15}$$

This implies

$$\langle \mathcal{F}(f), g \rangle_{L^2(\mathbb{R}^n,\mathbb{C})} \overset{(4.15)}{=} \langle f, \overline{\mathcal{F}(\bar{g})} \rangle_{L^2(\mathbb{R}^n,\mathbb{C})} \overset{4.4.21}{=} \langle f, \mathcal{R}(\mathcal{F}(g)) \rangle_{L^2(\mathbb{R}^n)}. \tag{4.16}$$

Applying this to all the component functions yields the statement for general r. □

4.4.25 Definition (Inverse Fourier Transform)**.** Let $g \in \mathscr{S} = \mathscr{S}(\mathbb{R}^n, \mathbb{C})$. Then

$$\begin{aligned} \check{g} : \mathbb{R}^n &\to \mathbb{C} \\ x &\mapsto \textstyle\int_{\mathbb{R}^n} e^{i\langle x,\xi \rangle} g(\xi) d\xi \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}^{-1}(g) : \mathbb{R}^n &\to \mathbb{C} \\ x &\mapsto (2\pi)^{-\frac{n}{2}} \check{g} \end{aligned}$$

are called *inverse Fourier transform of* g. For $g \in \mathscr{S}(\mathbb{R}^n, \mathbb{C}^r)$ this is again defined component wise.

4.4.26 Theorem (Inversion Theorem)**.** The Fourier transform is a linear homeomorphism

$$\mathcal{F} : \mathscr{S} \to \mathscr{S}$$

and its inverse is given by

$$\mathcal{F}^{-1} = \mathcal{F} \circ \mathcal{R}.$$

Here $\mathscr{S} = \mathscr{S}(\mathbb{R}^n, \mathbb{C}^r)$.

Proof. It suffices to check this for $r = 1$. The linearity of $\mathcal{F}$ is clear.

STEP 1 (range): We have to show that for any $f \in \mathscr{S}$, $\mathscr{F}(f) \in \mathscr{S}$ as well. Therefore let $\alpha, \beta \in \mathbb{N}^n$ and calculate

$$\xi^\alpha D^\beta \mathcal{F}(f) \overset{4.4.22}{=} \xi^\alpha \mathcal{F}(-x^\beta f)(\xi) \overset{4.4.22}{=} \mathcal{F}(D_x^\alpha(-x^\beta f))(\xi).$$

Now $D_x^\alpha(-x^\beta f) \in \mathscr{S} \subset L^1$ by 4.4.13 and therefore

$$\begin{aligned} \forall \xi \in \mathbb{R}^n : |\xi^\alpha (D^\beta \mathcal{F}(f))(\xi)| &= |\mathcal{F}(D_x^\alpha(-x^\beta f))(\xi)| \\ &\overset{4.4.18}{\leq} (2\pi)^{-\frac{n}{2}} \|D_x^\alpha(-x^\beta f)\|_{L^1(\mathbb{R}^n)} < \infty. \end{aligned} \tag{4.17}$$

STEP 2 (bijectivity): It is clear from the Definition 4.4.25 that the inverse Fourier transform $\mathcal{F}^{-1}$ satisfies $\mathcal{F}^{-1} = \mathcal{F} \circ \mathcal{R}$. We have to show that it really is an inverse to $\mathcal{F}$. So let $f \in \mathscr{S}$ and consider

$$\mathcal{F}^{-1}(\mathcal{F}(f))(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi \rangle} \mathcal{F}(f)(\xi) d\xi. \tag{4.18}$$

Denote by φ the fixed point of $\mathcal{F}$ described in Theorem 4.4.23. Let $\varepsilon > 0$, $x \in \mathbb{R}^n$ and define

$$\begin{aligned} g : \mathbb{R}^n &\to \mathbb{C} \\ \xi &\mapsto m_x(s_\varepsilon(\varphi))(\xi) = e^{i\langle x,\xi\rangle - \frac{\varepsilon^2|\xi|^2}{2}} \end{aligned}$$

and notice that by 4.4.21

$$\begin{aligned} \mathcal{F}(g)(\eta) &= \tau_{-x}(\mathcal{F}(s_\varepsilon(\varphi)))(\eta) = \tau_{-x}(\varepsilon^{-n}s_{1/\varepsilon}(\mathcal{F}(\varphi)))(\eta) \\ &\overset{4.4.23}{=} \tau_{-x}(\varepsilon^{-n}s_{1/\varepsilon}(\varphi))(\eta) = \varepsilon^{-n}e^{-\frac{|\eta-x|^2}{2\varepsilon^2}}. \end{aligned} \tag{4.19}$$

Therefore

$$\begin{aligned} I_\varepsilon(x) &:= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} g(\xi)\mathcal{F}(f)(\xi)d\xi \overset{4.4.24}{=} (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\mathcal{F}(g)(\xi)f(\xi)d\xi \\ &\overset{(4.19)}{=} (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\varepsilon^{-n}e^{-\frac{|\xi-x|^2}{2\varepsilon^2}}f(\xi)d\xi = (2\pi)^{-\frac{n}{2}}(f*\varphi_\varepsilon)(x), \end{aligned} \tag{4.20}$$

where $\varphi_\varepsilon(x) := \varepsilon^{-n}\varphi(x/\varepsilon)$ is a Dirac sequence. By Theorem 4.4.6

$$I_\varepsilon = (2\pi)^{-\frac{n}{2}} f*\varphi_\varepsilon \xrightarrow[L^1]{\varepsilon\to 0} (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}}dx f \overset{(4.13)}{=} f\ .$$

By the Theorem of Riesz/Fischer there is a subsequence $\varepsilon_k > 0$ such that ref

$$I_{\varepsilon_k} \xrightarrow[a.e.]{k\to\infty} f\ .$$

Thus for almost every $x \in \mathbb{R}^n$, Lebesgue's dominated convergence Theorem implies

$$\begin{aligned} f(x) &= \lim_{k\to\infty} I_{\varepsilon_k}(x) = (2\pi)^{-\frac{n}{2}}\lim_{k\to\infty}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle - \frac{\varepsilon_k^2|\xi|^2}{2}}\mathcal{F}(f)(\xi)d\xi \\ &= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\mathcal{F}(f)(\xi)d\xi \overset{(4.18)}{=} \mathcal{F}^{-1}(\mathcal{F}(f))(x). \end{aligned}$$

Since both sides are continuous, equality holds for all $x \in \mathbb{R}^n$.
By Theorem 4.4.21, the operator $\mathcal{R}$ commutes with $\mathcal{F}$. This implies

$$\mathrm{id} = \mathcal{F}^{-1}\circ\mathcal{F} = \mathcal{F}\circ\mathcal{R}\circ\mathcal{F} = \mathcal{F}\circ\mathcal{F}\circ\mathcal{R} = \mathcal{F}\circ\mathcal{F}^{-1}.$$

Therefore $\mathcal{F}^{-1}$ is indeed the inverse of $\mathcal{F}$ and $\mathcal{F}$ is bijective as claimed.

STEP 3 (continuity): It is clear that $\mathcal{R} : \mathscr{S} \to \mathscr{S}$ is a linear homeomorphism with inverse $\mathcal{R}^{-1} = \mathcal{R}$. Therefore it suffices to check that $\mathcal{F}$ is continuous. So let $f_j \in \mathscr{S}$, such that

$$f_j \xrightarrow[\mathscr{S}]{} 0\ .$$

Using the Leibniz this implies that for any $\alpha, \beta \in \mathbb{N}^n$

$$D_x^\alpha((-x^\beta)f_j) \xrightarrow[\mathscr{S}]{} 0\ .$$

By Theorem 4.4.13 this implies

$$D_x^\alpha((-x^\beta)f_j) \xrightarrow[L^1]{} 0\ .$$

Now the claim follows from

$$p^{\mathscr{S}}_{\alpha,\beta}(\mathcal{F}(f_j)) = \sup_{\xi\in\mathbb{R}^n}|\xi^\alpha D_\xi^\beta\mathcal{F}(f_j)(\xi)| \overset{(4.17)}{\leq} (2\pi)^{-\frac{n}{2}}\|D_x^\alpha((-x^\beta)f_j)\|_{L^1(\mathbb{R}^n)} \to 0.$$

□

4.4.27 Theorem (Plancherel)**.** The Fourier transform $\mathcal{F} : \mathscr{S} \to \mathscr{S}$ is an L^2-isometry. Therefore it extends to an L^2-isometry

$$\mathcal{F} : L^2 \to L^2.$$

Here $L^2 = L^2(\mathbb{R}^n, \mathbb{C}^r)$.

Proof. By Theorem 3.2.1, we have to check

$$\forall f \in \mathscr{S} : \|\mathcal{F}(f)\|_{L^2} = \|f\|_{L^2}.$$

We calculate

$$\begin{aligned}\langle \mathcal{F}(f), \mathcal{F}(f)\rangle_{L^2} &\overset{4.4.24}{=} \langle f, (\mathcal{R} \circ \mathcal{F} \circ \mathcal{F})(f)\rangle_{L^2} \overset{4.4.21}{=} \langle f, (\mathcal{F} \circ \mathcal{R} \circ \mathcal{F})(f)\rangle_{L^2} \\ &\overset{4.4.26}{=} \langle f, (\mathcal{F}^{-1} \circ \mathcal{F})(f)\rangle_{L^2} = \langle f, f\rangle_{L^2}.\end{aligned}$$

□

4.4.28 Lemma (Riemann/Lebesgue Lemma)**.**

4.5. Symbols and their asymptotic expansions

4.5.1 Definition (Symbol)**.** Let $U \subset \mathbb{R}^m$ and $k \in \mathbb{R}$. A function $\sigma \in \mathscr{C}^\infty(U \times \mathbb{R}^n, \mathbb{C}^{r' \times r})$ is *a symbol of order* k, if

$$\forall \alpha, \beta \in \mathbb{N}^n : \exists C_{\alpha,\beta} > 0 : \forall (x,\xi) \in U \times \mathbb{R}^n : |\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{k-|\beta|}. \quad (4.21)$$

The space of all those symbols is denoted by $\mathcal{S}^k := \mathcal{S}^k(U \times \mathbb{R}^n, \mathbb{C}^{r' \times r})$. Define

$$\mathcal{S}^{+\infty} := \bigcup_{k \in \mathbb{R}} \mathcal{S}^k, \qquad \mathcal{S}^{-\infty} := \bigcap_{k \in \mathbb{R}} \mathcal{S}^k.$$

Furthermore if $\sigma \in \mathcal{S}^k$ has compact x-support, we say $\sigma \in \mathcal{S}_c^k$. Define

$$p_{\alpha,\beta}^k(\sigma) := \sup_{(x,\xi) \in U \times \mathbb{R}^n} |\partial_x^\alpha \partial_x^\beta \sigma(x,\xi)|(1+|\xi|)^{|\beta|-k}.$$

4.5.2 Theorem (Elementary Properties of Symbols)**.** Let $k \in \mathbb{R}$.

(i) $\mathcal{S}^k$ is a complex vector space.

(ii) For any $k \in \mathbb{R}$ and the family

$$\{p_{\alpha,\beta}^k \mid \alpha, \beta \in \mathbb{N}^n\}$$

is a countable separating family of semi-norms, which induce a Frechét space topology on $\mathcal{S}^k$.

(iii) If $k_1 \leq k_2$, then $\mathcal{S}^{k_1} \subset \mathcal{S}^{k_2}$ and the inclusion is a bounded linear operator.

(iv) If $\sigma_1 \in \mathcal{S}^{k_1}(U \times \mathbb{R}^n, \mathbb{C}^{s \times r})$, $\sigma_2 \in \mathcal{S}^{k_2}(U \times \mathbb{R}^n; \mathbb{C}^{t \times s})$, then $\sigma_2\sigma_1 \in \mathcal{S}^{k_1+k_2}(\mathbb{R}^n; \mathbb{C}^{t \times r})$ and multiplication

$$\begin{aligned} \mathcal{S}^{k_1} \times \mathcal{S}^{k_2} &\to \mathcal{S}^{k_1+k_2} \\ (\sigma_1, \sigma_2) &\mapsto \sigma_1\sigma_2 \end{aligned}$$

is bilinear and continuous.

(v) If $\sigma \in \mathcal{S}^k = \mathcal{S}^k(U \times \mathbb{R}^n; \mathbb{C}^{s \times r})$, $\alpha, \beta \in \mathbb{N}^n$, then $\partial_x^\alpha \partial_\xi^\beta(\sigma) \in \mathcal{S}^{k-|\beta|}$ and differentiation

$$\begin{aligned} \mathcal{S}^k &\to \mathcal{S}^{k-|\beta|} \\ \sigma &\mapsto \partial_x^\alpha \partial_\xi^\beta(\sigma) \end{aligned}$$

is a bounded linear operator.

(vi) If $\sigma \in \mathcal{S}^k(U \times \mathbb{R}^n, \mathbb{C}^{s \times r})$ and $f \in \mathscr{S}(\mathbb{C}^r)$, then

$$\begin{aligned} \tau : U \times \mathbb{R}^n &\to \mathbb{C}^s \\ (x,\xi) &\mapsto \sigma(x,\xi)f(\xi) \end{aligned}$$

satisfies $\sigma \in \mathcal{S}^{-\infty}$.

Proof.

(i) If $\sigma, \sigma' \in \mathcal{S}^k$, $\lambda \in \mathbb{C}$, $\alpha, \beta \in \mathbb{N}^n$ and C, C' are the constants from (4.21), we simply calculate for any $x \in U, \xi \in \mathbb{R}^n$

$$\begin{aligned} |D_x^\alpha D_\xi^\beta(\sigma_1 + \lambda\sigma_2)(x,\xi)| &\leq C(1+|\xi|)^{k-|\beta|} + |\lambda|C'(1+|\xi|)^{k-|\beta|} \\ &\leq (C + |\lambda|C')(1+|\xi|)^{k-|\beta|}. \end{aligned}$$

(ii) It is clear that this is a family of semi-norms and that it is countable. It is separating through $p^k_{0,0}$. By Theorem 3.1.20 it is a locally convex space. To check completeness, assume that σ_i is a Cauchy sequence in $\mathcal{S}^k$, i.e. in all the $p^k_{\alpha,\beta}$-seminorms. This implies that $\partial^\alpha_x \partial^\beta_\xi(\sigma_i)(1+|\xi|)^{|\beta|-k}$ is a Cauchy sequence in $\mathscr{C}^0(U \times \mathbb{R}^n, \mathbb{C}^{s\times r})$. Since this space is complete

$$\exists \sigma_{\alpha,\beta} \in \mathscr{C}^0(U \times \mathbb{R}^n, \mathbb{C}^{s\times r}) : \ \partial^\alpha_x \partial^\beta_\xi(\sigma_i)(1+|\xi|)^{|\beta|-k} \xrightarrow[\mathscr{C}^0]{i\to\infty} \sigma_{\alpha,\beta} \,. \tag{4.22}$$

Define $\sigma := \sigma_{0,0} \cdot (1+|\xi|)^{|\beta|-k}$. Remember that a sequence of differentiable functions, which converges pointwise, and whose derivatives converge uniformly, has a limit that is differentiable and the limit and differentiation may be interchanged. Since the convergence above holds for all α and $(1+|\xi|)$ is obviously independent of x, this implies $\partial_{x_\nu}\sigma_{\alpha,\beta} = \sigma_{\alpha+e_\nu,\beta}$. This defines functions σ_β, such that $\partial^\alpha_x \sigma_\beta = \sigma_{\alpha,\beta}$. To obtain the statement for β, we have to ensure the derivatives converges uniformly. Therefore take any $0 \le \nu \le n$ and calculate

$$\begin{aligned}
&|\partial_{\xi_\nu}\left(\partial^\beta_\xi \partial^\alpha_x \sigma_i(x,\xi)(1+|\xi|)^{-k+\rho|\beta|-\delta|\alpha|}\right)| \\
&\le \partial^{\beta+e_\nu}_\xi \partial^\alpha_x \sigma_i(x,\xi)(1+|\xi|)^{-k+\rho|\beta+e_\nu|-\delta|\alpha|}| \\
&+ |\partial^\beta_\xi \partial^\alpha_x \sigma_i(x,\xi)(1+|\xi|)^{-k+\rho|\beta|-\delta|\alpha|} \underbrace{(-k+\rho|\beta|-\delta|\alpha|)\frac{\xi_\nu}{|\xi|}(1+|\xi|)^{-1}}_{\le \text{const}}|.
\end{aligned}$$

By (4.22) both summands converge uniformly. Therefore, we obtain a function σ, such that $\partial^\beta_\xi \sigma = \sigma_\beta$. Alltogether this implies

$$\sigma_i \xrightarrow[\|\_\|_{\mathcal{S}^{k,\rho,\delta}_{\alpha,\beta,K_j}}]{i\to\infty} \sigma(1+|\xi|)^{k-\rho|\beta|+\delta|\alpha|} \,.$$

(iii) We simply remark that for any $\lambda \in \mathbb{R}$, $\lambda \ge 1$, the map

$$\begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \lambda^x \end{array}$$

is monotonously increasing. Therefore, if $\sigma \in \mathcal{S}^{k_1}$, then

$$\forall x \in U : \forall \xi \in \mathbb{R}^n : |\partial^\alpha_x \partial^\beta_\xi \sigma(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{k_1-|\beta|} \le C_{\alpha,\beta}(1+|\xi|)^{k_2-|\beta|},$$

thus $\sigma \in \mathcal{S}^{k_2}$.

(iv) Let $\alpha, \beta \in \mathbb{N}^n$. By hypothesis there exist $C_1, C_2 > 0$ such that for any $x \in U, \xi \in \mathbb{R}^n$

$$\begin{aligned}
|\partial^\alpha_x \partial^\beta_\xi(\sigma_1)(x,\xi)| &\le C_1(1+|\xi|)^{k_1-|\beta|}, \\
|\partial^\alpha_x \partial^\beta_\xi(\sigma_2)(x,\xi)| &\le C_2(1+|\xi|)^{k_2-|\beta|}.
\end{aligned} \tag{4.23}$$

We calculate

$$\begin{aligned}
|\partial_x^\alpha \partial_\xi^\beta (\sigma_2\sigma_1)(x,\xi)| &\overset{\text{A.1.5}}{=} \Big|\partial_x^\alpha \sum_{\gamma\le\beta} \binom{\beta}{\gamma} \partial_\xi^{\beta-\gamma}(\sigma_1)\partial_\xi^\gamma(\sigma_2)\Big| \\
&\overset{\text{A.1.5}}{\le} \sum_{\gamma\le\beta}\sum_{\gamma'\le\alpha} \binom{\beta}{\gamma}\binom{\alpha}{\gamma'} |\partial_x^{\alpha-\gamma'}\partial_\xi^{\beta-\gamma}(\sigma_1)\partial_x^\alpha\partial_\xi^\gamma(\sigma_2)| \\
&\overset{(4.23)}{\le} \sum_{\gamma\le\beta}\sum_{\gamma'\le\alpha} \binom{\beta}{\gamma}\binom{\alpha}{\gamma'} C_1(1+|\xi|)^{k_1-|\beta-\gamma|}C_2(1+|\xi|)^{k_2-|\gamma|} \\
&= (1+|\xi|)^{k_1+k_2-|\beta|}\sum_{\gamma\le\beta}\sum_{\gamma'\le\alpha} \binom{\beta}{\gamma}\binom{\alpha}{\gamma'} C_1C_2.
\end{aligned}$$

(v) We just calculate for any α', β'

$$|\partial_x^{\alpha'}\partial_\xi^{\beta'}(\partial_x^\alpha\partial_\xi^\beta\sigma)(x,\xi)| = |\partial_x^{\alpha+\alpha'}\partial_\xi^{\beta+\beta'}\sigma(x,\xi)| \le C_{\alpha+\alpha',\beta+\beta'}(1+|\xi|)^{k-|\beta|-|\beta'|}.$$

(vi) Since $f \in \mathscr{S}(\mathbb{C}^r)$, we obtain from 4.4.9

$$\forall m \in \mathbb{N} : \forall \beta \in \mathbb{N}^n : \sup_{\xi\in\mathbb{R}^n} (1+|\xi|)^m |\partial^\beta(f)(\xi)| =: C_{m,\beta}(f) < \infty.$$

Those constants exist also for all $l \in \mathbb{R}_{>0}$. Choose any such l and calculate

$$\begin{aligned}
|\partial_x^\alpha\partial_\xi^\beta\tau(x,\xi)| = |\partial_x^\alpha\partial_\xi^\beta(\sigma(x,\xi)f(\xi))| &\le \sum_{\gamma\le\beta}\binom{\beta}{\gamma}|\partial_x^\alpha\partial_\xi^\gamma(\sigma)(x,\xi)\partial_\xi^{\beta-\gamma}(f)(\xi)| \\
&\le \sum_{\gamma\le\beta}\binom{\beta}{\gamma} p^k_{\alpha,\gamma}(\sigma)(1+|\xi|)^{k-|\gamma|}|\partial_\xi^{\beta-\gamma}(f)(\xi)| \\
&\le \sum_{\gamma\le\beta}\binom{\beta}{\gamma} p^k_{\alpha,\gamma}(\sigma)(1+|\xi|)^{k-l-|\beta|}(1+|\xi|)^l|\partial_\xi^{\beta-\gamma}(f)(\xi)| \\
&\le \sum_{\gamma\le\beta}\binom{\beta}{\gamma} p^k_{\alpha,\gamma}(\sigma)(1+|\xi|)^{k-l-|\beta|}C_{l,\beta-\gamma}(f) \\
&\le C(1+|\xi|)^{k-l-|\beta|},
\end{aligned}$$

thus $\tau \in \mathcal{S}^{k-l}$. Since $l \in \mathbb{R}_{>0}$ was arbitrary, this proves the result (remember that $\mathcal{S}^k \subset \mathcal{S}^{k+l}$ anyway.)

□

4.5.3 Definition (positively homogenous)**.** Let X be a real vector space. A function $f : X \setminus \{0\} \to \mathbb{R}$ is *positively homogenous of degree* $k \in \mathbb{R}$, if

$$\forall t > 0 : \forall x \in X \setminus \{0\} : f(tx) = t^k f(x).$$

4.5.4 Lemma (Properties of potivitely homogeonous functions)**.** Let $f, g : X \setminus \{0\} \to \mathbb{R}$ be positively homogenous of degree k and k'.

(i) If $k = k'$ and $\lambda \in \mathbb{R}$, then $f + \lambda g$ is positively homogenous of degree k.

(ii) The function fg is positively homogenous of degree $k + k'$.

(iii) A function $f \in \mathscr{C}^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ is positively homogenous of degree k if and only if

$$\forall x \in \mathbb{R}^n \setminus \{0\} : \nabla f(x)x = kf(x).$$

(iv) If $f \in \mathscr{C}^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ is positively homogenous of degree k, any $\partial_j f$ is positively homogenous of degree $k-1$.

(v) If $f \in \mathcal{C}^0(\mathbb{R}^n\{0\}, \mathbb{R})$ is positively homogenous of degree k, then

$$\forall x \in \mathbb{R}^n \setminus \{0\} : |f(x)| \leq C|x|^k, \qquad C := \max_{y \in \S^{n-1}} |f(y)|.$$

(vi) If $f \in \mathcal{C}^l(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ is positively homogenous of degree k, then for any $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq l$, $\partial_x^\alpha f$ is positively homogenous of degree $k - |\alpha|$ and

$$\exists C > 0 : \forall x \in \mathbb{R}^n \setminus \{0\} : |\partial_x^\alpha f(x)| \leq C_\alpha |x|^{k-|\alpha|} \tag{4.24}$$

Proof.

(i) We just calculate

$$\forall x \in X \setminus \{0\} : \forall t > 0 : (f + \lambda g)(tx) = f(tx) + \lambda g(tx) = t^k(f(x) + \lambda g(x)).$$

(ii) This is also very simple:

$$\forall x \in X \setminus \{0\} : \forall t > 0 (fg)(tx) = f(tx)g(tx) = t^k f(x) t^{k'} g(x) = t^{k+k'}(fg)(x).$$

(iii) "$\Rightarrow$": By differentiating, we obtain

$$\forall x \in \mathbb{R}^n \setminus \{0\} : 0 = \partial_t(f(tx) - t^k f(x))|_{t=1} = (\nabla f|_{tx} x - kt^{k-1} f(x))|_{t=1} = \nabla f|_x x - kf(x).$$

"$\Leftarrow$": Let $x \in \mathbb{R}^n$ and define $F : \mathbb{R}_{>0} \to \mathbb{R}$, $t \mapsto t^{-k} f(tx)$. Clearly $F(1) = f(x)$. We calculate

$$\begin{aligned}\partial_t F &= \partial_t(f(tx))t^{-k} - kt^{-k-1} f(tx) = \nabla f|_{tx} xt \cdot t^{-k-1} - kt^{-k-1} f(tx) \\ &= kf(tx)t^{-k-1} - kt^{-k-1} f(tx) = 0.\end{aligned}$$

Therefore $F \equiv f(x)$.

(iv) Consider any $x \in \mathbb{R}^n \setminus \{0\}$, $t > 0$ and calculate

$$\begin{aligned}\partial_j f|_{tx} &= \lim_{h \searrow 0} \frac{f(tx + he_j) - f(tx)}{h} = \lim_{h \searrow 0} \frac{f(tx + the_j) - f(tx)}{th} \\ &= \lim_{h \searrow 0} t^k \frac{f(x + he_j) - t^k f(x)}{th} = t^{k-1} \lim_{h \searrow 0} \frac{f(x + he_j) - (x)}{h} = t^{k-1} \partial_j f|_x.\end{aligned}$$

(v) We calculate for any $x \in \mathbb{R}^n \setminus \{0\}$

$$|f(x)| = \left| f\left(\tfrac{x}{\|x\|}\|x\|\right)\right| = \|x\|^k = \left| f\left(\tfrac{x}{\|x\|}\right)\right| \leq C\|x\|^k.$$

(vi) This follows from the previous claims.

□

4.5.5 Theorem (Famous Symbols).

(i) "Any symbol of a bounded PDO is a symbol", i.e. if

$$\sigma(x, \xi) = \sum_{|\alpha| \leq k} P_\alpha(x) \xi^\alpha, \qquad \forall |\alpha| \leq k : P_\alpha \in \mathcal{C}_b^\infty(U, \mathbb{C}^{r' \times r}),$$

then $\sigma \in \mathcal{S}^k(U \times \mathbb{R}^n; \mathbb{C}^{r' \times r})$.

(ii) "Schwarz functions", i.e. if $\sigma \in \mathscr{C}^\infty(U \times \mathbb{R}^n, \mathbb{C}^{r' \times r})$ satisfies

$$\forall \alpha, \beta \in \mathbb{N}^n : \forall K \Subset U : \forall d \in \mathbb{N} : \sup_{x \in K} |\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)|(1+|\xi|^m) < \infty,$$

then $\sigma \in \mathcal{S}^{-\infty}$. In particular, if σ has compact ξ-support, $\sigma \in \mathcal{S}^{-\infty}$.

(iii) "A positively homogenous function", i.e. let $\sigma \in \mathscr{C}^\infty(U \times \mathbb{R}^n, \mathbb{C}^{r' \times r})$, which depends only on ξ and which is positively homogenous of degree $k \in \mathbb{R}$ in ξ. Then $\sigma \in \mathcal{S}^k$.

(iv) The function $\mathbb{R}^n \times \mathbb{R}^n$, $(x,\xi) \mapsto (1+|\xi|^2)^{\frac{k}{2}}$, $k \in \mathbb{R}$ is a symbol in $\mathcal{S}^k$.

Proof.

(i) Since σ is a polynomial of degree k, $\partial_\xi^\beta \sigma = 0$, if $|\beta| > k$ (see Lemma A.1.6). For any $|\beta| \leq k$ and any $\alpha \in \mathbb{N}^n$ we calculate

$$\begin{aligned}
|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| &\leq \sum_{|\gamma| \leq k} |\partial_x^\alpha (P_\gamma)(x)||\partial_\xi^\beta \xi^\gamma| \overset{\text{A.1.6}}{\leq} \sum_{|\gamma| \leq k} |\partial_x^\alpha (P_\gamma)(x)||\beta! \binom{\gamma}{\beta} \xi^{\gamma-\beta}| \\
&\overset{\text{A.2.1}}{\leq} \sum_{|\gamma| \leq k} \beta! \binom{\gamma}{\beta} \|P_\gamma\|_{\mathscr{C}^{|\alpha|}} |\xi|^{|\gamma|-|\beta|} \\
&\leq \max_{|\gamma| \leq k} \|P_\gamma\|_{\mathscr{C}^{|\alpha|}} \sum_{|\gamma| \leq k} \beta! \binom{\gamma}{\beta} (1+|\xi|)^{k-|\beta|} \leq C_{\alpha,\beta}(1+|\xi|)^{k-|\beta|}.
\end{aligned}$$

(ii) This follows directly from the definition of the Schwarz space and the symbols.

(iii) This follows from Lem:PropPosHomo.

(iv) The function

$$\begin{aligned} f : \mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n &\to \mathbb{R} \\ (a,\xi) &\mapsto (a^2+|\xi|^2)^{\frac{k}{2}} \end{aligned}$$

is smooth on $\mathbb{R}^{n+1} \setminus \{0\}$ and positively homogenous of degree k. Therefore by (4.24)

$$\forall \alpha \in \mathbb{N}_0^{n+1} : \exists C_\alpha > 0 : \forall (a,\xi) \in \mathbb{R} \times \mathbb{R}^{n+1} : |\partial_\xi^\alpha f(a,\xi)| \leq C_\alpha \|(a,\xi)\|^{k-|\alpha|}$$

Specifying to those α satisfying $\alpha_0 = 0$ and to $a = 1$, we obtain

$$|\partial_\xi^\alpha ((1+|\xi|^2)^{\frac{k}{2}})| = |\partial_\xi^\alpha f(1,\xi)| \leq C_\alpha \|(1,\xi)\|^{k-|\alpha|} \leq C_\alpha (1+\|\xi\|)^{k-|\alpha|}$$

□

4.5.6 Definition (Exhaustion function)**.** Let $0 < c_1 < c_2$. A function $\chi = \chi_{c_1,c_2} \in \mathscr{C}^\infty(\mathbb{R}^n)$, such that

$$\forall \xi \in \mathbb{R}^n : \chi(\xi) = \begin{cases} 0 & , |\xi| \leq c_1 \\ 1 & , |\xi| \geq c_2 \end{cases}$$

is an *exhaustion function.*

4.5.7 Lemma (Family of exhaustion functions)**.** Let χ be an exhaustion function and $\chi_\varepsilon : \mathbb{R}^n \to \mathbb{R}$, $\xi \mapsto \chi(\varepsilon\xi)$. Then

$$\forall \alpha \in \mathbb{N}^n : \exists C_\alpha > 0 : \forall 0 < \varepsilon \leq 1 : |\partial_\xi^\alpha \xi_\varepsilon(\xi)| \leq C_\alpha (1+|\xi|)^{-|\alpha|}.$$

So the functions χ_ε are symbols of order zero with symbol estimates that are independent of ε. In other words $\{\chi_\varepsilon | 0 < \varepsilon \leq 1\} \subset \mathcal{S}^0$ is bounded.

Proof. Let $\alpha \in \mathbb{N}^n$ be arbitrary.
CASE 1 ($|\alpha| \geq 1$): If $|\alpha| \geq 1$, then $\partial_\xi^\alpha \chi_\varepsilon$ is compactly supported and smooth. Therefore

$$\sup_{\xi \in \mathbb{R}^n} |\partial_\xi^\alpha \chi(\xi)(1+|\xi|)^{|\alpha|}| =: C_\alpha < \infty.$$

Since $1 + |\xi| \neq 0$, this implies

$$|\partial_\xi^\alpha (\chi_\varepsilon)(\xi)| = |\partial_\xi^\alpha (\chi)(\xi)| \varepsilon^{|\alpha|} \leq \varepsilon^{|\alpha|} C_\alpha (1+|\xi|)^{-|\alpha|}.$$

Since $\varepsilon \leq 1$, this implies the statement.
CASE 2 ($\alpha = 0$): We just have to show that $\{\chi_\varepsilon | 0 \leq \varepsilon \leq 1\}$ is uniformly bounded. Define

$$C' := \max_{c_1 \leq |\xi| \leq c_2} |\chi(\xi)|.$$

By definition of an exhaustion function

$$\forall \xi \in \mathbb{R}^n : |\chi(\xi)| \leq \max(1, C') =: C_0.$$

Now by construction for any $\xi \in \mathbb{R}^n$

$$|\xi| \leq \varepsilon^{-1} c_1 \Rightarrow |\varepsilon \xi| \leq c_1 \Rightarrow |\chi_\varepsilon(\xi)| = 0$$
$$|\xi| \geq \varepsilon^{-1} c_2 \Rightarrow |\varepsilon \xi| \geq c_2 \Rightarrow |\chi_\varepsilon(\xi)| = 1$$
$$\varepsilon^{-1} c_1 \leq \xi \leq \varepsilon^{-1} c_2 \Rightarrow c_1 \leq \varepsilon \xi \leq c_2 \Rightarrow |\chi_\varepsilon| \leq C'.$$

In all cases $|\chi_\varepsilon| \leq C_0$.

□

4.5.8 Definition (Asymptotic expansion)**.** Let $\sigma \in \mathcal{S}^k$. Suppose (k_j) is a real sequence, which diverges monotonously to $-\infty$. Assume there are $\sigma_j \in \mathcal{S}^{k_j}$, such that

$$\forall N \in \mathbb{N} : \sigma - \sum_{j=0}^{N-1} \sigma_j \in \mathcal{S}^{m_N}.$$

Then we call $\sum_{j=0}^\infty \sigma_j$ an *asyptotic expansion of* σ (some authors call it a *formal development*). We denote this by

$$\sigma \sim \sum_{j=0}^\infty \sigma_j.$$

4.5.9 Theorem (Asyptotic expansion)**.** Let (k_j) be a real monotonous sequence such that $k_j \to -\infty$, and let $\sigma_j \in \mathcal{S}^{k_j}$. Then there exists $\sigma \in \mathcal{S}^{k_0}$, such that

$$\sigma \sim \sum_{j=0}^\infty \sigma_j.$$

Moreover, any two symbols σ, τ with the same asyptotic expansion satisfy

$$\sigma - \tau \in \mathcal{S}^{-\infty}.$$

Proof.

Step 1 (Construction of σ): Let $\chi = \chi_{1,2}$ be an exhaustion function as in 4.5.6 and define $\chi_\varepsilon(\xi) := \chi(\varepsilon\xi)$, $0 < \varepsilon \leq 1$. By 4.5.7 for any $\beta \in \mathbb{N}^n$ there exists a constant $C_\beta > 0$, such that

$$\forall \xi \in \mathbb{R}^n : |\partial_\xi^\beta(\chi_\varepsilon)(\xi)| \leq C_\alpha(1+|\xi|)^{-|\beta|}.$$

The Leibniz rule implies

$$\begin{aligned}\exists C_{j,\alpha,\beta} > 0 : \forall x \in U : \forall \xi \in \mathbb{R}^n : &|\partial_x^\alpha \partial_\xi^\beta(\chi_\varepsilon p_j)(x,\xi)| \leq C_{j,\alpha,\beta}(1+|\xi|)^{k_j-|\beta|} \\ &= C_{j,\alpha,\beta}(1+|\xi|)^{-1}(1+|\xi|)^{k_j+1-|\beta|}. \end{aligned} \tag{4.25}$$

Choose a monotone sequence (ε_j), such that $0 < \varepsilon_j \leq 1$, $\varepsilon_j \to 0$ and

$$\forall j \in \mathbb{N} : \forall \alpha, \beta \in \mathbb{N}^n : |\alpha| + |\beta| \leq j \Rightarrow \varepsilon_j \leq 2^{-j} C_{j,\alpha,\beta}^{-1}. \tag{4.26}$$

Define $\chi_j := \chi_{\varepsilon_j}$ and

$$\sigma(x,\xi) := \sum_{j=0}^{\infty} \chi_j(\xi)\sigma_j(x,\xi).$$

Since $|\xi| \leq \varepsilon_j^{-1} \Rightarrow \chi_j(\xi) = 0$, this sum is locally finite. Hence σ is a well-defined function.

Step 2 (Estimates): Now let $N \in \mathbb{N}$ be arbitrary. We may decompose

$$\sigma - \sum_{j=0}^{N-1} \sigma_j = \underbrace{\sum_{j=0}^{N-1} (\chi_j - 1)\sigma_j}_{=:p_N} + \underbrace{\sum_{j=N}^{\infty} \chi_j \sigma_j}_{=:q_N}.$$

For any ξ, such that $|\xi| \geq 2\varepsilon_{N-1}^{-1}$, we obtain $p_N(\xi) = 0$ by construction. Consequently p_N has compact support and therefore $p_N \in \mathcal{S}^{-\infty}$ by 4.5.5. Therefore it suffices to analyse q_N: By construction, we obtain

$$\begin{aligned}&\forall (x,\xi) \in U \times \mathbb{R}^n : \forall |\alpha| + |\beta| \leq j : |\partial_x^\alpha \partial_\xi^\beta(\chi_j\sigma_j)(x,\xi)| \leq C_{j,\alpha,\beta}(1+|\xi|)^{-1}(1+|\xi|)^{k_j+1-|\beta|} \\ &\overset{(4.26)}{\leq} 2^{-j}\varepsilon_j(1+|\xi|)^{-1}(1+|\xi|)^{k_j+1-|\beta|} \leq 2^{-j}(1+|\xi|)^{k_j+1-|\beta|}\end{aligned} \tag{4.27}$$

Now choose any fixed $\alpha, \beta \in \mathbb{N}^n$. Choose $j_0 \in \mathbb{N}$, such that

$$j_0 \geq \max(N, |\alpha| + |\beta|), \qquad k_{j_0} + 1 \leq k_N. \tag{4.28}$$

Since $\chi_j p_j \in \mathcal{S}^{k_j}$ and since $k_j \searrow -\infty$

$$q_N = \underbrace{\sum_{j=N}^{j_0-1} \chi_j\sigma_j}_{\in \mathcal{S}^{k_N}} + \underbrace{\sum_{j=j_0}^{\infty} \chi_j\sigma_j}_{=:q_{j_0}}.$$

By the choice of j_0 in (4.28) and (4.27), we calculate

$$|\partial_x^\alpha \partial_\xi^\beta q_{j_0}(x,\xi)| \leq \sum_{j=j_0}^{\infty} 2^{-j}(1+|\xi|)^{k_j+1-|\beta|} \leq \sum_{j=j_0}^{\infty} 2^{-j}(1+|\xi|)^{k_N-|\beta|} \leq (1+|\xi|)^{k_N-|\beta|}.$$

Consequently $q_N \in \mathcal{S}^{k_N}$. For $N = 0$, we obtain in particular

$$\sigma = \sum_{j=0}^{\infty} \chi_j\sigma_j = q_0 \in \mathcal{S}^{k_0}.$$

Step 3 (Uniqueness): Assume that

$$\sigma, \tau \sim \sum_{j=0}^{\infty} \sigma_j.$$

By construction

$$\forall N \in \mathbb{N} : \sigma - \tau = \Big(\sigma - \sum_{j=0}^{N-1} \sigma_j\Big) - \Big(\tau - \sum_{j=0}^{N-1} \sigma_j\Big) \in \mathcal{S}^{k_N}.$$

Since $k_j \searrow -\infty$, this implies $\sigma - \tau \in \mathcal{S}^{-\infty}$.

$\square$

5. Distribution Theory

"We Magog know the Divine exists. We know it created the stars, and the planets, the soft winds, and the gentle rains. We also know He created nightmares, because He created us."

Rev Bem, 10087 CY

In the previous section we described spaces of functions, which are the heaven of analysis. The dual of this is hell, which we describe in this chapter.

5.1. Basic Definitions

5.1.1 Remark (Reminder of convergence)**.** For those of you who just tuned in here is an overview of the most important function spaces in distribution theory (introduced in detail in section 4): Let $U \overset{\circ}{\subseteq} \mathbb{R}^n$ and let

$$\mathscr{D}(U) := \mathscr{C}^\infty_c(U, \mathbb{C})$$

endowed with the following notion of convergence: We say ϕ_j *converges to* ϕ *in* $\mathscr{D}(U)$, if there exists a compact $K \Subset U$ such that

$$\forall j \in \mathbb{N} : \operatorname{supp} \phi_j \subset K$$

and for any $k \in \mathbb{N}$

$$\phi_j \xrightarrow[\mathscr{C}^k(K)]{} \phi \, .$$

We denote this by

$$\phi_j \xrightarrow[\mathscr{D}]{} \phi \, .$$

Let

$$\mathscr{S} := \{\phi \in \mathscr{C}^\infty(U, \mathbb{C}) \mid \forall \alpha, \beta \in \mathbb{N}^n : p_{\alpha,\beta}(f) := \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta(f)(x)| < \infty\}$$

endowed with the following notion of convergence: We say ϕ_j *converges to* ϕ, if

$$\forall \alpha, \beta \in \mathbb{N}^n : \ \phi_j \xrightarrow[p_{\alpha,\beta}]{} \phi \, .$$

We denote this by

$$\phi_j \xrightarrow[\mathscr{S}]{} \phi \, .$$

Let

$$\mathscr{E}(U) := \mathcal{C}^\infty(U, \mathbb{C})$$

endowed with the following notion of convergence: We say a sequence ϕ_j *converges to* ϕ *in* $\mathscr{E}(U)$, if

$$\forall K \Subset U : \forall k \in \mathbb{N} : \ \phi_j \xrightarrow[\mathscr{C}^k(K)]{} \phi \, .$$

We denote this by

$$\phi_j \xrightarrow[\mathscr{E}]{} \phi \, .$$

In the last chapter, we constructed topologies on $\mathscr{D}$, $\mathscr{S}$, $\mathscr{E}$ and showed in excruciating detail that these topologies induce this notion of convergence. For many applications you can just forget about topology and take this as a definition. Whenever the continuity statements are involved, you can read them as sequential continuity and the convergence of a sequence was just defined.

5.1.2 Definition (Distribution)**.** Let $U \subset \mathbb{R}^n$ be open. The topological dual space, i.e. the space of continuous linear functionals,

(i) $\mathscr{D}'(U)$ is the *space of distributions.*

(ii) $\mathscr{S}'$ is the *space of tempered distributions.*

(iii) $\mathscr{E}'(U)$ is the *space of distributions having compact support.*

All these spaces are topologized by Theorem 3.1.27, i.e. they are endowed with the weak-*-topology, i.e. the topology of pointwise convergence. More explicitely, if $\mathscr{H} \in \{\mathscr{D}(U), \mathscr{E}(U), \mathscr{S}\}$, then

$$T_j \xrightarrow[\mathscr{H}']{} T \iff \forall \varphi \in \mathscr{H} : \; T_j(\varphi) \xrightarrow[\mathbb{C}]{} T(\varphi)$$

By the remark 5.1.1 above, a linear map $T : \mathscr{H}(U) \to \mathbb{C}$ is a distribution, if

$$\forall (\phi_j) \in \mathscr{H}^{\mathbb{N}} \; : \; \phi_j \xrightarrow[\mathscr{H}]{} 0 \implies T(\phi_j) \xrightarrow[\mathbb{C}]{} 0 \,.$$

On the other hand, if we use the topology on $\mathscr{D}(U)$, we can give another characterization of distributions.

5.1.3 Theorem and Definition (order)**.** A linear form $T : \mathscr{D}(U) \to \mathbb{C}$ is a distribution (i.e. is continuous) if and only if for every $K \Subset U$ there exist constants $C > 0$, $k \in \mathbb{N}$, such that

$$\forall \varphi \in \mathscr{D}(U) : |T(\varphi)| \leq C\|u\|_{\mathscr{C}^k(K)}.$$

If the constant k may be chosen independently of K, we call the smallest such k the *order of* u.

Proof. The existence of k is a direct consequence of 4.2.7 and the definition of the topology on $\mathscr{D}_K$. □

From 3.1.27 we also obtain:

5.1.4 Theorem. All the distribution spaces $\mathscr{D}'(U)$, $\mathscr{E}'(U)$ and $\mathscr{S}'$ are complete.

The following is completely trivial and therefore often a source of confusion.

5.1.5 Lemma (inclusions and restrictions)**.**

(i) $\mathscr{D}(U) \subset \mathscr{E}(U)$, the inclusion

$$\iota : \mathscr{D}(U) \hookrightarrow \mathscr{E}(U)$$

is continuous and hase dense image. The dual operator gives a continuous map

$$\begin{aligned} \iota' : \mathscr{E}'(U) &\to \mathscr{D}'(U) \\ T &\mapsto \iota'(T) = T|_{\mathscr{D}(U)} \end{aligned}$$

(ii) In case $U = \mathbb{R}^n$, $\mathscr{D} := \mathscr{D}(\mathbb{R}^n)$, $\mathscr{E} := \mathscr{E}(\mathbb{R}^n)$, we have the relations

$$\mathscr{D} \subset \mathscr{S} \subset \mathscr{E}$$

and the inclusions

$$i : \mathscr{D} \hookrightarrow \mathscr{S}, \qquad\qquad j : \mathscr{S} \hookrightarrow \mathscr{E}$$

are continuous. Their dual operators give continuous maps

$$\begin{aligned} i' : \mathscr{S}' &\to \mathscr{D}' \\ T &\mapsto i'(T) = T|_{\mathscr{D}(U)} \end{aligned} \qquad\qquad \begin{aligned} j' : \mathscr{E}' &\to \mathscr{S}' \\ T &\mapsto j'(T) = T|_{\mathscr{S}}. \end{aligned}$$

Proof. It suffices to check the statements for ι, i, j, since refThm:DualOperator implies the statements for the dual operatrs.

(i) The relation $\mathscr{D}(U) \subset \mathscr{E}(U)$ follows from the definition. So it is clear that the inclusion maps between the right spaces. To check continuity assume

$$\phi_j \xrightarrow[\mathscr{D}(U)]{} 0 \, .$$

By definition there exists $\tilde{K} \Subset U$ such that

$$\forall j \in \tilde{K} : \operatorname{supp} \phi_j \subset \tilde{K}$$

and

$$\forall k \in \mathbb{N} \, : \; \phi_j \xrightarrow[\mathscr{C}^k(\tilde{K})]{} 0 \, .$$

Now clearly, for any other $K \Subset U$

$$\forall k \in \mathbb{N} : \|\phi_j\|_{\mathscr{C}^k(K)} = \|\phi_j\|_{\mathscr{C}^k(K \cap \tilde{K})} \leq \|\phi_j\|_{\mathscr{C}^k(\tilde{K})},$$

thus

$$\phi_j \xrightarrow[\mathscr{C}^k(K)]{} 0$$

as well. By definition, this implies

$$\iota(\phi_j) \xrightarrow[\mathscr{E}(U)]{} 0 \, .$$

(ii) For $i : \mathscr{D} \to \mathscr{S}$ we argue as in the first part and check for any $\alpha, \beta \in \mathbb{N}^n$

$$p_{\alpha,\beta}(\phi_j) = \sup_{x \in \tilde{K}} |x^\alpha D^\beta(\phi_j)| \leq C \|\phi_j\|_{\mathscr{C}^{|\beta|}(\tilde{K})} \to 0.$$

For j we assume that

$$\phi_j \xrightarrow[\mathscr{S}]{} 0 \, .$$

For any $K \Subset \mathbb{R}^n$ this implies in particular

$$\forall \beta \in \mathbb{N}^n : \sup_{x \in K} |D^\beta \phi_j(x)| \leq p_{0,\beta}(\phi_j) \to 0.$$

Thus

$$\phi_j \xrightarrow[\mathscr{E}]{} 0 \, .$$

□

5.1.6 Remark. We have not yet shown that the dual operators ι', i', j' are injective as well. Of course they are, but it will be much more convenient to derive this statement later, c.f. 5.4.6.

How does a typical distribution look like?

5.1.7 Theorem and Definition (regular distributions)**.** A distribution $T \in \mathscr{D}'(U)$ is *regular*, if there exists $f \in L^1_{\text{loc}}(U, \mathbb{C})$ such that

$$\forall \varphi \in \mathscr{D}(U) : T(\varphi) = \langle f \rangle(\varphi) := \int_U f(x)\varphi(x) dx.$$

This defines an injection $\langle \_ \rangle : L^1_{\text{loc}}(U) \hookrightarrow \mathscr{D}'(U)$. (This is why some people, in particular physisics, do not distinguish between f and $\langle f \rangle$, but we will do so.) The inverse map on the image will be denoted by $\rangle \_ \langle : \langle L^1_{\text{loc}}(U) \rangle \subset \mathscr{D}'(U) \to L^1_{\text{loc}}(U)$.

Proof. By definition for any $\phi \in \mathscr{D}(U)$ there exists a $K \Subset U$ such that $\operatorname{supp}\phi \subset K$. Since f is integrable over any compact subset

$$|\langle f\rangle(\phi)| \leq \int_U |f(x)||\phi(x)|dx \leq \|\phi\|_{\mathscr{C}^0(K)}\|f\|_{L^1(K)}.$$

Therefore $\langle f\rangle \in \mathscr{D}'(U)$ is a distribution of order 0 by Lemma 5.1.3.
So the map $\langle \_ \rangle : L^1_{\text{loc}}(U) \to \mathscr{D}'(U)$ is well-defined. Its injectivity is a direct consequence of the stronger statement 5.1.8 below. □

5.1.8 Theorem (Fundamental Lemma of the Calculus of Variations)**.** Let $U \subset \mathbb{R}^n$ be open and $f \in [f] \in L^1(U)$ be a representative of an L^1-class. The following are equivalent:

(i) For any $\varphi \in \mathscr{D}(U)$: $\int_U f(x)\varphi(x)dx = 0$.

(ii) For any measurable bounded subset $M \Subset U$: $\int_U f(x)dx = 0$.

(iii) $f = 0$ a.e.

Proof. □ ref

5.2. Algebraic Properties

5.2.1. Module Structure

5.2.1 Definition (Multiplication by functions)**.** Let $T \in \mathscr{D}'(U)$ and $f \in \mathscr{E}(U)$. Then

$$\begin{aligned} fT : \mathscr{D}(U) &\to \mathbb{C} \\ \phi &\mapsto T(f\phi). \end{aligned}$$

is the *multiplication of f and T*.

5.2.2 Lemma. For any $f \in \mathscr{E}(U)$ the functional fT satisfies $fT \in \mathscr{D}'(U)$. Therefore $\mathscr{D}'(U)$ is a module over $\mathscr{E}(U)$.

Proof. Follows from the Leibniz rule. □

5.2.2. Sheaf Structure

5.2.3 Definition (extensions and restrictions)**.** Let $V \subset U \subset \mathbb{R}^n$ be open. Any function $\phi \in \mathscr{D}(V)$ can be extended by zero to a function $\phi_0 \in \mathscr{D}(U)$ (since $\operatorname{supp}\phi \Subset V$, ϕ_0 is still smooth).
This defines a restriction

$$\begin{aligned} \rho^U_V : \mathscr{D}'(U) &\to \mathscr{D}'(V) \\ T &\mapsto T|_V, \end{aligned}$$

where

$$\begin{aligned} T|_V : \mathscr{D}(V) &\to \mathbb{C} \\ \phi &\mapsto T(\phi_0). \end{aligned}$$

5.2.4 Theorem (Sheaf Structure)**.** $\mathscr{D}'$ is a sheaf of $\mathbb{C}$-vector spaces on $\mathbb{R}^n$ (c.f. **??**) (hence on any $U \overset{\circ}{\subseteq}$). In particular it satisfies the sheaf axioms

(i) For any $T \in \mathscr{D}'(\mathbb{R}^n)$ and any open cover $\mathbb{R}^n = \bigcup_{i\in I} U_i$

$$\forall i \in I : T|_{U_i} = 0 \Longrightarrow T = 0.$$

(ii) For any open cover $\mathbb{R}^n = \bigcup_{i\in I} U_i$ and any system $T_i \in \mathscr{D}'(U_i)$ such that

$$\forall i,j \in I : T_i|_{U_i\cap U_j} = T_j|_{U_i\cap U_j}$$

there exists $T \in \mathscr{D}'(\mathbb{R}^n)$ such that

$$\forall i \in I : T|_{U_i} = T_i.$$

This T is unique by (i).

Proof. Since $\mathbb{R}^n$ is paracompact, we may assume I to be countable.
STEP 1 (Presheaf Structure): Clearly if $T \in \mathscr{D}'(U)$ is of order k and $K \Subset U$

$$\forall \phi \in \mathscr{D}(V) : |T|_V(\phi)| = |T(\phi_0)| \le C\|\phi\|_{\mathscr{C}^k},$$

thus $T|_V$ really is a distribution (of order $\le k$). By construction $\mathscr{D}$ satisfies the presheaf axioms.
STEP 2 (First Sheaf Axiom): Let $U \subset \mathbb{R}^n$ and assume $U = \bigcup_{i\in I} U_i$ is an open cover and $T \in \mathscr{D}'(U)$ satisfies

$$\forall i \in I : T|_{U_i} = 0. \tag{5.1}$$

Let $\phi \in \mathscr{D}(U)$ be arbitrary. Since $\operatorname{supp}\phi \subset K \subset U$, where K is compact, there exists a finite subset $I' \subset I$, such that $K \subset \bigcup_{i\in I'} U_i$. By Theorem 4.2.10 for any $i \in I'$, there exists $\phi_i \in \mathscr{D}(U_i)$, such that

$$\phi = \sum_{i\in I'} \phi_i. \tag{5.2}$$

By linearity this implies

$$T(\phi) = \sum_{i\in I'} T(\phi_i) = \sum_{i\in I'} T|_{U_i}(\phi_i) \overset{(5.1)}{=} 0.$$

STEP 3 (Second Sheaf Axiom): Again let $U = \bigcup_{i\in I} U_i$ be an open cover and assume for any $i \in I$, there exists $T_i \in \mathscr{D}'(U_i)$ such that

$$\forall i,j \in I : T_i|_{U_i\cap U_j} = T_j|_{U_i\cap U_j}. \tag{5.3}$$

Let $\phi \in \mathscr{D}(U)$ be arbitrary again decompose it into $\phi = \sum_{i\in I'} \phi_i$ as in (5.2). Define

$$T(\phi) := \sum_{i\in I'} T_i(\phi_i).$$

We have to show that T is well-defined, i.e. that it does not depend on the chosen decomposition. It suffices to check

$$\sum_{i\in J} \phi_j = 0 \in \mathscr{D}(U) \Longrightarrow \sum_{j\in J} T_j(\phi_j) = 0,$$

where J is any finite index set and $\phi_j \in \mathscr{D}(U_j)$. Define $K := \bigcup_{j\in J} \operatorname{supp}\phi_j \subset U$ compact. By Theorem ... (!ToDoRef) there exist functions $\psi_k \in \mathscr{D}(U_k)$, $k \in J'$, J' finite, such that $\sum_{k\in J'} \psi_k = 1$. Then $\psi_k\phi_j \in \mathscr{D}(U_j \cap U_k)$ Therefore

$$\begin{aligned}\sum_{j\in J} T_j(\phi_j) &= \sum_{j\in J} T_j\Big(\sum_{k\in J'} \psi_k\phi_j\Big) = \sum_{j\in J}\sum_{k\in J'} T_j(\psi_k\phi_j) \overset{(5.3)}{=} \sum_{j\in J}\sum_{k\in J'} T_k(\psi_k\phi_j)\\ &= \sum_{k\in J'}\sum_{j\in J} T_k(\psi_k\phi_j) = \sum_{k\in J'} T_k\Big(\psi_k \sum_{j\in J}\phi_j\Big) = 0\end{aligned}$$

by hypothesis.
Consequently T is a well-defined map, which is obviously linear.
To see that it is continuous let $\phi \in \mathscr{D}(U)$ again let $\operatorname{supp}\phi \subset K$ as in Step 2 and $\psi_i \mathscr{D}(U_i)$, $i \in I'$, such that $\sum_{i\in I'} \psi_i = 1$. This implies

$$|T(\phi)| = \Big|T\Big(\sum_{i\in I'} \psi_i \phi\Big)\Big| \leq \sum_{i\in I'} |T_i(\psi_i\phi)| \leq \sum_{i\in I'} C_i \|\psi_i\phi\|_{\mathscr{C}^{k_i}(K)} \leq C\|\phi\|_{\mathscr{C}^k(K)}$$

(!ToDo noch etwas unpräzise).

□

5.3. Differentiation

Distributions are a perfect setting for differential operators.

5.3.1 Definition (Derivatives of Distributions)**.** Let $T \in \mathscr{D}'(U)$ and $\alpha \in \mathbb{N}^n$. Then

$$\begin{array}{rcl} \partial^\alpha T : \mathscr{D}(U) & \to & \mathbb{C} \\ \varphi & \mapsto & (-1)^{|\alpha|} T(\partial^\alpha \varphi) \end{array}$$

is a *derivative of* T. Analogously we define $D^\alpha T := (-i)^{|\alpha|}\partial^\alpha T$.
A linear combination

$$P = \sum_{|\alpha|\leq k} P_\alpha D^\alpha,$$

where $P_\alpha \in \mathscr{E}(U)$ is a *distributional differential operator.*

5.3.2 Lemma (Properties of Differentiation)**.**

(i) Let $T \in \mathscr{D}'(U)$. Then $\partial^\alpha T \in \mathscr{D}'(U)$. If T is of order k and $|\alpha| \leq l$. Then $\partial^\alpha T \in \mathscr{D}'(U)$ is of order $k + l$.

(ii) Any distributional differential operator P is a continuous operator

$$P : \mathscr{D}'(U) \to \mathscr{D}'(U).$$

(iii) For any k and $|\alpha| \leq k$ the following diagram commutes

$$\begin{array}{ccc} \mathscr{C}^k(U) & \xrightarrow{\langle\_\rangle} & \mathscr{D}'(U) \\ \downarrow{\scriptstyle \partial^\alpha} & & \downarrow{\scriptstyle \partial^\alpha} \\ \mathscr{C}^0(U) & \xrightarrow{\langle\_\rangle} & \mathscr{D}'(U) \end{array}$$

i.e. $\partial^\alpha\langle f\rangle = \langle \partial^\alpha f\rangle$.

Proof.

(i) Let $K \Subset U$. By definition there exists $C > 0$ and $k \in \mathbb{N}$, such that

$$\forall \varphi \in \mathscr{D}(U) : |T(\varphi)| \leq C\|\varphi\|_{\mathscr{C}^k(K)}$$

This implies

$$\forall \varphi \in \mathscr{D}(U) : |\partial^\alpha T(\varphi)| = |T(\partial^\alpha \varphi)| \leq C\|\partial^\alpha\varphi\|_{\mathscr{C}^k(K)} \leq C\|\varphi\|_{\mathscr{C}^{k+l}(K)}.$$

This shows $\partial^\alpha \in \mathscr{D}'(U)$ and the statement about the orders.

(ii) Follows from (i) and Lemma 5.2.2.

(iii) This is a direct application of partial integration: Let $f \in \mathscr{C}^k(U)$ and $\varphi \in \mathscr{D}(U)$ with support $\operatorname{supp}\varphi =: K \Subset U$. We extend $\varphi, f\partial_i\varphi, \partial_i f\varphi \in \mathscr{C}_c^{k-1}(\mathbb{R}^n)$ by zero, choose $R > 0$ such that $K \Subset B_R(0)$ and denote by ν the outward pointing unit normal field on $B_R(0)$. By Green's formula this implies for any $1 \leq j \leq n$ ref

$$\begin{aligned} \partial_j\langle f\rangle(\varphi) &= -\langle f\rangle(\partial_j\varphi) = -\int_U f(x)\partial_j(\varphi)(x)dx = -\int_{B_R(0)} f(x)\partial_j(\varphi)(x)dx \\ &= -\int_{\partial B_R(0)} f(x)\varphi(x)\nu^i(x)dx + \int_{B_R(0)} \partial_j(f)(x)\varphi(x)dx = \langle \partial_j f\rangle(\varphi), \end{aligned}$$

since $\operatorname{supp}\varphi \Subset K$. By induction we obtain the statement for arbitrary differentials ∂^α.

□

This explains the mysterious sign convention: If we had not introduced the factor $(-1)^{|\alpha|}$, the diagram were only commutative up to sign.
Also notice that there is no notion of "differentiability" for distributions: The are all differentiable of arbitrary order. Nevertheless it would not make sense to call them smooth since they are the most irregular objects in analysis. Differentiation of distributions works so well, because a distributional derivative is one of the weakest possible forms of differentiation.
Nevertheless there are several rules from classical calculus, which still hold.

5.3.3 Theorem (local constancy)**.** Let $U \subset \mathbb{R}^n$ be connected and for any $c \in \mathbb{C}$ denote by

$$\begin{aligned} f_c : U &\to \mathbb{C} \\ x &\mapsto c \end{aligned}$$

the constant function. Then for any $T \in \mathscr{D}'(U)$

$$\forall 1 \leq i \leq n : \partial_i T = 0 \iff \exists c \in \mathbb{C} : T = \langle f_c\rangle.$$

Proof.
"$\Rightarrow$": In classical calculus this is proven by the mean value theorem, which we do not have at our disposal. Therefore this is the hard direction.

Step 1 (reduction to local problem): We will check that $T|_V$ is generated by a constant function, where V is of the form

$$\emptyset \neq V = \tilde{V} \times I \mathring{\subseteq} U, \qquad \emptyset \neq \tilde{V} \mathring{\subseteq} \mathbb{R}^{n-1}, \qquad \emptyset \neq I \mathring{\subseteq} \mathbb{R}.$$

It is clear that if $T|_V = \langle f_c\rangle$ and $T|_W = \langle f_{c'}\rangle$, where $V \cap W \neq$, this implies $c = c'$, since $\mathscr{D}'(U)$ is a sheaf (c.f. Theorem 5.2.4). 5.2.4 also implies that it suffices to check the statement for $T := T|_V$.

Step 2 (vanishing criterion): We claim

$$\forall f \in \mathscr{D}(I) : \int_I f(t)dt = 0 \Rightarrow T(f) = 0. \tag{5.4}$$

The argument for this is the following: Extend f to a function $f \in \mathscr{D}(\mathbb{R})$ by zero. By assumption the function

$$\begin{aligned} F : I &\to \mathbb{C} \\ x &\mapsto \int_{-\infty}^x f(t)dt \end{aligned}$$

satisfies

$$F \in \mathscr{D}(I), \qquad\qquad F' = f.$$

Consequently, we obtain

$$T(f) = T(F') = -T'(F) = 0.$$

Step 3 (dimensional reduction): For any $f \in \mathscr{D}(V)$, define

$$\begin{array}{rcl} \tilde{f} : \tilde{V} & \to & \mathbb{C} \\ \tilde{x} & \mapsto & \int_I f(\tilde{x}, x_n) dx_n. \end{array}$$

We will now prove that for any $T \in \mathscr{D}'(V)$

$$\partial_n T = 0 \Longrightarrow \exists \tilde{T} \in \mathscr{D}'(\tilde{V}) : \forall f \in \mathscr{D}(V) : T(f) = \tilde{T}(\tilde{f}), \tag{5.5}$$

Choose a function

$$\psi \in \mathscr{D}(I), \qquad\qquad \int_I \psi(t)dt = 1 \tag{5.6}$$

Let $f \in \mathscr{D}(V)$ be arbitrary (again extended by zero to a function $f \in \mathscr{D}(\tilde{V} \times \mathbb{R})$ and define

$$\begin{array}{rcl} g : V & \to & \mathbb{C} \\ x = (\tilde{x}, x_n) & \mapsto & \int_{-\infty}^{x_n} f(\tilde{x}, s) - \int_I f(\tilde{x}, t)dt \; \psi(s)ds. \end{array}$$

Since for any $\tilde{x} \in \tilde{V}$

$$\int_I g(\tilde{x}, x_n)dx_n = \tilde{f}(\tilde{x}) - \tilde{f}(\tilde{x}) \int_I \psi(s)ds = 0,$$

this implies $g \in \mathscr{D}(V)$ and

$$\forall x = (\tilde{x}, x_n) \in V : \partial_n g(x) = f(x) - \tilde{f}(\tilde{x})\psi(x_n) =: f(x) - (\tilde{f} \otimes \psi)(x) \tag{5.7}$$

Consequently by defining the distribution $\tilde{T} \in \mathscr{D}'(\tilde{V})$ by

$$\begin{array}{rcl} \tilde{T} : \mathscr{D}(\tilde{V}) & \to & \mathbb{C} \\ h & \mapsto & T(\tilde{h} \otimes \psi), \end{array}$$

we obtain

$$0 = -\partial_n(T)(g) = T(\partial_n g) \overset{(5.7)}{=} T(f) - T(\tilde{f} \otimes \psi) = T(f) - \tilde{T}(\tilde{f}).$$

Step 4: We will prove the statement by induction over n.

Step 4.1 ($n = 1$): Define

$$c := T(\psi),$$

where ψ is from (5.6). Let $\phi \in \mathscr{D}(I)$ be arbitrary. Define the function $f \in \mathscr{D}(I)$ by

$$f(x) := \phi(x) - \psi(x) \int_I \phi(t)dt.$$

We obtain

$$\int_I f(x)dx = \int_I \phi(x)dx - \int_I \psi(x)dx \int_I \phi(t)dt = 0$$

and therefore

$$0 \overset{(5.4)}{=} T(f) = T\Big(\phi - \psi \int_I 1\phi(t)dt\Big) = T(\phi) - T(\psi)\langle f_1 \rangle(\phi) = T(\phi) - \langle f_c \rangle(\phi).$$

Step 4.2 ($n-1 \to n$): By hypothesis there exists $\tilde{c} \in \mathbb{R}$ such that

$$\forall \phi \in \mathscr{D}(V) : T(\phi) \overset{(5.5)}{=} \tilde{T}(\tilde{\phi}) = \langle f_{\tilde{c}} \rangle(\tilde{\phi}).$$

We define $c := \tilde{c}$ and claim that $T = \langle f_c \rangle \in \mathscr{D}'(V)$: We calulate for any $\phi \in \mathscr{D}(V)$

$$\begin{aligned} T(\phi) = \langle f_{\tilde{c}} \rangle(\tilde{\phi}) &= \int_{\tilde{V}} \tilde{c}\tilde{\phi}(\tilde{x}) d\tilde{x} = \int_{\tilde{V}} \tilde{c} \int_I \phi(\tilde{x}, x_n) dx_n d\tilde{x} \\ &= \int_{\tilde{V} \times I} c\phi(\tilde{x}, x_n) d(\tilde{x}, x_n) = \int_V c\phi(x) dx = \langle f_c \rangle(\phi). \end{aligned}$$

"$\Leftarrow$": We simply calculate

$$\partial_i T = \partial_i \langle f_c \rangle \overset{5.3.2,\text{(iii)}}{=} \langle \partial_i f_c \rangle = \langle 0 \rangle = 0.$$

□

5.4. Supports

You might have been wondering why the space $\mathscr{E}'$ is called the *distributions with compact support*. This will be apparent in a moment.

5.4.1 Definition (support)**.** Let $T \in \mathscr{D}'(U)$ be a distribution. Then

$$\operatorname{supp} T := U \setminus \{x \in U \mid \exists V \mathring{\subseteq} U : T|_V = 0\}$$

is the *support of* T. We say T is *compactly supported*, if $\operatorname{supp} T \subset U$ is compact.

5.4.2 Lemma (Properties of supports)**.** Let $T \in \mathscr{D}'(U)$.

(i) $\operatorname{supp} T \subset U$ is closed.

(ii) $T|_{U \setminus \operatorname{supp} T} = 0$.

(iii) For any $\varphi \in \mathscr{D}(U) : \operatorname{supp} \varphi \cap \operatorname{supp} T = \emptyset \implies T(\varphi) = 0$.

Proof.

(i) Follows from the definition.

(ii) Follows from 5.2.4,(ii).

(iii) Since $V := U \setminus \operatorname{supp} T$ is open, $\operatorname{supp} \varphi \subset V$ implies by definition

$$T(\varphi) = T|_V(\varphi) = 0.$$

□

The following Lemma should convice you that the notion of a support of a distribution is reasonable.

5.4.3 Lemma.

(i) If $f \in \mathscr{C}^0(U)$, then

$$\operatorname{supp}\langle f \rangle = \operatorname{supp} f.$$

(ii) In case $f \in L^1_{\text{loc}}(U)$ a point $x \in U$ is in $\operatorname{supp}\langle f \rangle$ if and only if for all sufficiently small $\varepsilon > 0$

$$\int_{B_\varepsilon(0)} |f(x)| > 0.$$

Proof.

(i) Let $x \in U$. By definition

$$\begin{aligned} x \notin \operatorname{supp}\langle f\rangle &\iff \exists x \in V \mathring{\subseteq} U : \langle f\rangle|_V = 0 \\ &\iff \exists x \in V \mathring{\subseteq} U : \forall \phi \in \mathscr{D}(V) : 0 = \langle f\rangle|_V(\phi) = \int_V f(x)\phi(x)dx \\ &\overset{5.1.8}{\iff} \exists x \in V \mathring{\subseteq} U : f|_V = 0 \\ &\iff x \notin \operatorname{supp} f. \end{aligned}$$

(ii) Assume $x \notin \operatorname{supp}\langle f\rangle$. Then there exists $\delta > 0$ such that $\langle f\rangle|_{B_\delta(x)} = 0$. By 5.1.8 this implies $f|_{B_\delta(x)} = 0$ a.e. Consequently for any $0 < \varepsilon < \delta$

$$\int_{B_\varepsilon(x)} |f(x)| = 0.$$

Conversely assume

$$\int_{B_\varepsilon(x)} |f(x)| = 0$$

for all sufficiently small $\varepsilon > 0$. Take any such ε. By 5.1.8 again, $f|_{B_\varepsilon(x)} = 0$ a.e. and

$$\forall \phi \in \mathscr{D}(B_\varepsilon(x)) : 0 = \int_{B_\varepsilon(x)} f(x)\phi(x)dx = \langle f\rangle(\phi),$$

thus $x \notin \operatorname{supp}\langle f\rangle$.

□

5.4.4 Theorem (Distributions with compact support)**.** Let $U \subset \mathbb{R}^n$ be open. Denote by $\iota : \mathscr{D}(U) \hookrightarrow \mathscr{E}(U)$ the canonical inclusion from 5.1.5.

(i) We claim

$$\iota'(\mathscr{E}'(U)) =: \mathscr{D}'_c(U) = \{T \in \mathscr{D}'(U) \mid T \text{ has compact support}\} \subset \mathscr{D}'(U)$$

and ι' is a linear homeomorphism $\mathscr{E}'(U) \to \mathscr{D}'_c(U)$. The inverse $\iota'^{-1} : \mathscr{D}'_c(U) \to \mathscr{E}'(U)$ may be explicitely computed as follows: Let $T \in \mathscr{D}'_c(U)$ with compact support $\operatorname{supp} T =: K$, let $K \Subset V \mathring{\subseteq} U$, $\psi \in \mathscr{D}(U)$ such that $\psi|_V = 1$ and $\phi \in \mathscr{E}(U)$. Then

$$\iota'^{-1}(T)(\phi) = T(\psi\phi).$$

(ii) The set $\mathscr{E}'(U) \cong \mathscr{D}'_c(U) \subset \mathscr{D}'(U)$ is dense. For any $T \in \mathscr{D}'(U)$ there exists a sequence $T_j \in \mathscr{E}'(U)$ such that

$$T_j \xrightarrow[\mathscr{D}']{} T$$

and for any $K \Subset U$ there exists $j(K) \in \mathbb{N}$ such that

$$\forall j \geq j(K) : \forall \varphi \in \mathscr{D}_K(U) : T_j(\varphi) = T(\varphi).$$

Proof.

Step 1 ($\iota'(\mathscr{E}'(U)) \subset \mathscr{D}'_c(U)$): Let $T \in \mathscr{E}'(U)$ be arbitrary. Since T is continuous, Theorem 3.1.22 implies, there exist constants $C > 0$, $k \in \mathbb{N}$, and a $K \Subset U$, such that

$$\forall \phi \in \mathscr{E}(U) : |T(\phi)| \leq C\|\phi\|_{\mathscr{C}^k(K)}$$

We claim that $\operatorname{supp} \iota'(T) = \operatorname{supp} T|_{\mathscr{D}(U)} \subset K$: Let $x \in U \setminus K$. Since this is open, there exists an open neighbourhood $V \subset U \setminus K$ of x. For any $\varphi \in \mathscr{D}(V)$, we obtain

$$|T(\varphi)| \leq C\|\varphi\|_{\mathscr{C}^k(K)} = 0.$$

Thus $T|_V = 0$. Therefore $x \notin \operatorname{supp} T$ and consequently $\operatorname{supp} \iota'(T) \subset K$. At this point we have established that

$$\iota' : \mathscr{E}'(U) \to \mathscr{D}'_c(U)$$

is a continuous linear map.

Step 2 (ι'^{-1} is well-defined): Assume $\psi_1, \psi_2 \in \mathscr{D}(U)$, $K \Subset V_1 \mathring{\subseteq} U$, $K \Subset V_2 \mathring{\subseteq} U$ and $\psi|_{V_1} = \psi|_{V_2} = 1$. This implies

$$\forall \phi \in \mathscr{E}(U) : \operatorname{supp}((\psi_1 - \psi_2)\varphi) \cap K = \emptyset.$$

Since $(\psi_1 - \psi_2)\varphi \in \mathscr{D}(U)$, we obtain

$$0 \overset{5.4.2}{=} T((\psi_1 - \psi_2)\phi) = T(\psi_1\phi) = T(\psi_2\phi).$$

Thus $\iota'^{-1} : \mathscr{D}'_c(U) \to \mathscr{E}'(U)$ is a well-defined linear map.

To see that it is continuous, let $T \in \mathscr{D}'_c(U)$, $K := \operatorname{supp} T$, $\psi \in \mathscr{D}(U)$ such that $\psi|_V = 1$. By 5.1.3 T has some order on K, i.e. there exist constants $C_1 > 0$, $k \in \mathbb{N}$, such that

$$\forall \phi \in \mathscr{E}(U) : |\iota'^{-1}(T)(\phi)| = |T(\psi\phi)| \leq C_1\|\psi\phi\|_{\mathscr{C}^k(K)} \leq C_2\|\phi\|_{\mathscr{C}^k(K)},$$

where the last inequality follows from the Leibniz rule. Therefore $\iota'^{-1}(T) \in \mathscr{E}'(U)$ by Theorem 3.1.22.

Step 3: Let T, K, ψ be as above. For any $\phi \in \mathscr{D}(U)$ the function $\psi\phi - \phi = (\psi - 1)\phi$ satisfies $\operatorname{supp}(\psi - 1)\phi \cap K = \emptyset$. Therefore

$$0 \overset{5.4.2}{=} T((\psi - 1)\phi) = T(\psi\phi) - T(\phi).$$

We obtain

$$\iota'(\iota'^{-1}(T))(\phi) = \iota'^{-1}(T)(\iota(\phi)) = T(\psi\phi) = T(\phi).$$

On the other hand, for any $T \in \mathscr{E}'(U)$, $\phi \in \mathscr{E}(U)$

$$\iota'^{-1}(\iota'(T))(\phi) = \iota'(T)(\psi\phi) = T(\psi\phi) = T(\phi)$$

by the same reasoning.

Step 4 (dense image): Let $T \in \mathscr{D}'(U)$. We have to show that there are $T_j \in \mathscr{D}'_c(U)$ such that

$$T_j \xrightarrow[\mathscr{D}'(U)]{} T .$$

Define the compact exhaustion

$$K_j := \{x \in U \mid |x| \leq j, d(x, \mathbb{R}^n \setminus U) \geq \frac{1}{j}\}.$$

(In case $U = \mathbb{R}^n$, just drop the second condition and set $K_j := B_j(0)$.) Chose $\psi_j \in \mathscr{D}(U)$ such that

$$\psi_j|_{K_j} = 1, \qquad \operatorname{supp} \psi_j \subset K_{j+1}.$$

Define $T_j \in \mathscr{D}'(U)$ by

$$\forall \phi \in \mathscr{D}(U) : T(\psi_j \varphi).$$

Then $\operatorname{supp} T_j \subset K_{j+1}$ and thus $T_j \in \mathscr{D}'_c(U)$. Let $\phi \in \mathscr{D}(U)$ and $\operatorname{supp} \phi =: K$. This implies

$$\forall j \in \mathbb{N} : \operatorname{supp} \psi_j \varphi \subset K.$$

For almost every j, we obtain $\psi_j|_K = 1$, thus

$$\psi_j \varphi \xrightarrow[\mathscr{D}(U)]{} \varphi \, .$$

Since T is continuous

$$T_j(\varphi) = T(\psi_j \varphi) \xrightarrow[\mathbb{C}]{} T(\varphi) \, .$$

By definition, this implies

$$T_j \xrightarrow[\mathscr{D}'(U)]{} T \, .$$

□

5.4.5 Convention. From now on we will no longer distinguish between $\mathscr{D}'_c(U)$ and $\mathscr{E}'(U)$.

5.4.6 Corollary. Lemma 5.1.5 may now be restated by saying that there are continuos inclusions

$$\mathscr{D} \hookrightarrow \mathscr{S} \hookrightarrow \mathscr{E},$$

$$\mathscr{E}' \hookrightarrow \mathscr{S}' \hookrightarrow \mathscr{D}',$$

all of which have dense image. (We consider these spaces on $U = \mathbb{R}^n$. In case $U \neq \mathbb{R}^n$ one has to ignore $\mathscr{S}$ and $\mathscr{S}'$.)

5.4.7 Theorem (point support)**.** Let $T \in \mathscr{E}'(U)$ be of order k and suppose there exists $a \in U$, such that $\operatorname{supp} T = \{a\}$. Then there exists $c_\alpha \in \mathbb{C}$, such that

$$T = \sum_{|\alpha| \leq k} c_\alpha \partial^\alpha \delta_a.$$

5.4.8 Lemma. The map $\langle \_ \rangle : \mathscr{E}(U) \to \mathscr{D}'(U)$ is an embedding.

Proof. We already know that it is injective.
Step 1: !ToDo
Step 2: To see that the inverse is continuous, we have to show that for a sequence $u_j \in \mathscr{E}(U)$, such that

$$\langle u_j \rangle \xrightarrow[\mathscr{D}'(U)]{} 0 \, ,$$

this implies

$$u_j \xrightarrow[\mathscr{E}(U)]{} 0 \, .$$

Assume to the contrary there exists $K \Subset U$ and infinitely many $j \in \mathbb{N}$ (for notational convenience we will assume all $j \in \mathbb{N}$), an $\varepsilon > 0$ and an $\alpha \in \mathbb{N}^n$, such that

$$\forall j \in \mathbb{N} : \forall x \in K : |\partial_x^\alpha u_j| \geq \varepsilon > 0.$$

We may assume that K is connected, thus $\partial_x^\alpha u_j$ is either stricly positive or negative. We assume the further and choose $\varphi \in \mathscr{D}(U)$, $0 \leq \varphi \leq 1$, such that $\varphi|_K \equiv 1$. This implies

$$0 = |\lim_{j\to\infty} \langle u_j\rangle(\partial_x^\alpha \varphi)| = |(-1)^\alpha \lim_{j\to\infty} \langle \partial_x^\alpha u_j\rangle(\varphi)| = \int_U \partial_x^\alpha u_j(x)\varphi(x)dx \geq \int_K \partial_x^\alpha u_j(x)\varphi(x)dx \geq \varepsilon\mu(K),$$

which is a contradiction.

□

5.5. Convolutions

Remember the notation from 4.4.20.

5.5.1 Definition (Convolution)**.** Let $T \in \mathscr{D}'(\mathbb{R}^n)$ and $\varphi \in \mathscr{D}(\mathbb{R}^n)$. For any $x \in \mathbb{R}^n$ define $\check{\tau}_x : \mathbb{R}^n \to \mathbb{R}^n$ by $y \mapsto \mathcal{R}(\tau_{-x}) = x - y$. Then $T * \varphi : \mathbb{R}^n \to \mathbb{C}$ defined by

$$x \mapsto T(\varphi \circ \check{\tau}_x) = T(y \mapsto \varphi(x-y))$$

is the *convolution of T with φ*.

5.5.2 Lemma (paramatrized Test functions)**.** Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be open, $\Phi \in \mathscr{E}(U \times V)$, $K \subset U$ compact, such that $\operatorname{supp}\Phi \subset K \times V$, $T \in \mathscr{D}'(U)$. Then $T_V : V \to \mathbb{C}$, defined by

$$y \mapsto T(\Phi(\_, y))$$

is smooth and

$$\partial_y^\alpha T_V = T(\partial_y^\alpha \Phi(\_, y)).$$

5.5.3 Theorem (Properties of convolutions)**.**

(i) Convolution defines a bilinear map $* : \mathscr{D}'(\mathbb{R}^n) \times \mathscr{D}(\mathbb{R}^n) \to \mathscr{E}(\mathbb{R}^n)$ and $* : \mathscr{E}'(\mathbb{R}^n) \times \mathscr{E}(\mathbb{R}^n) \to \mathscr{E}(\mathbb{R}^n)$.

(ii) In case $T = \langle f\rangle$, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$

$$\langle f\rangle * \varphi = f * \varphi.$$

(iii) The supports satisfy

$$\operatorname{supp}(T * \varphi) \subset \operatorname{supp} T + \operatorname{supp}\varphi.$$

(iv) The differentiation theorem holds analogously as

$$\partial^\alpha(T * \varphi) = \partial^\alpha T * \varphi = T * \partial^\alpha \varphi.$$

(v) $*$ is continuous in both factors.

Proof.

(i) Bilinearity follows from the definitions and the smoothness from Lemma 5.5.2.

(ii) We calculate

$$\forall x \in \mathbb{R}^n : (\langle f \rangle * \varphi)(x) = \langle f \rangle(\varphi \circ \check{\tau}_x)) = \int_{\mathbb{R}^n} f(y)\varphi(x-y)dy = (f * \varphi)(x).$$

(iii) Let $x \in \mathbb{R}^n$ such that

$$0 \neq (T * \varphi)(x) = T(\varphi \circ \check{\tau}_x)).$$

This implies $T \neq 0$ and there exists some $y \in \operatorname{supp} T \cap \operatorname{supp} \varphi$ (by 5.4.2). By definition there exist $y_j \in \{z \in \mathbb{R}^n | (\varphi \circ \tau_{-x})(z) \neq 0\}$ such that $y_j \to y$. This implies

$$\forall j \in \mathbb{N} : 0 \neq (\varphi \circ \tau_{-x})(y_j) = \varphi(x - y_j) \Longrightarrow \forall j \in \mathbb{N} : x - y_j \in \operatorname{supp} \varphi \Longrightarrow x - y \in \operatorname{supp} \varphi.$$

Alltogether

$$x \in \operatorname{supp} T + \operatorname{supp} \varphi.$$

By taking the closure of all such x this implies the statement (since $+$ is continuous).

(iv) We calculate

$$\partial_x^\alpha (T * \varphi)(x) = \partial_x^\alpha (T(\varphi \circ \check{\tau}_x))) \overset{5.5.2}{=} T(\partial_x^\alpha (\varphi \circ \check{\tau}_x)) = T(\partial_x^\alpha \varphi \circ \check{\tau}_x)),$$

which on the one hand equals $(T * \partial^\alpha \varphi)(x)$ and on the other hand

$$\begin{aligned} &T(\partial_x^\alpha (\varphi \circ \check{\tau}_x)) = T(\partial_x^\alpha (y \mapsto \varphi(x-y))) = (-1)^\alpha T(\partial_y^\alpha (y \mapsto \varphi(x-y))) \\ &= \partial^\alpha T(\varphi \circ \check{\tau}_x) = (\partial^\alpha T * \varphi)(x). \end{aligned}$$

(v) Laut Hörmi ist immerhin T* stetig. (p. 101) !ToDo Assume

$$\varphi_j \xrightarrow[\mathscr{D}]{} 0 .$$

This implies there exists a compact $K \subset \mathbb{R}^n$, such that $\varphi_j \subset K$. By definition T has some order k on K. Thus

$$\forall x \in \mathbb{R}^n : |T * \varphi_j|(x) = |T(\varphi_j \circ \check{\tau}_x)| \leq C \|\varphi_j \circ \check{\tau}_x)\|_{\mathscr{C}^k(K)} = C\|\varphi_j\|_{\mathscr{C}^k(K)} \to 0.$$

Clearly this implies

$$\forall x \in \mathbb{R}^n : \; \varphi_j \circ \check{\tau}_x \xrightarrow[\mathscr{D}]{} 0$$

and therefore

$$\forall x \in \mathbb{R}^n : \; (T * \varphi_j)(x) = T(\varphi_j \circ \check{\tau}_x) \xrightarrow[\mathbb{C}]{} 0$$

$$T_j \xrightarrow[\mathscr{D}']{} 0$$

and let $\varphi \in \mathscr{D}(\mathbb{R}^n)$. Let $K \subset \mathbb{R}^n$ be any compact subset. We obtain

$$\|T_j * \varphi\|_{\mathscr{C}^k(K)} = \max_{x \in K} \max_{|\alpha| \leq k} |\partial^\alpha (T_j * \varphi)|(x)$$

□

5.5.4 Lemma. Let $\varphi \in \mathscr{C}_0^j(\mathbb{R}^n)$, $\psi \in \mathscr{C}_c^0(\mathbb{R}^n)$. Then the Riemann sum satisfies

$$\sum_{k \in \mathbb{Z}^n} \varphi \circ \check{\tau}_{kh} h^n \psi(kh) \xrightarrow[\mathscr{C}_0^j]{h \to 0} \varphi * \psi$$

5.5.5 Theorem (Associativity)**.** Let $T \in \mathscr{D}'(\mathbb{R}^n)$, $\varphi, \psi \in \mathscr{D}(\mathbb{R}^n)$. This implies

$$(T * \varphi) * \psi = T * (\varphi * \psi).$$

Proof. !ToDo □

5.5.6 Lemma. For any $f \in \mathscr{D}(\mathbb{R}^n)$ we obtain

$$\forall \varphi \in \mathscr{D}(\mathbb{R}^n) : \langle T * f \rangle(\varphi) = T(\check{f} * \varphi).$$

Proof. We just calculate

$$\langle T * f \rangle(\varphi) = \int_{\mathbb{R}^n} (T * f)(y)\varphi(y)dy = \int_{\mathbb{R}^n} (T * f)(-y)\varphi(-y)dy = \int_{\mathbb{R}^n} (T * f)(0 - y)\check{\varphi}(y)dy$$
$$= ((T * f) * \check{\varphi})(0) \overset{5.5.5}{=} (T * (f * \check{\varphi}))(0) = T(f * \check{\varphi} \circ \check{\tau}_0) = T(\check{f} * \varphi),$$

where in the last step we used that

$$((f*\check{\varphi})\circ\check{\tau}_0)(x) = (f*\check{\varphi})(-x) = \int_{\mathbb{R}^n} f(-x-y)\varphi(-y)dy = \int_{\mathbb{R}^n} \check{f}(x+y)\varphi(-y)dy = (\check{f}*\varphi)(x).$$

□

5.5.1. Singular Support, Regularity, Regularization

Approximation is a standard application of the convolution in classical calculus. We are now in a position to further develop this theory in the context of distributions. In 5.1.7 we considered a distribution T to be regular if it may be identified with a function $f \in L^1_{\mathrm{loc}}$. The general idea behind this is, that a distribution is something of "worse" regularity than a function. However this way of thinking has two serious disadvantages: First of all L^1_{loc} is not a space of particularly "nice" functions. So even a "regular distribution" in the above sense is still a rather "irregular" object. The second problem is, that this point of view is rather rigid: Just as a function it may be very regular somewhere and very irregular somewhere else. The following definition makes this idea precise.

5.5.7 Definition (Singular Support)**.** Let $T \in \mathscr{D}'(U)$. Then

$$\operatorname{sing-supp} T := U \setminus \{x \in U \mid \exists V \mathring{\subseteq} U : \exists f \in \mathscr{E}(V) : T|_U = \langle f \rangle\}$$

is the *singular support of* T. We say T *is smooth* if $T = \langle f \rangle$ for some $f \in \mathscr{E}(U)$.

Notice the similarity to the definition of the support of T in 5.4.1.

5.5.8 Lemma (Smooth Approximation)**.** Let $T \in \mathscr{D}'(\mathbb{R}^n)$ and let η_ε be the standard mollifier from 4.4.5 (or any other Dirac sequence). This implies

$$\langle T * \eta_\varepsilon \rangle \xrightarrow[\mathscr{D}']{\varepsilon \searrow 0} T\ .$$

Proof. We already showed in Lemma 5.5.6 that

$$\forall \varphi \in \mathscr{D}(\mathbb{R}^n) : \langle T * \eta_\varepsilon \rangle(\varphi) = T(\check{\eta}_\varepsilon * \varphi).$$

Theorem 4.4.6 from classical calculus implies

$$\check{\eta}_\varepsilon * \varphi \xrightarrow[\mathscr{D}]{r \searrow 0} \varphi ,$$

and thus

$$T(\check{\eta}_\varepsilon * \varphi) \xrightarrow[\mathbb{C}]{r \searrow 0} T(\varphi) .$$

Therefore

$$\langle T * \eta_\varepsilon \rangle \xrightarrow[\mathscr{D}']{} T .$$

□

5.5.9 Theorem (Smooth Approximation)**.** The map $\langle \_ \rangle : \mathscr{D}(U) \to \mathscr{D}'(U)$ has dense image. So for any distribution $T \in \mathscr{D}'(U)$ there exists $\Phi_j \in \mathscr{D}(U)$, such that

$$\langle \Phi_j \rangle \xrightarrow[\mathscr{D}']{} T .$$

Proof. Let $T \in \mathscr{D}'(U)$. We already showed in Theorem 5.4.4, that $\mathscr{E}(U) \subset \mathscr{D}(U)$ is dense. Therefore there exist $T_j \in \mathscr{E}(U)$, such that

$$T_j \xrightarrow[\mathscr{D}']{} T .$$

Let $K \subset U$ be compact. We also showed in 5.4.4, that there exists $j(K) \in \mathbb{N}$, such that

$$\forall j \geq j(K) : \forall \varphi \in \mathscr{D}_K(U) : T_j(\varphi) = T(\varphi). \tag{5.8}$$

Let

$$K_j := \{x \in U \mid |x| \leq j \text{ and } d(x, \mathbb{R}^n \setminus U) \geq \frac{1}{j}\},$$

where the second condition is dropped in case $U = \mathbb{R}^n$. Let η_ε be the standard mollifier. Define

$$\Phi_j := T_{j(K_{2j})} * \eta_{\varepsilon(j)},$$

where $\varepsilon(j) := \frac{1}{3j}$. By Theorem 5.5.3 we obtain $\Phi_j \in \mathscr{D}(U)$. We have to show, that

$$\forall \varphi \in \mathscr{D}(U) : \ \langle \Phi_j \rangle(\varphi) \xrightarrow[\mathbb{C}]{} T(\varphi) .$$

Therefore let $\varphi \in \mathscr{D}(U)$. There exists $l \in \mathbb{N}$, such that $\operatorname{supp} \varphi \subset K_l$. This implies

$$\forall j \geq l : \langle \Phi_j \rangle(\varphi) = \langle T_{j(K_{2j})} * \eta_{\varepsilon(j)} \rangle(\varphi) \overset{5.5.6}{=} T_{j(K_{2j})}(\check{\eta}_{\varepsilon(j)} * \varphi) \overset{(5.8)}{=} T(\check{\eta}_{\varepsilon(j)} * \varphi). \tag{5.9}$$

Now

$$\forall j \geq l : \operatorname{supp}(\check{\eta}_{\varepsilon(j)}) * \varphi) \subset \operatorname{supp} \check{\eta}_{\varepsilon(j)}) + \operatorname{supp} \varphi \subset B_{1/(3j)}(0) + K_j \subset K_{2j},$$

which is compact. By classical calculus (!ToDo ref)

$$\forall k \in \mathbb{N} : \ \check{\eta}_{\varepsilon(j)} * \varphi \xrightarrow[\mathscr{C}^k(U)]{j \to \infty} \varphi .$$

This implies

$$\check{\eta}_{\varepsilon(j)} * \varphi \xrightarrow[\mathscr{D}(U)]{j \to \infty} \varphi$$

and therefore

$$T(\check{\eta}_{\varepsilon(j)} * \varphi) \xrightarrow[\mathbb{C}]{j \to \infty} T(\varphi) .$$

By (5.9) this implies the statement. □

5.5.2. Distributional Convolution

We are in a rather assymetric situation so far since we only defined the convolution of a distribution with a function. In this subsection we will go one step further and define the convolution of two distributions. Unfortunately this is not always possible. Before we can start, we require the following technical lemma.

5.5.10 Lemma (Translation invariance)**.**

(i) The convolution $* : \mathscr{D}'(\mathbb{R}^n) \times \mathscr{D}(\mathbb{R}^n) \to \mathscr{E}(\mathbb{R}^n)$ commutes with all translations, i.e.

$$\forall h \in \mathbb{R}^n : (T * \varphi) \circ \tau_h = T * (\varphi \circ \tau_h).$$

(ii) Conversely, let $F : \mathscr{D}(\mathbb{R}^n) \to \mathscr{E}(\mathbb{R}^n)$ is a continuous linear map, that commutes with all translations, there exists a unique distribution $T \in \mathscr{D}'(\mathbb{R}^n)$, such that

$$\forall \varphi \in \mathscr{D}(\mathbb{R}^n) : F(\varphi) = T * \varphi.$$

(iii) In particular: Two distributions $T_1, T_2 \in \mathscr{D}'(\mathbb{R}^n)$ are equal if and only if

$$\forall \varphi \in \mathscr{D}(\mathbb{R}^n) : T_1 * \varphi = T_2 * \varphi.$$

Proof.

(i) Let $h, x \in \mathbb{R}^n$ be arbitrary. The simple calculation

$$\forall y \in \mathbb{R}^n : (\tau_h \circ \check{\tau}_x)(y) = \tau_h(x - y) = x - y + h = x + h - y = \check{\tau}_{x+h}(y)$$

directly implies

$$((T * \varphi) \circ \tau_h)(x) = (T * \varphi)(x + h) = T(\varphi \circ \check{\tau}_{x+h}) = T(\varphi \circ \tau_h \circ \check{\tau}_x) = (T * (\varphi \circ \tau_h))(x).$$

(ii) The hypothesis can be expressed more precisely by

$$\forall h \in \mathbb{R}^n : \forall \varphi \in \mathscr{D}(\mathbb{R}^n) : F(\varphi) \circ \tau_h = F(\varphi \circ \tau_h). \tag{5.10}$$

STEP 1 (Uniqueness): Assume there exists a distribution $T \in \mathscr{D}'(\mathbb{R}^n)$, such that $F(\varphi) = T * \varphi$. This implies

$$T(\varphi) = T(\varphi \circ \check{\tau}_0) = (T * \varphi)(0) = F(\varphi)(0).$$

STEP 2 (Existence): So we have no choice but to define $T : \mathscr{D}(\mathbb{R}^n) \to \mathbb{C}$ by

$$\varphi \mapsto F(\varphi)(0) = \delta_0(F(\varphi)),$$

which is a distribution. By hypothesis it satisfies

$$F(\varphi)(x) = (F(\varphi) \circ \tau_x)(0) \overset{(5.10)}{=} F(\varphi \circ \tau_x)(0) = T(\varphi \circ \tau_x) = (T * \varphi)(x).$$

(iii) This follows by combining (i) with (ii).

□

5.5.11 Theorem and Definition. Let $T_1, T_2 \in \mathscr{D}'(\mathbb{R}^n)$ and let at least one of them be compactly supported. The map $F : \mathscr{D}(\mathbb{R}^n) \to \mathscr{E}(\mathbb{R}^n)$, $F(\varphi) = T_1 * (T_2 * \varphi)$ is linear, translation invariant and continuous. The unique distribution T, such that

$$\forall \varphi \in \mathscr{D}(\mathbb{R}^n) : T_1 * (T_2 * \varphi) = T * \varphi$$

is the *convolution of* T_1 *and* T_2. We define

$$T_1 * T_2 := T.$$

By construction the convolution between distributions is associative with the convolution of a distribution and a function, i.e.

$$\forall \varphi \in \mathscr{D}(\mathbb{R}^n) : (T_1 * T_2) * \varphi = T_1 * (T_2 * \varphi). \tag{5.11}$$

Proof. The map F is linear and continuous by Theorem 5.5.3 and translation invariant by 5.5.10,(i). Hence by Lemma 5.5.10,(ii) there exists a unique distribution T, such that

$$\forall \varphi \in \mathscr{D}(u) : T * \varphi = F(\varphi) = T_1 * (T_2 * \varphi).$$

□

Before we prove anything about this distributional convolution, we notice, that we obtain another equality criterion for distributions.

5.5.12 Lemma. Two distributions $T_1, T_2 \in \mathscr{D}'(\mathbb{R}^n)$ are equal if and only if

$$\forall \varphi, \psi \in \mathscr{D}(\mathbb{R}^n) : T_1 * (\varphi * \psi) = T_2 * (\varphi * \psi).$$

Proof. By Theorem 5.5.5 the hypothesis implies

$$\forall \varphi, \psi \in \mathscr{D}(\mathbb{R}^n) : (T_1 * \varphi) * \psi = (T_2 * \varphi) * \psi.$$

By Theorem 5.5.3,(ii) this is equivalent to

$$\forall \varphi, \psi \in \mathscr{D}(\mathbb{R}^n) : \langle T_1 * \varphi \rangle * \psi = \langle T_2 * \varphi \rangle * \psi.$$

By Lemma 5.5.10,(iii), this implies

$$\forall \varphi \in \mathscr{D}(\mathbb{R}^n) : \langle T_1 * \varphi \rangle = \langle T_2 * \varphi \rangle.$$

Since $\langle \_ \rangle$ is injective by 5.1.7, this implies

$$\forall \varphi \in \mathscr{D}(\mathbb{R}^n) : T_1 * \varphi = T_2 * \varphi.$$

Using 5.5.10,(iii) again, this implies the statement. □

5.5.13 Theorem (Properties of Convolutions)**.** Let $T, T_1, T_2, T_3 \in \mathscr{D}'(\mathbb{R}^n)$, $f \in \mathscr{D}(\mathbb{R}^n)$.

(i) If one of the three distributions has compact support

$$(T_1 * T_2) * T_3 = T_1 * (T_2 * T_3).$$

(ii) The convolution may be explicitely computed by

$$(T_1 * T_2)(\varphi) = T_1(\check{T}_2 * \varphi).$$

(iii) The convolution is commutative, i.e.

$$T_1 * T_2 = T_2 * T_1.$$

(iv) The delta distribution is a neutral element, i.e.

$$\forall a \in \mathbb{R}^n : \delta_a * f = f \text{ and } T * \delta_a = T.$$

(v) The support containment remains valid, i.e.

$$\operatorname{supp}(T_1 * T_2) \subset \operatorname{supp} T_1 + \operatorname{supp} T_2.$$

Proof.

(i)

(ii)

(iii) The idea is to use the equality criterion 5.5.12. For any $\varphi, \psi \in \mathscr{D}(\mathbb{R}^n)$ we calculate

$$(T_1 * T_2) * (\varphi * \psi) \overset{(5.11)}{=} T_1 * (T_2 * (\varphi * \psi)) \overset{5.5.5}{=} T_1 * ((T_2 * \varphi) * \psi) \overset{4.4.2,(\text{iv})}{=} T_1 * (\psi * (T_2 * \varphi))$$
$$\overset{5.5.5}{=} (T_1 * \psi) * (T_2 * \varphi) \overset{4.4.2,(\text{iv})}{=} (T_2 * \varphi) * (T_1 * \psi) = (T_2 * T_1) * (\varphi * \psi),$$

where in the last step we applied all the others before in reversed order and with the roles of T_1, T_2 interchanged.

(iv)

(v) Let η_ε be the standard mollifier. We obtain

$$\operatorname{supp}((T_1 * T_2) * \eta_\varepsilon) = \operatorname{supp}(T_1(T_2 * \eta_\varepsilon)) \overset{5.5.3,(\text{iii})}{\subset} \operatorname{supp} T_1 + \operatorname{supp}(T_2 * \eta_\varepsilon) \subset \operatorname{supp} T_1 + \operatorname{supp} T_2 + B_\varepsilon(0).$$

This holds for any $\varepsilon > 0$ and therefore the statement follows from 5.5.8.

□

5.5.14 Theorem (Singular Support)**.** Let $T_1, T_2 \in \mathscr{D}'(\mathbb{R}^n)$. Then

$$\operatorname{sing-supp}(T_1 * T_2) \subset \operatorname{sing-supp} T_1 + \operatorname{sing-supp} T_2.$$

Proof. Hörmi 4.2.5 □

5.6. Products

!ToDo Hier unbedingt nochmal in den Friedlander / Joshi schauen We already encountered distributions over product spaces as some technical issues in proofs. In this chapter we will systematically introduce this topic and prove Schwartz' celebrated Kernel Theorem. In this section let $U_1 \mathring{\subseteq} \mathbb{R}^{n_1}$ and $U_2 \mathring{\subseteq} \mathbb{R}^{n_2}$.

5.6.1 Definition (Tensor product)**.** For any two functions $\varphi \in \mathscr{D}(U)$, $\psi \in \mathscr{D}(V)$, the function $\varphi \otimes \psi : U \times V \to \mathbb{C}$ defined by

$$(x, y) \mapsto \varphi(x)\psi(y),$$

is the *tensor product of* φ *and* ψ.

Tensor products behave fantastically with respect to integration.

5.6.2 Lemma (Properties of tensor products)**.**

(i) We obtain $\varphi \otimes \psi \in \mathscr{D}(U \times V)$,

$$\forall K \Subset U \times V : \forall k \in \mathbb{N} : \|\varphi \otimes \psi\|_{\mathscr{C}^k(K)} \leq \|\varphi\|_{\mathscr{C}^k(\pi_U(K))} \|\psi\|_{\mathscr{C}^k(\pi_V(K))}$$

and

$$\operatorname{supp} \varphi_1 \otimes \varphi_2 = \operatorname{supp} \varphi_1 \times \operatorname{supp} \varphi_2.$$

(ii) For any $\varphi_1 \in \mathscr{D}(U_1)$, $\varphi_2 \in \mathscr{D}(U_2)$

$$\int_{U_1 \times U_2} (\varphi_1 \otimes \varphi_2)(x, y) d(x, y) = \int_{U_1} \varphi_1(x) dx \int_{U_2} \varphi_2(y) dy.$$

(iii) Furthermore the generated regular distributions satisfy

$$\langle \varphi_1 \otimes \varphi_2 \rangle (\psi_1 \otimes \psi_2) = \langle \varphi_1 \rangle (\psi_1) \cdot \langle \varphi_2 \rangle (\psi_2).$$

(iv) The tensor product defines a continuous bilinear map $\otimes : \mathscr{D}(U) \times \mathscr{D}(V) \to \mathscr{D}(U \times V)$.

Proof. (i)

(ii)

(iii)

(iv) We show, that $\otimes$ is continuous in both factors. Since the situation is symmetric it suffices to check, that it is continuous in the first factor. Therefore assume there is a sequence $\varphi_j \in \mathscr{D}(U)$, such that

$$\varphi_j \xrightarrow[\mathscr{D}(U)]{} 0$$

and let $\psi \in \mathscr{D}(V)$ be arbitrary. By definition there exists a compact $K \Subset U$, such that all j satisfy $\varphi_j \subset K$ and

$$\forall k \in \mathbb{N} : \varphi_j \xrightarrow[\mathscr{C}^k(K)]{} 0 .$$

By definition there exists a compact $L \Subset V$, such that $\operatorname{supp} \psi \subset L$. Therefore

$$\forall j \in \mathbb{N} : \operatorname{supp} \varphi_j \otimes \psi \subset K \times L \Subset U \times V$$

and

$$\forall k \in \mathbb{N} : \|\varphi_j \otimes \psi\|_{\mathscr{C}^k(K \times L)} \leq \|\varphi_j\|_{\mathscr{C}^k(K)} \|\psi\|_{\mathscr{C}^k(L)} \to 0.$$

□

The fact that regular distributions are dense gives rise to the hope, that the distribution space over a product behaves equally nice.

5.6.3 Theorem and Definition. Let $T_1 \in \mathscr{D}'(U_1)$, $T_2 \in \mathscr{D}'(U_2)$. There exists a distribution $T \in \mathscr{D}'(U_1 \times U_2)$, such that

$$\forall \varphi_1 \in \mathscr{D}(U_1) : \forall \varphi_2 \in \mathscr{D}(U_2) : T(\varphi_1 \otimes \varphi_2) = T_1(\varphi_1) T_2(\varphi_2) \tag{5.12}$$

and any two distributions $T, T' \in \mathscr{D}'(U_1 \times U_2)$ which agree on all tensor products are equal. This distribution satisfies

$$\forall \psi \in \mathscr{D}(U \times V) : T(\psi) = T_1(x_1 \mapsto T_2(x_2 \mapsto \psi(x_1, x_2))) = T_2(x_2 \mapsto T_1(x_1 \mapsto \psi(x_1, x_2))) \tag{5.13}$$

and is called the *tensor product of* T_1 *and* T_2. We define

$$T_1 \otimes T_2 := T.$$

Analogous statements hold if $\mathscr{D}'$ is replaced by $\mathscr{E}'$.

Proof.
STEP 1 (Uniqueness): It suffices to check the following: Let $T \in \mathscr{D}'(U_1 \times U_2)$ be a distribution satisfying

$$\forall \varphi_1 \in \mathscr{D}(U_1) : \forall \varphi_2 \in \mathscr{D}(U_2) : T(\varphi_1 \otimes \varphi_2) = 0, \tag{5.14}$$

then $T = 0$. In 4.4.5 we constructed the function η for an arbitrary $\mathbb{R}^n$. Let $\eta_j \in \mathscr{C}^\infty(R^{n_j})$, $j = 1, 2$, be these functions and $(\eta_1)_\varepsilon$, $(\eta_2)_\varepsilon$ be the associated dirac sequences as in 4.4.5. Then $\eta_\varepsilon := (\eta_1)_\varepsilon \otimes (\eta_2)_\varepsilon$ is a dirac sequence in $\mathbb{R}^{n_1+n_2}$. Therefore by 5.5.8

$$\langle T * \eta_\varepsilon \rangle \xrightarrow[\mathscr{D}'()]{\varepsilon \searrow 0} T \ .$$

Now for any $(x_1, x_2) \in U_1 \times U_2$

$$(T * \eta_\varepsilon)(x_1, x_2) = T(\eta_\varepsilon \circ \check{\tau}_{x_1,x_2}) = T(((\eta_1)_\varepsilon \circ \check{\tau}_{x_1}) \otimes ((\eta_2)_\varepsilon \circ \check{\tau}_{x_2})) \overset{(5.14)}{=} 0.$$

!ToDo: Hörmander argumentiert hier irgendwie anders. Kann sein, dass das Problem darin besteht, dass T eigentlich keine Distribution auf $\mathbb{R}^n$ is.
STEP 2 (Existence): We wil define the distribution $T : \mathscr{D}(U_1 \times U_2) \to \mathbb{C}$ as follows: Let $\varphi \in \mathscr{D}(U_1 \times U_2)$ and first assume there are compact subsets $K_j \subset U_j$, $j = 1, 2$, such that

$$\operatorname{supp} \varphi \subset K_1 \times K_2. \tag{5.15}$$

Since T_j is a distribution

$$\exists C_j > 0 : \forall \varphi_j \in \mathscr{D}_{K_j}(U_j) : |T(\varphi_j)| \leq C_j \|\varphi_j\|_{\mathscr{C}^{k_j}(K_j)}. \tag{5.16}$$

Define $I_\varphi : U_1 \to \mathbb{C}$ by

$$x_1 \mapsto T_2(x_2 \mapsto \varphi(x_1, x_2)).$$

By Lemma 5.5.2 $I_\varphi \in \mathscr{D}_{K_1}(U_1)$ and

$$\forall \alpha \in \mathbb{N}^{n_2} : \partial^\alpha_{x_1} I_\varphi(x_1) = T_2(x_2 \mapsto \partial^\alpha_{x_1} \varphi(x_1, x_2)).$$

Therefore

$$|T_1(I_\varphi)| \overset{(5.16)}{\leq} C_1 \|I_\varphi\|_{\mathscr{C}^{k_1}(K_1)} \overset{(5.16)}{\leq} C_1 C_2 \|\varphi\|_{\mathscr{C}^{\max(k_1,k_2)}(K_1 \times K_2)}.$$

So by defining $T(\varphi) := T_1(I_\varphi)$, we obtain a continuous functional on the subspace of all $\varphi \in \mathscr{D}(U_1 \times U_2)$ satisfying (5.15). Clearly all tensor products belong to this space and satisfy

$$\begin{aligned}
&\forall \varphi_j \in \mathscr{D}(U_j) : T(\varphi_1 \otimes \varphi_2) = T_1(I_{\varphi_1 \otimes \varphi_2}) = T_1(x_1 \mapsto T_2(x_2 \mapsto (\varphi_1 \otimes \varphi_2)(x_1, x_2))) \\
&= T_1(x_1 \mapsto T_2(x_2 \mapsto \varphi_1(x_1)\varphi_2(x_2))) = T_1(x_1 \mapsto \varphi_1(x_1) T_2(x_2 \mapsto \varphi_2(x_2))) = T_1(\varphi_1) T_2(\varphi_2),
\end{aligned}$$

thus (5.12) is satisfied on this subspace as well as the first part of (5.13) (by construction). Now suppose $\varphi \in \mathscr{D}(U_1 \times U_2)$ and $\operatorname{supp} \varphi \Subset K \subset U_1 \times U_2$ is arbitrary. Then K admits a finite cover $\{U_i\}_{i=1,\dots,N}$ of product open sets. Take a partition of unity $\{\psi_i\}_{i=1,\dots,N}$

subordinate to this cover. Then every $\psi_i\varphi$ has a support contained in a compact set that is contained in the product of two compact set in U_1 and U_2. Therefore

$$T(\varphi) := \sum_{i=1}^{N} \psi_i\varphi$$

is defined by what we have already constructed and satisfies (5.12) and the first part of (5.13) by what we have already proven. By uniqueness this definition does not depend on the chosen partitions of unity. If we interchange the roles of $j = 1$ and $j = 2$ in this proof, we obtain an operator T' satisfying the same properties, but the second part of (5.13) instead of the first one. Again by uniqueness they have to agree.

□

The following is the most important and most famous theorem concerning distributions over product spaces.

5.6.4 Theorem and Definition (Schwartz kernel theorem)**.**

(i) Let $K \in \mathscr{D}'(U_1 \times U_2)$. Then $\mathscr{K} : \mathscr{D}(U_2) \to \mathscr{D}'(U_1)$ defined by

$$\psi \mapsto (\varphi \mapsto K(\varphi \otimes \psi)) \tag{5.17}$$

is a continuous linear operator.

(ii) For any continuous linear operator $\mathscr{K} : \mathscr{D}(U_2) \to \mathscr{D}'(U_1)$ there exists a unique $K \in \mathscr{D}'(U_1 \times U_2)$, such that (5.17) holds. We call K the *Schwartz kernel of* $\mathscr{K}$.

Proof.

(i) The linearity is obvious.

Step 1 ($\mathscr{K}(\psi)$ is continuous): We have to show, that for any $\psi \in \mathscr{D}(U_2)$, we obtain $\mathscr{K}(\psi) \in \mathscr{D}'(U_1)$. To that end let $\varphi_j \in \mathscr{D}(U_1)$, such that

$$\varphi_j \xrightarrow[\mathscr{D}(U_1)]{} 0 .$$

By 5.6.2 $\otimes$ is continuous in both factors. Therefore, this implies

$$\varphi_j \otimes \psi \xrightarrow[\mathscr{D}(U_1 \times U_2)]{} 0$$

and since K is continuous, this implies

$$\mathscr{K}(\psi)(\varphi_j) = K(\varphi_j \otimes \psi) \xrightarrow[\mathbb{C}]{} 0 .$$

Step 2 ($\mathscr{K}$ is continuous): Now assume that

$$\psi_j \xrightarrow[\mathscr{D}'(U_2)]{} 0 .$$

Now let $\varphi \in \mathscr{D}(U_1)$ be arbitrary. Again since $\otimes$ is continuous by 5.6.2, we obtain

$$\psi_j \otimes \varphi \xrightarrow[\mathscr{D}(U_1 \times U_2)]{} 0$$

and since K is continuous, we obtain

$$K(\psi_j)(\varphi) = K(\psi_j \otimes \varphi) \xrightarrow[\mathbb{C}]{} 0 .$$

Since φ was arbitrary,

$$K(\psi_j) \xrightarrow[\mathscr{D}'(U_2)]{} 0 .$$

(ii) We split the proof into two parts.

STEP 1 (Uniqueness): This is the easy part: Assume there are two distributions $K_1, K_2 \in \mathscr{D}'(U_1 \times U_2)$, such that

$$\forall \varphi \in \mathscr{D}(U_1) : \forall \varphi_2 \in \mathscr{D}(U_2) : K_1(\varphi \times \psi) = \mathscr{K}(\psi)(\varphi) = K_2(\varphi \otimes \psi).$$

Then the uniqueness part of 5.6.3 immediately implies $K_1 = K_2$.

STEP 2 (Existence): This is the hard part. Let $j = 1, 2$.

STEP 2.1: By definition there are constants $C > 0$, k_j, such that

$$\forall \varphi \in \mathscr{D}_{K_1}(U_1) : \forall \psi \in \mathscr{D}_{K_2}(U_2) : |\mathscr{K}(\varphi)(\psi)| \leq C\|\varphi\|_{\mathscr{C}^{k_1}(K_1)}\|\psi\|_{\mathscr{C}^{k_2}(K_2)}. \tag{5.18}$$

Thus $\beta : \mathscr{D}_{K_1}(U_1) \times \mathscr{D}_{K_2}(U_2) \to \mathbb{C}$, $(\varphi, \psi) \mapsto \mathscr{K}(\varphi)(\psi)$, is continuous in both factors. Since the domains are Frechét spaces, this implies that β is continuous (c.f. 3.2.2).

STEP 2.2 (Construction of K_ε): Choose functions $\psi_j \in \mathscr{D}(\mathbb{R}^{n_j})$, $j = 1, 2$, satisfying

$$\psi_j \geq 0, \qquad \int_{\mathbb{R}^{n_j}} \psi_j(x)dx = 1, \qquad \operatorname{supp}\psi_j \subset B_1(0) \subset \mathbb{R}^{n_j},$$

for example the function η from 4.4.5 (for $n = n_1, n_2$). Notice that

$$\operatorname{supp}\Big(y_j \mapsto \psi\Big(\frac{x_j - y_j}{\varepsilon}\Big)\Big) \subset B_\varepsilon(x_j). \tag{5.19}$$

Assume that $K_j \subset U_j$ is a compact neighbourhood of the open sets $Y_j \subset K_j \subset U_j$ and that $0 < \varepsilon < d(Y_j, U_j \setminus K_j)$. Define $K_\varepsilon : Y_1 \times Y_2 \to \mathbb{C}$ by

$$(x_1, x_2) \mapsto \varepsilon^{-n_1-n_2}\mathscr{K}\Big(y_2 \mapsto \psi_2\Big(\frac{x_2 - y_2}{\varepsilon}\Big)\Big)\Big(y_1 \mapsto \psi_1\Big(\frac{x_1 - y_1}{\varepsilon}\Big)\Big)$$

This is well-defined by (5.19) and the choice of ε.

STEP 2.3 (Wait!): Let's make some explainatory (logically superflous) remarks here to clarify, why we chose K_ε as we have done: Assume we had already found our desired (such that (5.17) holds) $K \in \mathscr{D}'(Y_1 \times Y_2)$. Then by 5.5.8 this would imply

$$\langle K * \Psi_\varepsilon \rangle \xrightarrow[\mathscr{D}'(Y_1 \times Y_2)]{\varepsilon \searrow 0} K\ .$$

Now the definition of K_ε states precisely that

$$\begin{aligned}K_\varepsilon(x_1, x_2) &= \mathscr{K}\Big(y_2 \mapsto \varepsilon^{-n_2}\psi_2\Big(\frac{x_2 - y_2}{\varepsilon}\Big)\Big)\Big(y_1 \mapsto \varepsilon^{-n_1}\psi_1\Big(\frac{x_1 - y_1}{\varepsilon}\Big)\Big)\\ &\overset{(5.17)}{=} K((\psi_1)_\varepsilon \circ \check{\tau}_{x_1} \otimes (\psi_2)_\varepsilon \circ \check{\tau}_{x_2}) = K(\underbrace{(\psi_1)_\varepsilon \otimes (\psi_2)_\varepsilon}_{=:\Psi_\varepsilon} \circ \check{\tau}_{(x_1,x_2)}) = (K * \Psi_\varepsilon)(x_1, x_2).\end{aligned}$$

Thus $K_\varepsilon = K * \Psi_\varepsilon$. Of course this is not a proof, since we have not yet constructed K, but it outlines the way how to do it.

STEP 2.4 (K_ε has a limit): We would like to show, that $\langle K_\varepsilon \rangle$ has a limit $K \in \mathscr{D}'(Y_1 \times Y_2)$ and that this is the K we are looking for. By (5.18) we obtain

$$\forall (x_1, x_2) \in Y_1 \times Y_2 : |K_\varepsilon(x_1, x_2)| \leq C\|\psi_1 \circ \check{\tau}_{x_1}\|_{\mathscr{C}^{k_1}(K_1)}\|\psi_2 \circ \check{\tau}_{x_2}\|_{\mathscr{C}^{k_2}(K_1)} \leq C'\varepsilon^{-\mu},$$

where $\mu := n_1 + n_2 + k_1 + k_2$.
Now let $\psi \in \mathscr{E}(\mathbb{R}^n)$ be arbitrary and define $\psi_j(x) := -x_j\psi(x)$. We calculate

$$\begin{aligned}
&\varepsilon\partial_\varepsilon(\varepsilon^{-n}\psi(x/\varepsilon)) + \sum_{j=1}^{n} x_j\partial_{x_j}(\varepsilon^{-n}\psi(x/\varepsilon)) \\
&= -n\varepsilon^{n}\psi(x/\varepsilon) - \varepsilon^{-n+1}\sum_{j=1}^{n}\partial_j\psi(x/\varepsilon)\varepsilon^{-2} + \varepsilon^{-n-1}\sum_{j=1}^{n} x_j\partial_{x_j}\psi(x/\varepsilon) \\
&= -n\varepsilon^{n}\psi(x/\varepsilon)
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=1}^{n}\partial_{x_j}(\varepsilon^{-n}\psi_j(x/\varepsilon)) = -\varepsilon^{-n}\sum_{j=1}^{n}\partial_{x_j}(x_j\psi(x/\varepsilon)) = -\varepsilon^{-n}\sum_{j=1}^{n}\psi(x/\varepsilon) + x_j\partial_j(\psi)(x/\varepsilon)\varepsilon^{-1} \\
&= -n\varepsilon^{-n}\psi(x/\varepsilon) - \varepsilon^{-n-1}\sum_{j=1}^{n} x_j(\partial_j\psi)(x/\varepsilon) \\
&= \varepsilon\partial_\varepsilon(\varepsilon^{-n}\psi(x/\varepsilon)) + \sum_{j=1}^{n} x_j\partial_{x_j}(\varepsilon^{-n}\psi(x/\varepsilon)) - \varepsilon^{-n-1}\sum_{j=1}^{n} x_j(\partial_j\psi)(x/\varepsilon) \\
&= \varepsilon\partial_\varepsilon(\varepsilon^{-n}\psi(x/\varepsilon)) + \varepsilon^{-n-1}\sum_{j=1}^{n} x_j\partial_{x_j}(\psi)(x/\varepsilon)) - \varepsilon^{-n-1}\sum_{j=1}^{n} x_j\partial_{x_j}(\psi)(x/\varepsilon) \\
&= \varepsilon\partial_\varepsilon(\varepsilon^{-n}\psi(x/\varepsilon))
\end{aligned}$$

$$\partial_\varepsilon(\varepsilon^{-n}\psi(x/\varepsilon)) = -n\varepsilon^{-n-1}\psi(x/\varepsilon) - \varepsilon^{n-2}\sum_{j=1}^{n}\partial_j\psi(x/\varepsilon)$$

!ToDo

□

5.6.5 Theorem (Smooth Kernels)**.** Assume $K \in \mathscr{E}(U_1 \times U_2)$. Then $\mathscr{K}$ has a continuous extension $\mathscr{K} : \mathscr{E}'(U_2) \to \mathscr{E}(U_1)$. In case $\mathscr{K} : \mathscr{E}'(U_2) \to \mathscr{E}(U_1)$ is linear and continuous its Schwarz Kernel K satisfies $K \in \mathscr{E}(U_1 \times U_2)$

.

Proof. !ToDo □

5.6.6 Theorem. Let $U_1 \overset{\circ}{\subseteq} \mathbb{R}^{m_1}$, $U_2 \overset{\circ}{\subseteq} \mathbb{R}^{m_2}$ and $F : U_1 \to U_2$ be smooth. Then

$$\begin{array}{rcl} F^* : \mathscr{D}(U_2) & \to & \mathscr{D}'(U_1) \\ \psi & \mapsto & \langle F^*(\psi) \rangle \end{array}$$

is a continuous operator and its Schwarz Kernel is given by

$$\begin{array}{rcl} K : \mathscr{D}(U_1 \times U_2) & \to & \mathbb{C} \\ \Phi & \mapsto & \mapsto \int_{U_1} \Phi(x, F(x)) dx. \end{array}$$

Proof. The result will follow from the Schwarz Kernel Theorem 5.6.4 after we have proven the following.

Step 1: We calculate for any $\varphi \in \mathscr{D}(U_1)$, $\psi \in \mathscr{D}(U_2)$

$$\begin{aligned} \langle F^*(\psi) \rangle(\varphi) &= \int_{U_1} F^*(\psi)(x)\varphi(x) dx = \int_{U_1} \psi(F(x))\varphi(x) dx \\ &= \int_{U_1} (\varphi \otimes \psi)(x, F(x)) dx = K(\varphi \otimes \psi) \end{aligned} \tag{5.20}$$

Step 2 (F^* is continuous): Clearly F^* is linear. Assume

$$\psi_j \xrightarrow[\mathscr{D}(U_2)]{} 0$$

For any $\varphi \in \mathscr{D}(U_1)$

$$\forall x \in U_1 : \ \psi_j(F(x))\varphi(x) \xrightarrow[\mathbb{C}]{} \psi(F(x))\varphi(x)$$

and $|\psi_j(F(x))\varphi(x)| \leq C|\varphi(x)| \in L^1(U_1)$. Thus, by Lebesgue dominated convergence,

$$\int_{U_1} \psi_j(F(x))\varphi(x) dx \xrightarrow[\mathbb{C}]{} \int_{U_1} \psi(F(x))\varphi(x) dx \ .$$

By (5.20) this proves the claim.

Step 3 ($K \in \mathscr{D}'(U_1 \times U_2)$): Assume

$$\Phi_j \xrightarrow[\mathscr{D}(U_1 \times U_2)]{} 0$$

and $\operatorname{supp} \psi_j \subset L \Subset U_1 \times U_2$. Define $L_1 := \pi_1(L) \Subset U_1$ and calculate

$$|K(\Phi_j)| \leq \int_{U_1} |\Phi_j(x, F(x))| dx = \int_{L_1} |\Phi_j(x, F(x))| dx \leq |L_1| \|\Phi_j\|_{\mathscr{C}^0(U_1 \times U_2)} \to 0.$$

$\square$

5.7. Fourier Transform

Nach Hörmi Definition 7.1.1 D is Dense in S 7.1.8 Fourier is Iso S'->S' 7.1.10 Fourier is smooth 7.1.14 Distribuational Convolution Theorem 7.1.15 Dualization of Diagonalization Properties

5.7.1 Theorem (positive Distributions)**.** Let $u \in \mathscr{D}'(U)$ be *positive* , i.e.

$$\forall \varphi \in \mathscr{D}(U) : \varphi \geq 0 \Longrightarrow u(\varphi) \geq 0.$$

Then there exists a Borel-measure μ, such that

$$u(\varphi) = \int_U \varphi d\mu.$$

6. Sobolev Spaces

"We are Grey. We stand between the darkness and the light."

Delenn, 2259

We have gone to heaven, we have gone to hell. Now we are down to earth. Functions in a Sobolev space are beeings in between. The first major problem one encounters when starting Sobolev theory is the definition of the Sobolev spaces. In the literature you will find dozens of definitions on different levels of abstraction suited for a large variety of different purposes. Dealing with all these purposes and treating Sobolev spaces completely is far beyond the scope of this book. In a first step we will however introduce some common definitions and show that they agree whenever this is senseful. Then we will establish all the theorems suited for our purpose, namely the study of pseudo-differential operators on hermitian vector bundles.

6.1. Local Theory

6.1.1 Lemma. For any $s \in \mathbb{R}$ there exist constants $c_1, c_2 > 0$, such that

$$\forall \xi \in \mathbb{R}^n : c_1(1+|\xi|)^{2s} \leq (1+|\xi|^2)^s \leq c_2(1+|\xi|)^{2s}.$$

Proof. All expressions are positive, so taking the power to s is legitimate as well as taking the power to $1/s$. Therfore it suffices to check the statement for $s=1$.
By the binomic formulae

$$(1+|\xi|)^2 = 1+|\xi|^2+2|\xi|.$$

Therefore

$$\frac{(1+|\xi|)^2}{1+|\xi|^2} = 1+\frac{2|\xi|}{1+|\xi|^2} \xrightarrow{|\xi|\to\infty} 1 \; ,$$
$$\frac{1+|\xi|^2}{(1+|\xi|)^2} = \frac{1+|\xi|^2}{1+|\xi|^2+2|\xi|} \leq 1.$$

Since convergent series are bounded, the result follows. □

6.1.2 Lemma. For any $k \in \mathbb{N}$ there are constants $c_1, c_2 > 0$, such that

$$c_1(1+|\xi|)^k \leq \sum_{|\alpha|\leq k}$$

6.1.3 Definition (Sobolev Space)**.** For any $s \in \mathbb{R}$ define the *s scalar product* $\langle \_, \_ \rangle_s : \mathscr{S} \times \mathscr{S} \to \mathbb{C}$ by

$$\langle f, g\rangle_s := \langle (1+|\xi|)^s \mathcal{F}(f), (1+|\xi|)^s \mathcal{F}(g)\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1+|\xi|)^{2s} \langle \mathcal{F}(f), \mathcal{F}(g)\rangle_{\mathbb{C}^r} d\xi$$

and denote by $\|\_\|_s$ the induced norm. The completion of $(\mathscr{S}, \|\_\|_s)$ with respect to this norm is the *Sobolev space of order s*. We denote this space by

$$H^s := \overline{\mathscr{S}}^{\|\_\|_s}$$

The following facts are immediate.

6.1.4 Lemma. Let $s \in \mathbb{R}$.

(i) The norm $\|\_\|_s$ is equivalent to

$$|f|_s^2 := \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\mathscr{F}(f)(\xi)|^2 d\xi.$$

(ii) The space $(\mathscr{S}, \langle \_, \_ \rangle_s)$ is a pre-Hilbert space.

(iii) H^s is a Hilbert space.

(iv) $\mathscr{S} \subset H^s$ is dense.

(v) If $s' > s$, then $H^{s'} \subset H^s$ and the inclusion $\iota : H^{s'} \hookrightarrow H^s$ is continuous.

(vi) For any $s > 0$ there is an inclusion $H^s \hookrightarrow L^2$.

Proof.

(i) This follows directly from 6.1.1.

(ii) Since $\forall \xi \in \mathbb{R}^n : 1 + |\xi| \neq 0$, this is clear.

(iii) By definition.

(iv) By definition.

(v) ...

(vi) This follows from the fact, that $H^0 \cong L^2$.

□

6.1.5 Definition. Let $U \subset \mathbb{R}^n$ be any open. For any $s \in \mathbb{R}$, we define

$$H^s(U) := \overline{\mathscr{C}_c^\infty(U)}^{\|\_\|_s} i.e.$$

$H^s(U)$ is the closure of $\mathscr{C}_c^\infty(U)$ with respect to the $\|\_\|_s$-norm.

6.1.6 Lemma. $H^s(\mathbb{R}^n) = H^s$.

Proof. It suffices to check that $\mathscr{C}_c^\infty(\mathbb{R}^n) \subset \mathscr{S}$ is dense with respect to the $\|\_\|_s$-norm. Therefore let $f \in \mathscr{S}$, let $\mathbb{R}^n = \cup_{k \in \mathbb{N}} B_k(0)$ be an open cover, let $\{\psi_k\}_{k \in \mathbb{N}}$ be a partition of unity subordinate to this cover and $f_k := \psi_k f \in \mathscr{C}_c^\infty(\mathbb{R}^n)$. Clearly

$$f_k \xrightarrow[p.w.]{} f \ , \qquad |f_k| \leq |f|.$$

Thus by Lebesgue dominated convergence, we obtain

$$f_k \xrightarrow[L^2(\mathbb{R}^n)]{} f \ .$$

Since the Fourier transform is continuous in this topology by Plancharel's Theorem (c.f. **??**), we obtain

$$\mathcal{F}(f_k) \xrightarrow[L^2(\mathbb{R}^n)]{} \mathcal{F}(f) \ .$$

This implies

$$f_k \xrightarrow[\|\_\|_s]{} f \ .$$

□

6.1.7 Theorem (positive integer case)**.** Let $k \in \mathbb{N}_{>0}$. The norms $\|\_\|_k$ and

$$\|f\|^2_{W^{k,2}} = \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} |\partial^\alpha f(x)|^2 dx$$

are equivalent on $\mathscr{S}$.

Proof. By 6.1.4,(i) the norm $\|\_\|_k$ is equivalent to $|\_|_k$. It therefore suffices to check that $|\_|_k$ is equivalent to $\|\_\|_{W^{k,2}}$.
Step 1: We claim that there are constants $C_1, C_2 > 0$, such that

$$\forall \xi \in \mathbb{R}^n : C_1(1+|\xi|^2)^k \leq \sum_{|\alpha|\leq k} |\xi^\alpha|^2 \leq C_2(1+|\xi|^2)^k. \tag{6.1}$$

To see the left inequality let k' be Hölder conjugate to k. By the Hölder inequality on $\mathbb{R}^{n+1}$

$$1+|\xi|^2 = |\langle(1,\dots,1)(1,\xi_1^2,\dots,\xi_n^2)\rangle_{\mathbb{R}^{n+1}}| \leq \|(1,\dots,1)\|_{k'}\|1,\xi_1^2,\dots,\xi_n^2)\|_k,$$

which implies

$$(1+|\xi|^2)^k \leq \underbrace{\|(1,\dots,1)\|^k_{k'}}_{=:C_1^{-1}} \left(1+\sum_{j=1}^n \xi_j^{2k}\right) \leq C_1^{-1} \sum_{|\alpha|\leq k} |\xi^\alpha|^2,$$

where the last inequality holds since on the right and side we are summing positive numbers over a larger index set: In particular we sum over $\alpha = 0$, which corresponds to the summand $|\xi^0|^2 = 1$ on the left and side. And among all the $|\alpha| = k$ there are in particular all the ke_j, $1 \leq j \leq n$, which corresponds to the summands $|\xi^{ke_j}|^2 = \xi_j^{2k}$ on the left hand side.
On the other hand since $|\xi^\alpha| \leq |\xi|^{|\alpha|}$ by A.2.1, we obtain

$$\sum_{|\alpha|\leq k} |\xi^\alpha|^2 \leq \sum_{|\alpha|\leq k} (|\xi|^2)^{|\alpha|} \leq \sum_{|\alpha|\leq k} (1+|\xi|^2)^{|\alpha|} \leq \sum_{|\alpha|\leq k} (1+|\xi|^2)^k \leq (1+|\xi|^2)^k \underbrace{\sum_{|\alpha|\leq k} 1}_{=:C_2}.$$

Step 2: We calculate

$$|f|_k^2 = \int_{\mathbb{R}^n} |(1+|\xi|^2)^k|\mathscr{F}(f)(\xi)|^2 d\xi \overset{(6.1)}{\leq} C_1^{-1} \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} |\xi^\alpha \mathcal{F}(f)(\xi)|^2 d\xi \overset{4.4.22}{=} \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} |\mathcal{F}(\partial^\alpha f)(\xi)|^2 d\xi$$

$$\overset{??}{=} \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} |\partial^\alpha(f)(\xi)|^2 d\xi = \|f\|^2_{W^{k,2}}$$

and similar

$$\|f\|^2_{W^{k,2}} = \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} |\partial^\alpha f(\xi)|^2 d\xi \overset{??}{=} \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} |\mathcal{F}(\partial^\alpha f)(\xi)|^2 d\xi \overset{4.4.22}{=} \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} |\xi^\alpha \mathcal{F}(f)(\xi)|^2 d\xi$$

$$\overset{(6.1)}{\leq} C_2 \int_{\mathbb{R}^n} (1+|\xi|^2)^k |\mathcal{F}(f)(\xi)|^2 d\xi = C_2 |f|_k^2.$$

□

6.1.8 Lemma. For any $\alpha \in \mathbb{C}$, the map $\Lambda^\alpha : \mathscr{S} \to \mathscr{S}$, $f \mapsto \Lambda^\alpha(f)$, where $\Lambda^\alpha(f)(\xi) := \mathcal{F}^{-1}((1+|\xi|)^{2\alpha}(\mathcal{F}(f)(\xi)))$, exstends to an isometry $H^s \to H^{s-\operatorname{Re}\alpha}$ whith inverse $\Lambda^{-\alpha}$. For any $\beta \in \mathbb{C}$, $\Lambda^{\alpha+\beta} = \Lambda^\alpha + \Lambda^\beta$.

Proof. We calculate

$$\|\Lambda^\alpha(f)\|^2_{s-\operatorname{Re}\alpha} = \int_{\mathbb{R}^n} (1+|\xi|)^{2s-2\operatorname{Re}\alpha}|(1+|\xi|)^{2\alpha}(\mathcal{F}(f)(\xi))|$$
$$= \int_{\mathbb{R}^n} (1+|\xi|)^{2s-2\operatorname{Re}\alpha}(1+|\xi|)^{2\operatorname{Re}\alpha}|(\mathcal{F}(f)(\xi))| = \|f\|^2_s.$$

For any $\beta \in \mathbb{C}$

$$\Lambda^{\alpha+\beta}(f) = \mathcal{F}^{-1}((1+|\xi)^{\alpha+\beta}(\mathcal{F}(f)(\xi)) = \mathcal{F}^{-1}((1+|\xi)^{\alpha}\mathcal{F}(\mathcal{F}^{-1}((1+|\xi)^{\beta}(\mathcal{F}(f)(\xi)))) = (\Lambda^\alpha \circ \Lambda^\beta)(f).$$

and clearly $\Lambda^0(f) = f$. This implies the statement. $\square$

6.1.9 Theorem (Sobolev Interpolation)**.** Let $s, s', t, t' \in \mathbb{R}$, such that $s < t$ and $s' < t'$. Assume $T \in \mathscr{L}(H^s, H^{s'})$ and $T(H^t) \subset H^{t'}$. Then for any $\theta \in [0,1]$

$$T \in \mathscr{L}(H^{\theta t + (1-\theta)s}, H^{\theta t' + (1-\theta)s'}).$$

Proof. The case $\theta = 0$ holds by hypothesis. We will prove the continuity by the closed graph theorem.
Step 1 (Case $\theta = 1$): We have to show that $T \in \mathscr{L}(H^t, H^{t'})$. To that end assume

$$x_n \xrightarrow[H^t]{} x\ , \qquad\qquad Tx_n \xrightarrow[H^{t'}]{} y\ .$$

By 6.1.4, the inclusions $H^t \hookrightarrow H^s$ and $H^{t'} \hookrightarrow H^{s'}$ are continuous. Thus

$$x_n \xrightarrow[H^s]{} x\ , \qquad\qquad Tx_n \xrightarrow[H^{s'}]{} y\ .$$

By hypothesis $T : H^s \to H^{s'}$ is continuous. Therefore $T(x) = y$ in $H^{s'}$ and thus in $H^{t'}$.
Step 2 (Case $0 < \theta < 1$): Let $f, g \in \mathscr{S}$ and define $u : \bar{\Omega} :\to \mathbb{C}$, $z \mapsto \langle f, (\Lambda^{(1-z)s'+zt'} \circ T \circ \Lambda^{(z-1)s-zt})(g)\rangle$. Here we use the notation of Lemma 6.1.8 above. The function u is holomorphic on Ω. Let $z \in \bar{\Omega}$ and calculate

$$\begin{aligned}|u(z)| &= |\langle f, (\Lambda^{(1-z)s'+zt'}(T(\Lambda^{(z-1)s-zt})(g)))\rangle_{L^2}| = |\langle \Lambda^{\bar{z}(t'-s')}(f), \Lambda^{s'}(T(\Lambda^{-s}(\Lambda^{z(s-t)})(g)\rangle_{L^2}| \\ &\le \|\Lambda^{\bar{z}(t'-s')}(f)\|_{L^2}\|\Lambda^{s'}(T(\Lambda^{-s}(\Lambda^{z(s-t)})(g)\|_{L^2} \le \|\Lambda^{t'-s'}(f)\|_{L^2}\|T\|_{s->s'}\|\Lambda^{s-t}(g)\|_{L^2},\end{aligned} \tag{6.2}$$

where the last inequality can be seen as follows: Since $z \in \bar{\Omega}$, we obtain $\operatorname{Re} z \in [0,1]$. Since $t' - s' > 0$ by hypothesis, this implies

$$|(1+|\xi|)^{2(t'-s')\bar{z}}| = |(1+|\xi|)^{2(t'-s')\operatorname{Re} z}| \le |(1+|\xi|)^{2(t'-s')}|.$$

Thus Plancharel's Theorem implies

$$\|\Lambda^{\bar{z}(t'-s')}(f)\|^2_{L^2} = \|\mathcal{F}^{-1}((1+|\xi|)^{2\bar{z}(t'-s')}(\mathcal{F}((f))))\|^2_{L^2}$$
$$\le \|(1+|\xi|)^{\operatorname{Re} z(t'-s')}(\mathcal{F}((f)))\|^2_{L^2} = \|\Lambda^{t'-s'}(f)\|^2_{L^2}.$$

Similar, since $s-t<0$ by hypothesis

$$|(1+|\xi|)^{2(s-t)z}| = |(1+|\xi|)^{2(s-t)\operatorname{Re} z}| \leq 1.$$

Again by Plancharel's Theorem $\|\Lambda^{(s-t)z}(g)\|_{L^2} \leq \Lambda^{s-t}(g)\|_{L^2}$.
Furthermore,

$$\|\Lambda^{s'} \circ T \circ \Lambda^{-s}\|_{\mathscr{L}(L^2,L^2)} = \|T\|_{\mathscr{L}(H^s,H^{s'})} = \|T\|_{s->s'}.$$

by 6.1.8. The outcome of this is, that u is bounded on $\bar{\Omega}$ and holomorphic on Ω anyway. Now we want to apply 3.3.1 and calculate for any $y \in \mathbb{R}$

$$|u(iy)| \overset{(6.2)}{\leq} \|\Lambda^{-iy((t'-s'))}(f)\|_{L^2}\|T\|_{s->s'}\|\Lambda^{iy(s-t)} \leq \|f\|_{L^2}\|g\|_{L^2}|\underbrace{\|T\|_{s->s'}}_{=:M_0}$$

$$|u(1+iy)| = |\langle f, \Lambda^{(1-(1+iy))s'+(1+iy)s'}(T(\Lambda^{(1+iy-1)s-(1+iy)t}(g)))\rangle|$$
$$= |\langle f, (\Lambda^{t'} \circ T \circ \Lambda^{-t})(g)\rangle| \leq \|f\|_{L^2}\underbrace{\|T\|_{t->t'}}_{=:M_1}\|g\|$$

Thus by 3.3.1

$$\forall \theta \in [0,1] : |u(\theta+iy)| \leq M_0^{1-\theta}M_1^{\theta}\|f\|_{L^2}\|g\|_{L^2}.$$

In particular

$$|\langle f, \underbrace{\Lambda^{(1-\theta)s'+\theta t'}T\Lambda^{(\theta-1)s-\theta t}}_{=:T_\theta} g\rangle| \leq M_0^{1-\theta}M_1^{\theta}\|f\|_{L^2}\|g\|_{L^2}.$$

In particular $T_\theta \in \mathscr{L}(L^2, L^2)$, thus $T \in \mathscr{L}(H^{\theta t+(1-\theta)s}, H^{\theta t'+(1-\theta)s'})$.

□

6.1.10 Corollary. If $T \in \mathscr{L}(H^{s_1}, H^{s_1})$ and $T : H^{s_2} \to H^{s_2}$, then

$$\forall s \in [s_1, s_2] : T \in \mathscr{L}(H^s, H^s).$$

6.1.11 Theorem (Sobolev Embedding Theorem)**.** Let $s \in \mathbb{R}$, $k \in \mathbb{N}$, such that

$$s > \frac{n}{2} + k.$$

Then there exists a constant $K_s > 0$, such that

$$\forall f \in \mathscr{S} : \|f\|_{\mathscr{C}^k} \leq K_s\|f\|_s.$$

Thus there exists a constinuous embedding $H^s \hookrightarrow \mathscr{C}^k$. (Here $\mathscr{C}^k = \mathscr{C}^k(\mathbb{R}^n)$).

Proof.
Step 1: The hypothesis implies that

$$s > \frac{n}{2} + k \Rightarrow s - k > \frac{n}{2} \Rightarrow -(k-s) > \frac{n}{2}.$$

Therefore Lemma A.2.4 implies

$$(1+|\xi|^2)^{-\frac{s-k}{2}} = (1+|\xi|^2)^{\frac{k-s}{2}} \in L^2(\mathbb{R}^n). \tag{6.3}$$

STEP 2: Let $f \in \mathscr{S}$ and $\alpha \in \mathbb{N}^n$, $|\alpha| \leq k$. Using the Fourier Inversion Theorem 4.4.26 and the Cauchy/Schwarz inequality, we obtain for any $x \in \mathbb{R}^n$

$$
\begin{aligned}
&|\partial^\alpha f(x)| \leq \int_{\mathbb{R}^n} |e^{i\langle x,\xi\rangle} \mathcal{F}(\partial^\alpha f)(\xi)| d\xi \overset{4.4.22}{=} \int_{\mathbb{R}^n} |\xi^\alpha \mathcal{F}(f)(\xi)| d\xi \\
&\leq \int_{\mathbb{R}^n} (1+|\xi|^2)^{-\frac{s-k}{2}} (1+|\xi|^2)^{\frac{s-k}{2}} |\xi|^{|\alpha|} |\mathcal{F}(f)(\xi)| d\xi \leq |\langle (1+|\xi|^2)^{-\frac{s-k}{2}}, (1+|\xi|^2)^{\frac{s}{2}} |\mathcal{F}(f)(\xi)| \rangle_{L^2}| \\
&\leq \underbrace{\|(1+|\xi|^2)^{-\frac{s-k}{2}}\|_{L^2}}_{=:C} \|(1+|\xi|^2)^{\frac{s}{2}} |\mathcal{F}(f)(\xi)|\|_{L^2} = C\Big(\int_{\mathbb{R}^n} (1+|\xi|^2)^s\Big) |\mathcal{F}(f)(\xi)|^2 d\xi\Big)^{\frac{1}{2}} = C|f|_s.
\end{aligned}
$$

Using 6.1.4,(i) and summing over all such α yields the result.

□

6.1.12 Theorem (Rellich Lemma)**.** Let $t < s$, $K \Subset \mathbb{R}^n$ and $(f_j) \in H^s$ be a bounded sequence of functions such that $\operatorname{supp} f_j \subset K$. Then there exists a subsequence (f_{j_ν}) which converges in any H^t.

Proof. Assume that

$$\forall j \in \mathbb{N} : \|f_j\|_s \leq C.$$

STEP 1: In a first step, we will show, that the hypothesis implies, that $(\hat{f}_j)$ has a compactly convergent subsequence (i.e. a subsequence that converges on any compact subset $K' \Subset \mathbb{R}^n$ with respect to $\|\_\|_{\mathcal{C}^0(K')}$.)
Let $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^n, \mathbb{C})$, such that $\varphi|_K \equiv 1$. For any $j \in \mathbb{N}$, we obtain $f_j = \varphi f_j$ and therefore by Theorem 4.4.19

$$\hat{f}_j = \hat{\varphi} * \hat{f}_j$$

and by Theorem 4.4.3

$$\partial^\alpha(\hat{f}_j) = \partial^\alpha(\hat{\varphi}) * \hat{f}_j.$$

By the Cauchy/Schwarz inequality, this implies for any $\xi \in \mathbb{R}^n$

$$
\begin{aligned}
&|\partial^\alpha(\hat{f}_j)(\xi)| \leq \int_{\mathbb{R}^n} |\partial_\xi^\alpha(\hat{\varphi})(\xi-\eta)\hat{f}_j(\eta)| d\eta = \int_{\mathbb{R}^n} |\partial_\xi^\alpha(\hat{\varphi})(\xi-\eta)(1+|\eta|^2)^{-\frac{s}{2}}(1+|\eta|^2)^{\frac{s}{2}}\hat{f}_j(\eta)| d\eta \\
&= |\langle |\partial_\xi^\alpha(\hat{\varphi})(\xi-\eta)(1+|\eta|^2)^{-\frac{s}{2}}|, (1+|\eta|^2)^{\frac{s}{2}}|\hat{f}_j|(\eta)\rangle_{L^2}| \\
&\leq \underbrace{\|\partial_\xi^\alpha(\hat{\varphi})(\xi-\eta)(1+|\eta|^2)^{-\frac{s}{2}}\|_{L^2}}_{=:K_\alpha(\xi)} \|(1+|\eta|^2)^{\frac{s}{2}}\hat{f}_j(\eta)\|_{L^2} = K_\alpha(\xi)\|f_j\|_s \leq K_\alpha(\xi)C
\end{aligned}
$$

!ToDo Warum liegt die Funktion in $K_\alpha(\xi)$ überhaupt in L^2?
Since K_α is continuous, this implies that the sequence $(\partial^\alpha(\hat{f}_j))$ is uniformly bounded on any compact subset. By the mean value theorem, this implies that on any compact subset K' all the $\hat{f}_j$ are Lipschitz continuous with the same Lipschitz constant. In particular they are equicontinuous on K'. By the Arzelá-Ascoli Theorem (!ToDoRef) there exists a $\|\_\|_{\mathcal{C}^0(K')}$-convergent subsequence. By taking a compact exhaustion of $\mathbb{R}^n$ and a diagonal sequence argument, we may inductively construct a sequence that is compactly convergent on all of $\mathbb{R}^n$.
STEP 2: Now we prove the following claim: If $(f_j) \in H^s$ is bounded and $\hat{f}_j$ is compactly convergent, then for any $t < s$, (f_j) converges in H^t.

Since H^t is complete, it satisfies to check, that (f_j) is a H^t-Cauchy-sequence. So assume that $\varepsilon > 0$, $t < s$, fix any $r > 0$ and calculate

$$\begin{aligned} \|f_j - f_k\|_t^2 &= \int_{\mathbb{R}^n} (1+|\xi|^2)^t |\hat{f}_j(\xi) - \hat{f}_k(\xi)| d\xi \\ &= \underbrace{\int_{B_r(0)} (1+|\xi|^2)^t |\hat{f}_j(\xi) - \hat{f}_k(\xi)| d\xi}_{=:I_1(r)} + \underbrace{\int_{\mathbb{R}^n \setminus B_r(0)} (1+|\xi|^2)^t |\hat{f}_j(\xi) - \hat{f}_k(\xi)| d\xi}_{=:I_2(r)} \end{aligned}$$

STEP 2.1: Now if $|\xi| > r$, we may estimate

$$(1+|\xi|^2)^t = (1+|\xi|^2)^{t-s}(1+|\xi|^2)^s \leq (1+r^2)^{t-s}(1+|\xi|^2)^s,$$

since $t - s < 0$. Therefore we may bound $I_2(r)$ by

$$I_2(r) \leq (1+r^2)^{t-s} \|f_j - f_k\|_s^2.$$

Since (f_j) is bounded in H^s, there exists a constant, such that $\|f_j - f_k\|_s^2 \leq C$. Therefore we have archieved:

$$\exists r > 0 : \forall j, k \in \mathbb{N} : I_2(r) \leq \varepsilon/2.$$

(Notice, that this would not have been possible if $t = r$.)
STEP 2.2: Take such an r. Regardless how large it might be, $\bar{B}_r(0) \Subset \mathbb{R}^n$ is compact. Define the constant

$$C' := (1+|r|^2)^t) \operatorname{vol}(B_r(0)).$$

By hypothesis $(\hat{f}_j)$ is compactly convergent and therefore in particular compactly Cauchy. Consequently

$$\exists N \in \mathbb{N} : \forall j, k \geq N : \|\hat{f}_j - \hat{f}_k\|_{\mathcal{C}^0(\bar{B}_r(0))} \leq \frac{\varepsilon}{2C'}.$$

Therefore

$$\forall j, k \geq N : I_1(r) \leq (1+|r|^2)^t) \operatorname{vol}(B_r(0)) \|\hat{f}_j - \hat{j}_k\|_{\mathcal{C}^0(\bar{B}_r(0))} < \frac{\varepsilon}{2}.$$

□

6.1.13 Theorem (Sobolev Representation Theorem)**.** For any $s \in \mathbb{R}$ the L^2 scalar product $\mathscr{S} \times \mathscr{S} \to \mathbb{C}$

$$(f, g) \mapsto \int_{\mathbb{R}^n} \langle f, g \rangle_{\mathbb{C}^r}$$

has a continuous extension to $H^s \times H^{-s} \to \mathbb{C}$ that is non-degenerate.
The map $B : H^{-s} \to (H^s)'$, $f \mapsto (g \mapsto \langle g, \bar{f} \rangle_{L^2})$ is an isometry (notice that $(H^s)'$ is a Hilbert space as well.)

Proof.
STEP 1 (Continuity & Extension): For any $f, g \in \mathscr{S}$, we calculate

$$\begin{aligned} |\langle f, g \rangle_{L^2}| &\overset{??}{=} |\langle \mathcal{F}(f), \mathcal{F}(g) \rangle_{L^2}| = |\langle (1+|\xi|)^s \mathcal{F}(f), (1+|\xi|)^{-s} \mathcal{F}(g) \rangle_{L^2}| \\ &\leq \|(1+|\xi|)^s \mathcal{F}(f)\|_{L^2} \|(1+|\xi|)^{-s} \mathcal{F}(g)\|_{L^2} = \|f\|_s \|g\|_{-s}. \end{aligned} \tag{6.4}$$

Thus there exists a continuous extension $\langle \_, \_ \rangle : H^s \times H^{-s} \to \mathbb{C}$.

STEP 2 (Isometry): We now show that B is isometric. Of course we endow $(H^s)'$ with the operator norm. For any $f \in H^s$, we obtain

$$\|B(f)\|_{(H^s)'} = \sup_{\|g\|_s=1} |\langle g, \bar{f}\rangle| \overset{(6.4)}{\leq} \|\bar{f}\|_{-s} \overset{4.4.21}{=} \|f\|_{-s}.$$

To see that this is actually an equality, define $g_0 := \mathcal{F}^{-1}(\mathcal{F}(\bar{f})(1+|\xi|)^{-2s})$. This implies

$$|B(f)(g_0)| = |\langle g_0, \bar{f}\rangle_{L^2}| = |\langle \mathcal{F}(g_0), \mathcal{F}(\bar{f})\rangle_{L^2}| = |\langle \mathcal{F}(\bar{f})(1+|\xi|)^{-2s}, \mathcal{F}(\bar{f})\rangle_{L^2}| = \|f\|_{-s}^2.$$

On the other hand

$$\begin{aligned}\|g_0\|_s^2 &= \int_{\mathbb{R}^n} (1+|\xi|)^{2s}|\mathcal{F}(g_0)(\xi)|^2 d\xi = \int_{\mathbb{R}^n} (1+|\xi|)^{2s}|\mathcal{F}(\bar{f})(\xi)(1+|\xi|)^{-2s}|^2 d\xi \\ &\overset{4.4.21}{=} \int_{\mathbb{R}^n} (1+|-\xi|)^{-2s}|\overline{\mathcal{F}(f)}(-\xi)|^2 d\xi = \|f\|_{-s}^2.\end{aligned}$$

Combining this, we obtain

$$\|B(f)\|_{(H^s)'} \geq \left| B(f)\Big(\frac{g_0}{\|g_0\|_s}\Big)\right| = \|f\|_{-s}.$$

This also proves, that $\langle \_, \_\rangle_{L^2}$ is not degenerate on H^s.
STEP 3 (Surjectivity): By 3.2.3, B has closed image. Denote by $\Phi : H^s \to (H^s)'$, $f \mapsto \langle \_, \bar{f}\rangle$, the linear isometry from the Frechét-Riesz Representation theorem. Since $\mathscr{S} \subset H^s$ is dense, $\Phi(\mathscr{S}) \subset (H^s)'$ is dense as well by 3.2.4. Let $l = \Phi(f) \in \Phi(\mathscr{S})$. Define $h := \overline{\Lambda^s(\bar{f})} \in \mathscr{S} \subset H^{-s}$. We calculate for any $g \in H^s$:

$$\begin{aligned}B(h)(g) &= \langle g, \bar{h}\rangle_{L^2} = \langle g, \Lambda^s(\bar{f})\rangle_{L^2} = \langle \mathcal{F}(g), \mathcal{F}(\Lambda^s(\bar{f})\rangle_{L^2} \\ &= \int_{\mathbb{R}^n} \mathcal{F}(g)(\xi)(1+|\xi|)^{2s}\mathcal{F}(\bar{f})(\xi) d\xi = \langle g, \bar{f}\rangle_s = \Phi(f)(g) = l(g).\end{aligned}$$

This implies $\Phi(\mathscr{S}) \subset \operatorname{im} B$ and therefore

$$(H^s)' \overset{3.2.4}{=} \overline{\Phi(\mathscr{S})} \subset \overline{\operatorname{im}(B)} = \operatorname{im} B.$$

Thus $\operatorname{im} B = (H^s)'$.

□

6.1.14 Corollary. Let $T, T^* : \mathscr{S} \to \mathscr{S}$ be linear maps, such that

$$\forall f, g \in \mathscr{S} : \langle Tf, g\rangle = \langle f, T^* g\rangle.$$

Let $s \in \mathbb{R}$ and assume there exists $c > 0$, such that

$$\forall f \in \mathscr{S} : \|Tf\|_s \leq c\|f\|_s \tag{6.5}$$

then T^* satisfies

$$\forall g \in \mathscr{S} : \|T^* g\|_{-s} \leq c\|g\|_{-s}.$$

If (6.5) holds for any $k \in \mathbb{N}$, then T extends to a bounded linear map $T : H^k \to H^k$ and T^* extends to a bounded linear map $T^* : H^{-k} \to H^{-k}$.

Proof. Using 6.1.13, we calculate for any $f, g \in \mathscr{S}$

$$|\langle f, T^* g\rangle_{L^2}| = |\langle Tf, g\rangle_{L^2}| \leq \|Tf\|_s \|g\|_{-s} \leq c\|f\|_s\|g\|_{-s}.$$

Consequently using 6.1.13 again, we obtain

$$\begin{aligned} \|T^*g\|_{-s} &= \|\overline{T^*g}\|_{-s} = \|B(\overline{T^*g})\|_{(H^s)'} = \sup_{\|f\|_s=1} |B(\overline{T^*g})(f)| \\ &= \sup_{\|f\|_s=1} |\langle B(\overline{T^*g})(f)\rangle| = \sup_{\|f\|_s=1} |\langle f, T^*g, \rangle| \leq c\|g\|_{-s}. \end{aligned}$$

□

Proof. Using 6.1.13, we calculate for any $f, g \in \mathscr{S}$

$$|\langle T^*g, f\rangle| = |\langle g, Tf\rangle| \leq \|g\|_{-s}\|Tf\|_s \leq c\|f\|_s\|g\|_{-s}$$

Consequently using 6.1.13 again, we obtain

$$\|T^*g\|_{-s} = \|B(T^*g)\|_s = \|B(\overline{T^*g})\|_s = \| \sup_{\|f\|_s=1} |\langle T^*g, f\rangle| \leq c\|g\|_s$$

□

6.1.15 Theorem. Let $A \in \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^{n\times n})$ be a smooth matrix-valued function, such that for all $\alpha \in \mathbb{N}$, $|D^\alpha A|$ is bounded (here $|\_|$ denotes the operator norm). Then for any $s \in \mathbb{R}$, the map $T : \mathscr{S} \to \mathscr{S}$, $f \mapsto Af$, extends to a bounded linear map $T : H^s \to H^s$.

Proof. The calculation

$$\langle f, \bar{A}^t g\rangle_{L^2} = \int_{\mathbb{R}^n} \langle f, \bar{A}^t g\rangle_{\mathbb{C}^r} = \int_{\mathbb{R}^n} \langle Af, g\rangle_{\mathbb{C}^r} = \langle Tf, g\rangle$$

proves that $T^*f = \bar{A}^t f$. For any $s \in \mathbb{N}_0$

$$\|Tf\|^2_{W^{s,2}} = \sum_{|\alpha|\leq s} \int_{\mathbb{R}^n} |D^\alpha(Af)| \overset{\text{A.1.3}}{=} \sum_{|\alpha|\leq s}\sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}^n} |D^{\alpha-\beta}(A)D^\beta f)| \leq C\|f\|^2_{W^{s,2}},$$

thus T is bounded with respect to the $W^{s,2}$-norm. This norm is equivalent to $\|\_\|_s$ by 6.1.7. Therefore T is a bounded linear map $H^s \to H^s$ for all positive integers s. By 6.1.14 T is bounded for all negative integers as well. Consequently 6.1.10 implies the statement. □

6.1.16 Theorem (Diffeomorphism Invariance)**.** Let $U, V \subset \mathbb{R}^n$ be open sets with smooth boundary and let $\Phi : \bar{U} \to \bar{V}$ be a diffeomorphism. For any $s \in \mathbb{R}$, the map $T : \mathscr{C}^\infty_c(V) \to \mathscr{C}^\infty_c(U)$, $f \mapsto f \circ \Phi$, extends to a bounded linear map $T : H^s(V) \to H^s(U)$.
Let $\Phi : \mathscr{S} \to \mathscr{S}$ be a diffeomorphism which is linear outside a compact subset. For any $s \in \mathbb{R}$, the map $T : \mathscr{S} \to \mathscr{S}$, $f \mapsto f \circ \Phi$, extends to a bounded linear map $T : H^s \to H^s$.

Proof.
Step 1:

Step 2: Define $T^*: \mathscr{S} \to \mathscr{S}$, $g \mapsto \overline{\det(\nabla\Phi)} g \circ \Phi^{-1}$ and verify for all $f, g \in \mathscr{S}$

$$\langle Tf, g\rangle_{L^2} = \int_{\mathbb{R}^n} \langle Tf, g\rangle dx = \int_{\mathbb{R}^n} \langle f\circ\Phi, g\rangle dx = \int_{\mathbb{R}^n} \langle f\circ\Phi, g\circ\Phi^{-1}\circ\Phi\rangle \det(\nabla\Phi^{-1})\det(\nabla\Phi) dx$$
$$= \int_{\mathbb{R}^n} \langle f, g\circ\Phi^{-1}\rangle \det(\nabla\Phi^{-1}) dy = \langle f, T^* g\rangle_{L^2}.$$

Now we check for any $s \in \mathbb{N}$

$$\|Tf\|^2_{W^s} = \sum_{|\alpha|\leq s} \int_{\mathbb{R}^n} |D^\alpha(f\circ\Phi)|^2 dx \overset{\text{A.2.6}}{\leq} C\|f\|^2_{W^s}.$$

By 6.1.14 T^* is bounded as well, thus both maps extend to $T: H^s \to H^s$, $T^*: H^{-s} \to H^{-s}$ for any $s \in \mathbb{N}$. Consequently, the result follows from 6.1.10.

□

6.1.17 Theorem (PDOs on Sobolev spaces)**.** For any $s \in \mathbb{R}$, $D^\alpha: \mathscr{S} \to \mathscr{S}$ has a continuous extension to $D^\alpha: H^s \to H^{s-|\alpha|}$. For any $s \in \mathbb{N}$ and any PDO $P = \sum_{|\alpha|\leq k} P_\alpha D^\alpha \in \operatorname{Diff}^k(\mathbb{R}^n, \mathbb{C}^r, \mathbb{C}^s)$, such that $P_\alpha \in \mathscr{C}^k_b(\mathbb{R}^n)$, there is a continuous extension $P: H^s \to H^{s-k}$.

Proof.
Step 1: Let $f \in \mathscr{S}$. We calculate

$$\|D^\alpha f\|^2_{s-|\alpha|} = \int_{\mathbb{R}^n} (1+|\xi|)^{2(s-|\alpha|)} |\mathcal{F}(D^\alpha f)(\xi)|^2 d\xi = \int_{\mathbb{R}^n} (1+|\xi|)^{2s-2|\alpha|} |\xi^\alpha \mathcal{F}(f)(\xi)|^2 d\xi$$
$$\leq \int_{\mathbb{R}^n} (1+|\xi|)^{2s-2|\alpha|} |\xi|^{2\alpha} |\mathcal{F}(f)(\xi)|^2 d\xi = \|f\|^2_s.$$

Therefore $D^\alpha: (\mathscr{S}, \|\_\|_s) \to (\mathscr{S}, \|\_\|_{s-|\alpha|})$ is continuous and has a continuous extension $D^\alpha: H^s \to H^{s-|\alpha|}$.
Step 2: Now consider $P_\alpha D^\alpha$. We calculate

$$\|P_\alpha D^\alpha f\|^2_{s-|\alpha|} \overset{6.1.7}{\leq} C\|P_\alpha D^\alpha f\|^2_{W^{s-|\alpha|,2}} = C \sum_{|\beta|\leq s-|\alpha|} \int_{\mathbb{R}^n} |\partial^\beta P_\alpha D^\alpha f|^2 dx$$
$$\overset{\text{A.1.3}}{=} C \sum_{|\beta|\leq s-|\alpha|} \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \int_{\mathbb{R}^n} |\partial^{\beta-\gamma}(P_\alpha) D^{\alpha+\gamma}(f)|^2 dx$$
$$\leq C\|P_\alpha\|^2_{\mathscr{C}^k(\mathbb{R}^n)} \sum_{|\beta|\leq s-|\alpha|} \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \int_{\mathbb{R}^n} |D^{\alpha+\gamma}(f)|^2 dx$$
$$\leq C'\|P_\alpha\|^2_{\mathscr{C}^k(\mathbb{R}^n)} \sum_{|\delta|\leq s} \int_{\mathbb{R}^n} |D^{\alpha+\gamma}(f)|^2 dx \overset{6.1.7}{\leq} C''\|P_\alpha\|^2_{\mathscr{C}^k(\mathbb{R}^n)} \|f\|^2_s.$$

□

6.2. Globalization: Elementary approach

Let $E \to M$ be a hermitian vector bundle of rank r over a compact manifold M.

6.2.1 Definition (Good presentation)**.** A *good presentation of* E is given by the following data

(i) A finite system of maps $\{\tilde{\varphi}_j : \bar{U}_j \to \bar{\mathrm{B}}^m\}_{j=1,\dots,N}$, where U_j is open, such that $\tilde{\varphi}_j$ is a chart for M. We also require that the balls $B_j := \{p \in U_j \mid |\tilde{\varphi}_j(p)|^2 \leq 1/2\}$ still cover M.

(ii) A finite system of maps

$$\varphi_j := \frac{1}{\sqrt{1 - |\tilde{\varphi}_j|^2}} \tilde{\varphi}_j.$$

(iii) A finite system of local trivializations $\Phi_j : E|_{\bar{U}_j} \to \bar{U}_j \times \mathbb{C}^r$.

(iv) A partition of unity $\{\psi_j\}_{j=1,\dots,N}$ subordinate to the cover $\{B_j\}_{j=1,\dots,N}$.

6.2.2 Lemma (Properties of good presentations)**.** Any such vector bundle $E \to M$ admits a good presentation. Good presentations have the following additional properties:

(i) $\varphi_j : U_j \to \mathbb{R}^m$ is a diffeomorphism, $\varphi_j(B_j) = \mathrm{B}^m$.

(ii) For any $u \in \Gamma(E)$, let $\tilde{u}_j : \mathbb{R}^n \to \mathbb{C}^r$ be the push-forward associated to φ_j and Φ_j (c.f. 2.2.9). Then $\tilde{u}_j$ is bounded, in fact $\tilde{u}_j \in \mathscr{S}$.

(iii) Let $u_j := \psi_j u$, the section u may be decomposed into

$$u = \sum_{j=1}^{N} u_j,$$

where $u_j \in \Gamma_c(B_j)$. The map $({\varphi_j}_* \circ {\Phi_j}_*)(u_j)$ has compact support in B^m.

Proof. Existence is clear.

(i) Follows from the definition.

(ii) By definition $f := \tilde{u}_j = \mathrm{pr}_2 \circ \Phi_j \circ u \circ \varphi_j^{-1} : \mathbb{R}^m \to \mathbb{C}^r$.

□

6.2.3 Definition. Let $(\varphi_j, \Phi_j, \psi_j)_{j=1,\dots,N}$ be good presentation for E, let $u \in \Gamma(E)$ and $s \in \mathbb{R}$. Then

$$\|u\|_s := \sum_{j=1}^{N} \|({\varphi_j}_* \circ {\Phi_j}_*)(u_j)\|_{H^s(\mathrm{B}^m, \mathbb{C}^r)}$$

is the *Sobolev s-norm*. The completion of $\Gamma(E)$ with respect to this norm is the *Sobolev space of order s on E*, which we denote by

$$H^s(E) := \overline{\Gamma(E)}^{\|\_\|_s}.$$

6.2.4 Lemma. The space $H^s(E)$ is well-defined, i.e. the equivalence class of the Sobolev s-norm is indepent of the good presentation of E that was used to define it.

Proof. Let $(\tilde{\varphi}_j, \tilde{\Phi}_j, \tilde{\psi}_j)$ be another good presentation. We say two presentations are equivalent if their induced Sobolev s-norms are equivalent. Since this is transitive, it is sufficient to seperately check independence of the charts, the trivializations and the partition of unity.

STEP 1 ($(\tilde{\varphi}_j, \Phi_j, \psi_j) \sim (\varphi_j, \Phi_j, \psi_j)$): By Theorem 6.1.16 the operator $T_j := (\tilde{\varphi}_j \circ \varphi_j^{-1})_* : H^s(\mathrm{B}^m, \mathbb{C}^r) \to H^s(\mathrm{B}^m, \mathbb{C}^r)$ is continous. Define $C := \max_{1\leq j\leq N} \|T_j\|$ and calculate:

$$\|u\|_{s,(\tilde{\varphi}_j,\Phi_j,\psi_j)} = \sum_{j=1}^N \|(\tilde{\varphi}_{j*} \circ \Phi_{j*})(u_j)\|_{H^s(\mathrm{B}^m,\mathbb{C}^r)} = \sum_{j=1}^N \|(\tilde{\varphi}_{j*} \circ \varphi_{j*}^{-1} \circ \varphi_{j*} \circ \Phi_{j*})(u_j)$$
$$= \sum_{j=1}^N \|(\tilde{\varphi}_j \circ \varphi_j^{-1})_*(\varphi_{j*} \circ \Phi_{j*})(u_j))\|_{H^s(\mathrm{B}^m,\mathbb{C}^r)} \leq C \sum_{j=1}^N \|(\varphi_{j*} \circ \Phi_{j*})(u_j))\|_{H^s(\mathrm{B}^m,\mathbb{C}^r)}$$
$$= \|u\|_{s,(\varphi_j,\Phi_j,\psi_j)}.$$

STEP 2 ($(\varphi_j, \tilde{\Phi}_j, \psi_j) \sim (\varphi_j, \Phi_j, \psi_j)$): Since Φ_j, $\tilde{\Phi}_j$ are both trivializations of the bundle E, there exists a function $A_j \in \mathscr{C}^\infty(U_j, Gl(r))$, such that $\tilde{\Phi}_j \circ \Phi^{-1} = \mathrm{id} \times A$. Notice that for any $f \in \mathscr{C}^\infty(U, \mathbb{C}^r)$

$$(\tilde{\Phi}_{j*} \circ \Phi_{j*}^{-1})(f) = \tilde{\Phi}_{j*}(\Phi_j^{-1} \circ \mathrm{id} \times f) = \mathrm{pr}_2 \circ \mathrm{id} \times A_j \circ \mathrm{id} \times f = A_j f.$$

The operator $T_j : H^s(\mathrm{B}^m, \mathbb{C}^r) \to H^s(\mathrm{B}^m, \mathbb{C}^r)$, $f \mapsto A_j \varphi_{j*}(f)$ is continuous by Theorem 6.1.15. Define $C := \max_{1\leq j\leq N} \|T_j\|$ and calculate

$$\|u\|_{s,(\varphi_j,\tilde{\Phi}_j,\psi_j)} = \sum_{j=1}^N \|(\varphi_{j*} \circ \tilde{\Phi}_{j*})(u_j)\|_{H^s(\mathrm{B}^m,\mathbb{C}^r)} = \sum_{j=1}^N \|(\varphi_{j*} \circ \tilde{\Phi}_{j*}) \circ (\Phi_j^{-1})_* \circ \Phi_{j*}(u_j)\|_{H^s(\mathrm{B}^m,\mathbb{C}^r)}$$
$$= \sum_{j=1}^N \|(\varphi_{j*} \circ A_j \Phi_{j*}(u_j)\|_{H^s(\mathrm{B}^m,\mathbb{C}^r)} = \sum_{j=1}^N \|(\varphi_{j*}(A_j)(\varphi_{j*} \circ \Phi_{j*})(u_j)\|_{H^s(\mathrm{B}^m,\mathbb{C}^r)}$$
$$\leq C \sum_{j=1}^N \|(\varphi_{j*} \circ \Phi_{j*})(u_j)\|_{H^s(\mathrm{B}^m,\mathbb{C}^r)} = C\|u\|_{s,(\varphi_j,\Phi_j,\psi_j)}.$$

STEP 3 ($(\varphi_j, \Phi_j, \tilde{\psi}_j) \sim (\varphi_j, \Phi_j, \psi_j)$): (!ToDo so noch nicht ganz Cauchy)

$$\|u\|_{s,(\varphi,\Phi,\tilde{\psi})} = \sum_{j=1}^N \|(\varphi_{j*} \circ \Phi_{j*})(\tilde{\psi}_j u)\|_s = \sum_{j=1}^N \|(\varphi_{j*} \circ \Phi_{j*})(\tilde{\psi}_j \sum_{i=1}^N \psi_i u)\|_s$$
$$\leq \sum_{j=1}^N \sum_{i=1}^N \|(\varphi_{j*} \circ \Phi_{j*})(\tilde{\psi}_j \psi_i u)\|_s$$
$$= \sum_{j=1}^N \sum_{i=1}^N \|\varphi_{j*}(\tilde{\psi}_j)(\varphi_{j*} \circ \Phi_{j*})(\psi_i u)\|_s$$
$$\leq \sum_{1\leq i,j\leq N, i\neq j} \|(\varphi_{j*} \circ \Phi_{j*})(\psi_i u)\|_s + \sum_{1\leq j\leq N} \|(\varphi_{j*} \circ \Phi_{j*})(\psi_j u)\|_s$$
$$\leq C \sum_{1\leq i,j\leq N, i\neq j} \|\varphi_{j*}(\psi_j)(\varphi_{j*} \circ \Phi_{j*})(\psi_i u)\|_s + \|u\|_{s,(\varphi,\Phi,\psi)}$$
$$= C \sum_{1\leq i,j\leq N, i\neq j} \|\varphi_{i*}(\psi_i)(\varphi_{j*} \circ \Phi_{j*})(\psi_j u)\|_s + \|u\|_{s,(\varphi,\Phi,\psi)}$$
$$= C' \sum_{1\leq i,j\leq N, i\neq j} \|(\varphi_{j*} \circ \Phi_{j*})(\psi_j u)\|_s + \|u\|_{s,(\varphi,\Phi,\psi)} \leq C' N \|u\|_{s,(\varphi,\Phi,\psi)}$$

□

6.3. Globalization: Geometric approach

Instead of defining the Sobolev spaces locally, one can use methods from differential geometry defining them globally. We assume the reader to be familiar with connections on vector bundles.

6.3.1 Definition (Basic Sobolev norm)**.** Let $E \to M$ be a hermitian vector bundle, M be a compact manifold, and let ∇ be a connection. For any section $u \in \Gamma(E)$ let ∇u be the covariant derivative of u and

$$\nabla^j u := \underbrace{\nabla \ldots \nabla}_{j \text{ times}} u$$

be the j-fold covariant derivative of u. We assume that the connection is extended to all the tensor bundles $T^k E$. For any $k \in \mathbb{N}$ define

$$\|u\|_k^2 := \sum_{j=0}^{k} \int_M |\nabla^j u|^2,$$

where $|\_|$ is the extension of the fibre metric in E. We say $\|\_\|_k$ is the *basic Sobolev k-norm.*

6.3.2 Lemma. The basic Sobolev k-norm is independent of the choice of metrics and connection.

Proof. If ∇, $\tilde{\nabla}$ are two connections, their difference $\tilde{\nabla} - \nabla =: A$ is a tensor field $A \in \Gamma(T^*M \otimes E)$, i.e.

$$\forall X \in \mathcal{T}(M) : \forall u \in \Gamma(E) : \tilde{\nabla}_X u - \nabla_X u = A(X, u).$$

Now by definition $\nabla u \in \Gamma(T^*M \otimes E)$, $\nabla u(X) = \nabla_X u$. Therefore

$$|\tilde{\nabla} u| = |\nabla s - A(\_, u)| \leq |\nabla u| + |A(\_, u)| \leq |\nabla u| + |A||u|.$$

□

6.4. Globalization of the results

Regardless how we define the Sobolev spaces on bundles, the key results from the local theory globalize as well.

6.4.1 Theorem (Globalized Sobolev Spaces)**.** Let $E, F \to M$ be a hermitian bundles of rank r_E and r_F over a compact m-manifold and $s \in \mathbb{R}$.

(i) Embedding: For any integer $k \in \mathbb{N}$ and, such that $s > m/2 + k$, there is a continuous inclusion $H^s(E) \to \mathscr{C}^k(E)$.

(ii) Rellich: Any sequence (u_j), which is bounded in $H^s(E)$ has a subsequence, that converges in $\mathscr{C}^k(E)$.

(iii) For any Riemannian volume form dV on M, the bilinear pairing $\langle \_, \_ \rangle : \Gamma(E) \to \Gamma(E^*) \to \mathbb{C}$

$$(u, u') \mapsto \int_M u'(u) dV$$

extends to a perfect pairing $H^s(E) \times H^{-s}(E)$.

(iv) For any $A \in \mathrm{Hom}(E, F; M)$, the map $T_A : \Gamma(E) \to \Gamma(F)$, $u \mapsto Au$, extends to a bounded linear map $T_A : H^s(E) \to H^s(F)$.

(v) Any differential operator $P : \Gamma(E) \to \Gamma(F)$ of order k extends to a bounded linear map $P : H^s(E) \to H^s(F)$.

Proof.

(i) !ToDo

(ii) !ToDo

(iii) !ToDo

(iv) !ToDo

(v) Take a good presentation of E and F simultanously, i.e. assume that $(\varphi_j, \Phi_j, \psi_j)_{j=1,\dots,N}$ is a good presentation of E, such that $(\varphi_j, \Psi_j, \psi_j)_{j=1,\dots,N}$ is a good presentation of F. Assume $P_j := (\varphi_{j_*} \circ \Psi_{j_*}) \circ P \circ (\varphi_{j_*} \circ \Phi_{j_*})^{-1} : \mathscr{C}^\infty(\mathrm{B}^m, \mathbb{C}^{r_E}) \to \mathscr{C}^\infty(\mathrm{B}^m, \mathbb{C}^{r_F})$ is a local representation of P. By Theorem 6.1.17, P_j extends to a continuous operator $H^s(B^m, \mathbb{C}^{r_E}) \to H^{s-k}(\mathrm{B}^m, \mathbb{C}^{r_F})$. Define $C := \max_{1 \le j \le N} \|P_j\|$ and calculate for any $u \in \Gamma(E)$

$$\|Pu\|_s = \sum_{j=1}^N \|(\varphi_{j_*} \circ \Psi_{j_*})(\psi_j P(u))\|_{H^s(\mathrm{B}^m, \mathbb{C}^{r_F})} = \sum_{j=1}^N \|(\varphi_{j_*}(\psi_j)(\varphi_{j_*} \circ \Psi_{j_*})(P(u)))\|_{H^s(\mathrm{B}^m, \mathbb{C}^{r_F})}$$
$$= \sum_{j=1}^N \|(\varphi_{j_*}(\psi_j)(\varphi_{j_*} \circ \Psi_{j_*} \circ (\varphi_{j_*} \circ \Psi_{j_*})^{-1} \circ P_j \circ \varphi_{j_*} \circ \Phi_{j_*})(u)\|_{H^s(\mathrm{B}^m, \mathbb{C}^{r_F})}$$
$$= \sum_{j=1}^N \|(\varphi_{j_*}(\psi_j)(P_j \circ \varphi_{j_*} \circ \Phi_{j_*})(u)\|_{H^s(\mathrm{B}^m, \mathbb{C}^{r_F})} = \sum_{j=1}^N \|(P_j \circ \varphi_{j_*} \circ \Phi_{j_*})(\psi_j u)\|_{H^s(\mathrm{B}^m, \mathbb{C}^{r_F})}$$
$$\le C \sum_{j=1}^N \|(\varphi_{j_*} \circ \Phi_{j_*})(\psi_j u)\|_{H^{s-k}(\mathrm{B}^m, \mathbb{C}^{r_F})} = \|u\|_{s-k}.$$

□

Dualitätssatz, Lesch Übungszettel 3, Afg 1 Methode der komplexen Interpolation, Taylor PDE I S.275, Lesch Übungszettel 2 Globalisieren Definition über schwache Ableitungen und Äquivalenz zur gegebenen Definition

7. FIO: Fourier Integral Operators

"Up and down, up and down, I will lead them up and down.
I am fear'd in field and town, Goblin, lead them up and down. "

PUCK,

7.1. Motivation

Oscillatory integrals are motivated by the following observation.

7.1.1 Theorem. Let

$$P = \sum_{|\alpha| \leq k} P_\alpha D^\alpha \in \mathrm{Diff}^k(\mathbb{R}^n, \mathbb{C}^r, \mathbb{C}^{r'}), \qquad \forall |\alpha| \leq k : P_\alpha \in \mathscr{C}_b^\infty(\mathbb{R}^n, \mathbb{C}^{r' \times r})$$

be a differential operator with full symbol σ. Then for any $u \in \mathscr{S}(\mathbb{C}^r)$

$$Pu(x) = \int_{\mathbb{R}^n_\xi} e^{i\langle x, \xi \rangle} \sigma(x, \xi) \mathcal{F}(u)(\xi) đ\xi = \int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_y} e^{i\langle x-y, \xi \rangle} \sigma(x, \xi) u(y) đy đ\xi.$$

Proof. Let $x \in \mathbb{R}^n$ be arbitrary. By the Fourier Inversion formula (c.f. 4.4.26), we have

$$u(x) = \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} \mathcal{F}(u)(\xi) đ\xi$$

Since $u \in \mathscr{S}$, we may interchange differentiation and integration in order to obtain

$$\begin{aligned} P(u)(x) &= \int_{\mathbb{R}^n} \sum_{|\alpha| \leq k} P_\alpha(x) D_x^\alpha \left(e^{i\langle x, \xi \rangle} \right) \mathcal{F}(u)(\xi) đ\xi = \int_{\mathbb{R}^n} \sum_{|\alpha| \leq k} P_\alpha(x) \xi^\alpha e^{i\langle x, \xi \rangle} \mathcal{F}(u)(\xi) đ\xi \\ &= \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} \sigma(x, \xi) \mathcal{F}(u)(\xi) đ\xi. \end{aligned}$$

Inserting the definition of the Fourier transform implies the statement. □

7.1.2 Remark. The integral on the right hand side will be of our interest. We would like to write this integral as

$$\iint_{\mathbb{R}^n_\xi \times \mathbb{R}^n_y} e^{i\langle x-y, \xi \rangle} \sigma(x, \xi) u(y) d(y, \xi), \tag{7.1}$$

but we can't! Why? Because if this integral existed over the product space, the iterated integrals

$$\int_{\mathbb{R}^n_y} \int_{\mathbb{R}^n_\xi} e^{i\langle x-y, \xi \rangle} \sigma(x, \xi) u(y) d\xi dy = \int_{\mathbb{R}^n_y} \int_{\mathbb{R}^n_\xi} e^{i\langle x-y, \xi \rangle} \sigma(x, \xi) d\xi u(y) dy$$

existed as well by Tonellis theorem. Notice that in almost every senseful case, this makes absolutely no sense: A function is Lebesgue integrable if and only if its absolute value is. But look at the inner integral: If we try to integrate its absolute value, we obtain

$$\int_{\mathbb{R}^n_\xi} |e^{i\langle x-y, \xi \rangle} \sigma(x, \xi)| d\xi = \int_{\mathbb{R}^n_\xi} |\sigma(x, \xi)| d\xi = \|\sigma(x, \xi)\|_{L^1(\mathbb{R}^n_\xi)}.$$

But σ is a polynomial in ξ. In $\int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_y} \ldots$ the polynomial growth of σ was compensated by the decay of $\mathcal{F}(u)$, which is a Schwartz function. If we interchange the order of integration, the inner integral does not have anything to do with u. Only the polynomial growth of σ remains, there is no decay to compensate, and the integral explodes. This also implies that the Lebesgue integral (7.1) over the product space does not exist. There is no solution to this problem in general: There are iterated integrals, for which the corresponding integral over the product space does not exist and which massively depend on the order of integration. Nevertheless we may restrict our attention to a special class of functions and we can modify the notion of an integral itself, such that (7.1) exists as a well-defined *oscillatory integral*.

7.2. Phase functions

7.2.1 Definition (Phase function)**.** Define $\dot{\mathbb{R}}^n := \mathbb{R}^n \setminus \{0\}$ and remember that $U \subset \mathbb{R}^m$ is open. A function $\Phi : U \times \mathbb{R}^n \to \mathbb{R}$ is a *phase function*, if it satisfies the following properties:

(i) $\Phi \in \mathscr{C}^\infty(U \times \dot{\mathbb{R}}^n, \mathbb{R})$

(ii) Φ is positive homogenous of degree 1 on $\dot{\mathbb{R}}^n$, i.e.

$$\forall x \in U : \forall \theta \in \dot{\mathbb{R}}^n : \forall t > 0 : \Phi(x, t\theta) = t\Phi(x, \theta).$$

(iii) Φ has no critical points, i.e.

$$\forall x \in U : \forall \theta \in \dot{\mathbb{R}}^n : \nabla_{(x,\theta)}\Phi(x, \theta) \neq 0.$$

We explicitely allow the case $m = 0$, in which case the domain of definition becomes $\{0\} \times \mathbb{R}^n \cong \mathbb{R}^n$.

7.2.2 Definition (conical)**.** A subset $C \subset U \times \dot{\mathbb{R}}^n$ is *concial*, if

$$\forall t > 0 : (x, \theta) \in C \Rightarrow (x, t\theta) \in C.$$

If C is a conical set, a set D is a *conical neighbourhood of* C, if D is a conical set and $\bar{C} \subset D^\circ$ (where the closure $\bar{C}$ is taken in the subspace $U \times \dot{\mathbb{R}}^n$).

7.2.3 Definition. Let Φ be a phase function on $U \times \mathbb{R}^n$. Define $\pi : U \times \mathbb{R}^n \to U$ and

$$\begin{aligned} C_\Phi &:= \{(x, \theta) \in U \times \dot{\mathbb{R}}^n : \nabla_\theta(\Phi)(x, \theta) = 0\} \\ S_\Phi &:= \pi(C_\Phi), \quad R_\Phi := U \setminus C_\Phi. \end{aligned}$$

7.2.4 Lemma. Let Φ be a phase function.

(i) C_Φ is conical and closed in $U \times \dot{\mathbb{R}}^n$.

(ii) S_Φ is closed and R_Φ is open in U.

(iii) For any $x \in R_\Phi$, the function

$$\begin{aligned} \Phi_x : \mathbb{R}^n &\to \mathbb{R} \\ \theta &\mapsto \Phi(x, \theta) \end{aligned}$$

is a phase function on $\mathbb{R}^n$.

Proof.

(i) Since Φ is positively homogenous of order 1, this follows from 4.5.4. By continuity it is closed.

(ii) Therefore S_Φ is closed and R_Φ is open in U.

(iii) It is clear that Φ_x satisfies properties (i) and (ii) of the Definition 7.2.1 (even for any $x \in U$). Condition (iii) however is only satisfied if $x \in R_\Phi$.

□

7.3. Oscillatory Integrals

7.3.1 Theorem. Let Φ be a phase function on $U \times \mathbb{R}^n$. There exists

$$L = L_\Phi(U \times \mathbb{R}^n) \in \operatorname{Diff}^1(U \times \mathbb{R}^n, \mathbb{C})$$

such that

(i)

$$L = \sum_{\nu=1}^{n} a_\nu(x,\xi)\partial_{\theta_\nu} + \sum_{\mu=1}^{m} b_\mu(x,\theta)\partial_{x_\mu} + c(x,\theta), \tag{7.2}$$

(ii) whith coefficients

$$\begin{aligned} \forall 1 \leq \nu \leq n : a_\nu &\in \mathcal{S}^0(U \times \mathbb{R}^n, \mathbb{C}), \\ \forall 1 \leq \mu \leq m : b_\mu &\in \mathcal{S}^{-1}(U \times \mathbb{R}^n, \mathbb{C}), \\ c &\in \mathcal{S}^{-1}(U \times \mathbb{R}^n, \mathbb{C}), \\ \forall x \in U : \forall |\xi| \leq 1 : \forall 1 \leq \nu \leq n &: \forall 1 \leq \mu \leq m : a_\nu(x,\theta) = b_\mu(x,\theta) = 0, \end{aligned} \tag{7.3}$$

(iii) satisfying

$$L^* e^{i\Phi} = e^{i\Phi}. \tag{7.4}$$

(iv) In fact there are infinitely many such operators.

(v) We explicitely allow the case $m = 0$ here, i.e. $U \times \mathbb{R}^n \cong \mathbb{R}^n$. In that case L does not have any derivatives in any x_μ-direction.

(vi) In case $U = U_x \times U_y \subset \mathbb{R}^{m_x} \times \mathbb{R}^{m_y}$ is a subset of a product and if for any $y \in U_y$, $\nabla_{(x,\theta)}\Phi \neq 0$, the operator L has an expression

$$L = \sum_{\nu=1}^{n} a_\nu(x,y,\xi)\partial_{\theta_\nu} + \sum_{\mu=1}^{m_x} b_\mu(x,y,\theta)\partial_{x_\mu} + \sum_{\lambda=1}^{m_y} b_\lambda(x,y,\theta)\partial_{y_\lambda} + c(x,\theta),$$
$$\forall 1 \leq \lambda \leq m_y : b_\lambda \equiv 0.$$

(vii) The map

$$L : S^k(U \times \mathbb{R}^n, \mathbb{C}) \to S^{k-1}(U \times \mathbb{R}^n, \mathbb{C})$$

is continuous. The map

$$\begin{aligned} S^k(U \times \mathbb{R}^n) \times \mathscr{D}(U, \mathbb{C}) &\to S^{k-l}(U \times \mathbb{R}^n, \mathbb{C}) \\ (a, u) &\mapsto (x,\theta) \mapsto L^l(a(x,\theta)u(x)) \end{aligned}$$

is continuous as well.

Proof.

STEP 1 (Construction of L^*): Define

$$P := \sum_{\nu=1}^{m} \partial_{\theta_\nu}(\Phi)|\theta|^2 \partial_{\theta_\mu} + \sum_{\mu=1}^{n} \partial_{x_\mu}(\Phi)\partial_{x_\mu} \in \mathrm{Diff}^1(U \times \mathbb{R}^n, \mathbb{C})$$

We calculate for any $(x,\theta) \in U \times \dot{\mathbb{R}}^n$:

$$\partial_{\theta_\nu}(e^{i\Phi}) = i\partial_{\theta_\nu}(\Phi)e^{i\Phi}, \qquad \partial_{x_\mu} e^{i\Phi} = i\partial_{x_\mu}(\Phi)e^{i\Phi},$$

which implies

$$\begin{aligned} -iPe^{i\Phi} &= \Big(-i\sum_{\nu=1}^{m} \partial_{\theta_\nu}(\Phi)|\theta|^2\partial_{\theta_\mu} - i\sum_{\mu}^{n} \partial_{x_\mu}(\Phi)\partial_{x_\mu}\Big)e^{i\Phi} \\ &= \Big(\sum_{\nu=1}^{m} |\theta|^2 \partial_{\theta_\nu}(\Phi)^2 + \sum_{\mu=1}^{n} \partial_{x_\mu}(\Phi)^2\Big)e^{i\Phi}. \end{aligned} \tag{7.5}$$

Define

$$\begin{aligned} \psi : U \times \dot{\mathbb{R}}^n &\to \mathbb{C} \\ (x,\theta) &\mapsto \Big(\textstyle\sum_{\nu=1}^{m} |\theta|^2\partial_{\theta_\nu}(\Phi)^2 + \sum_{\mu=1}^{n} \partial_{x_\mu}(\Phi)^2\Big)^{-1}. \end{aligned}$$

By the properties of a phase function, the term in brackets is never zero. Therefore this is well-defined.
In case we are in situation (vi), the hypothesis ensures that if we leave out differentiation with respect to the y_λ directions, this term still is nonzero.
We obtain $\psi \in \mathscr{C}^\infty(U \times \dot{\mathbb{R}}^n)$. By Lemma 4.5.4, we obtain that ψ is positively homogenous of degree -2 in θ. By construction and (7.5)

$$-i\psi P e^{i\Phi} = e^{i\Phi}. \tag{7.6}$$

The problem is that ψ might blow up at $\theta = 0$ and thus does not admit a smooth extension to $U \times \mathbb{R}^n$. Therefore we must cut off this singularity: Let $\chi \in \mathscr{C}_c^\infty(\mathbb{R}^n, \mathbb{R})$ such that

$$\forall |\theta| \leq 1 : \chi(\theta) = 1, \qquad \forall |\theta| \geq 2 : \chi(\theta) = 0. \tag{7.7}$$

Define

$$M := -i\psi(1-\chi)P + \chi \in \mathrm{Diff}^1(U \times \mathbb{R}^n, \mathbb{C}).$$

Of course for $\theta = 0$, we interpret $(1-\chi(\theta))\psi(x,\theta) = 0$. Define $L := M^*$. This is a differential operator, which by construction satisfies

$$L^* e^{i\Phi} = Me^{i\Phi} = -i\psi(1-\chi)Pe^{i\Phi} + \chi e^{i\Phi} \overset{(7.6)}{=} (1-\chi)e^{i\Phi} + \chi e^{i\Phi} = e^{i\Phi}.$$

Since there are infinitely many such cut off functions χ, there are infinitely many such operators.

STEP 2 (symbols estimates): It remains to show (ii). By construction, the coefficients of M are given by

$$\begin{aligned} \forall 1 \leq \nu \leq m : \tilde{a}_\nu(x,\theta) &= -i|\theta|^2\psi(x,\theta)(1-\chi(\theta))\partial_{\theta_\nu}(\Phi)(x,\theta), \\ \forall 1 \leq \mu \leq n : \tilde{b}_\nu(x,\theta) &= -i\psi(x,\theta)(1-\chi(\theta))\partial_{x_\mu}(\Phi)(x,\theta), \\ \tilde{c}(x,\theta) &= \chi(\theta). \end{aligned}$$

In the following, we will use 4.5.4 and 4.5.5 several times.

STEP 2.1 ($\tilde{a}_\nu$): By hypothesis Φ is positively homogenous of order 1 in θ. Therefore $\partial_{\theta_\nu}(\Phi)$ is pos. hom. of order 0. The function ψ is pos. hom. of order -2 and clearly $\theta \mapsto |\theta|^2$ is pos. hom. of order 2. Therefore $|\theta|^2\psi\partial_{\theta_\nu}(\Phi)$ is pos. hom. of order 0 and hence a symbol of order 0. Since $1-\chi = 1$ outside $\bar{B}_2(0)$ this is unchanged, if we multiply with $1-\chi$.

STEP 2.2 ($\tilde{b}_\mu$): By the same token, we observe that $\psi\partial_{x_\mu}(\Phi)$ is pos. hom. of order $-2+1 = -1$. Therefore the product is a symbol of order $-1+0=-1$.

STEP 2.3 ($\tilde{c}$): Since χ has compact support, it is a symbol of any order. By 7.7, we obtain

$$\forall |\theta| \leq 1 : a_\nu(x,\theta) = 0 = b_\nu(x,\theta).$$

STEP 2.4 (representation for L): We have shown that M has a representation (7.2) with coefficients satisfying (7.3). By 2.3.16 $L = M^{**}$ and therefore it suffices to show that the coefficients of M^* satisfy (7.3) as well. By (2.21), we obtain for any $f \in \mathscr{C}^\infty(U \times \mathbb{R}^n, \mathbb{C})$

$$\begin{aligned} M^*(f) &= \sum_{\nu=1}^n \partial_{\theta_\nu}(\tilde{a}_\nu^* f) + \sum_{\mu=1}^m \partial_{x_\nu}(\tilde{b}_\nu^* f) + c^* f \\ &= \sum_{\nu=1}^n \partial_{\theta_\nu}(\tilde{a}_\nu^*) f + \tilde{a}_\nu^* \partial_{\theta_\nu}(f) + \sum_{\mu=1}^m \partial_{x_\nu}(\tilde{b}_\nu^*) f + \tilde{b}_\nu^* \partial_{x_\nu}(f) + c^* f \\ &= \sum_{\nu=1}^n \tilde{a}_\nu^* \partial_{\theta_\nu}(f) + \sum_{\mu=1}^m \tilde{b}_\nu^* \partial_{x_\nu}(f) + \underbrace{\Big(\sum_{\nu=1}^n \partial_{\theta_\nu}(\tilde{a}_\nu^*) + \sum_{\mu=1}^m \partial_{x_\nu}(\tilde{b}_\nu^*) + c^* \Big)}_{=:c \in \mathcal{S}^0} f. \end{aligned}$$

Clearly $a_\nu := \tilde{a}_\nu^* \in \mathcal{S}^0$, $\tilde{b}_\nu^* \in \mathcal{S}^{-1}$ as claimed. We have proven (7.3). This also implies (vi).

STEP 3 (continuity): By 4.5.2(i) the symbols are a complex vector space, so it suffices to check that the various summands have the desired mapping properties. By 4.5.2(iv) multiplication of symbols is continuous. By 4.5.2(v) differentiation of symbols is continuous. Putting this together implies that L is continuous. Since $\mathscr{D}(U) \subset S^0(U \times \mathbb{R}^n, \mathbb{C})$, we obtain the second statement by an analogous reasoning.

□

7.3.2 Corollary. Let $a \in \mathcal{S}^k(U \times \mathbb{R}^n, \mathbb{C})$, let Φ be a phase function and let L be the operator form Theroem 7.3.1 above. Then for any $l \in \mathbb{N}$ such that $k - l < -n$, and any $u \in \mathscr{D}(U, \mathbb{C})$, the Lebesgue integral

$$I_\Phi(a,u,l) := \iint\limits_{U \times \mathbb{R}^n} e^{i\Phi(x,\theta)} L^l(a(x,\theta)u(x)) d(x,\theta)$$

exists.

Proof. Since $(a,u) \in \mathcal{S}^k \times \mathscr{D}$, we obtain $L^l(au) \in \mathcal{S}^{k-l}$ by 7.3.1(vii). By definition this implies

$$\forall x \in K : \forall \xi \in \mathbb{R}^n : |L^l(au)(x,\xi)| \leq C(1+|\xi|)^{k-l},$$

where $K := \operatorname{supp} u$. By hypothesis $k - l < -n$. Therefore Lemma A.2.4 implies that $I_\Phi(a,u,l)$ exists. □

7.3.3 Remark. Since $e^{i\Phi} = (L^*)e^{i\Phi}$ one might be tempted to employ this corollary to define

$$\iint\limits_{U\times\mathbb{R}^n} e^{i\Phi(x,\theta)}a(x,\theta)u(x)d(x,\theta) := \iint\limits_{U\times\mathbb{R}^n} (L^*)^l e^{i\Phi(x,\theta)}a(x,\theta)u(x)d(x,\theta)$$
$$:= \iint\limits_{U\times\mathbb{R}^n} e^{i\Phi(x,\theta)}L^l(au)(x,\theta)d(x,\theta).$$

In fact this is what we are going to do in 7.3.5. The only problem is that we do not yet know, if the right hand side depends on l. This problem will be solved by considering an alternative approach to this definition: The problem, we want to solve at the moment is that the integral on the left hand side does not exist as a Lebesgue integral, because of the growth of the integrand in θ. So let's cut off the integrand in θ!

7.3.4 Lemma. Let $a \in \mathcal{S}^k(U\times\mathbb{R}^n)$, Φ be a phase function and $u \in \mathscr{D}(U,\mathbb{C})$. Let $\chi \in \mathscr{D}(\mathbb{R}^n,\mathbb{R})$ be any function, which equals 1 in a neighbourhood of 0. Define

$$I_{\Phi,\varepsilon}(a,u) := \iint\limits_{U\times\mathbb{R}^n} e^{i\Phi(x,\theta)}\chi(\varepsilon\theta)a(x,\theta)u(x)d(x,\theta).$$

Then for any l such that $k-l<-n$

$$\lim_{\varepsilon\searrow 0} I_{\Phi,\varepsilon}(a,u) = I_\Phi(a,u,l),$$

using the notation from 7.3.2. Consequently, the limit does not depend on the cut off function χ. In turn the expression $I_\Phi(a,u,l)$ does not depend on l (as long as $k-l<-n$) and L (as long as L is an operator satisfying the conditions from 7.3.1).

Proof. First of all notice that for any ε, $I_{\Phi,\varepsilon}(a,u)$ exists, since the integrand now has compact θ-support. It even has compact support in $U\times\mathbb{R}^n$. Clearly, we obtain the pointwise convergence

$$\forall (x,\theta)\in U\times\mathbb{R}^n:\ e^{i\Phi(x,\theta)}\chi(\varepsilon\theta)a(x,\theta)u(x) \xrightarrow[\mathbb{C}]{\varepsilon\searrow 0} e^{i\Phi(x,\theta)}a(x,\theta)u(x)\,. \tag{7.8}$$

Since χ has compact support, $\chi\in\mathcal{S}^0(U\times\mathbb{R}^n)$. Consequently

$$\forall \alpha,\beta\in\mathbb{N}^n : \exists C_\beta>0 : |\partial_x^\alpha\partial_\theta^\beta\chi(\theta)| \leq C_\beta(1+|\theta|)^{-|\beta|}.$$

Now for any $0<\varepsilon\leq 1$

$$|\partial_\theta^\beta(\chi(\varepsilon\theta))| \leq |\partial_\theta^\beta(\chi)(\theta)\varepsilon^{|\beta|}| \leq C_\beta|\varepsilon|^{|\beta|}(1+|\theta|)^{-|\beta|} \leq C_\beta(1+|\theta|)^{-|\beta|}. \tag{7.9}$$

Thus for any $l\in\mathbb{N}$, $k-l<-n$, we obtain

$$\begin{aligned}\lim_{\varepsilon\searrow 0} I_{\Phi,\varepsilon}(a,u) &= \lim_{\varepsilon\searrow 0}\iint\limits_{U\times\mathbb{R}^n} e^{i\Phi(x,\theta)}\chi(\varepsilon\theta)a(x,\theta)u(x)d(x,\theta)\\ &= \lim_{\varepsilon\searrow 0}\iint\limits_{U\times\mathbb{R}^n} (L^*)^l(e^{i\Phi(x,\theta)})\chi(\varepsilon\theta)a(x,\theta)u(x)d(x,\theta)\\ &\overset{2.3.16}{=} \lim_{\varepsilon\searrow 0}\iint\limits_{U\times\mathbb{R}^n} e^{i\Phi(x,\theta)}L^l(\chi(\varepsilon\theta)a(x,\theta)u(x))d(x,\theta)\\ &= \iint\limits_{U\times\mathbb{R}^n} e^{i\Phi(x,\theta)}L^l(a(x,\theta)u(x))d(x,\theta),\end{aligned}$$

by the Lebesgue dominated convergence theorem and (7.8), (7.9). □

7.3.5 Definition (Oscillatory integral)**.** For any $a \in \mathcal{S}^k(U \times \mathbb{R}^n, \mathbb{C})$, any phase function Φ and any $u \in \mathscr{D}(U, \mathbb{C})$, the expression (using notation from 7.3.2 and 7.3.4)

$$I_\Phi(a,u) := I_\Phi(a,u,l) := \lim_{\varepsilon \searrow 0} I_{\Phi,\varepsilon}(a,u),$$

where l and χ are chosen as in 7.3.4 above, is called an *oscillatory integral*. Somewhat more explicitely

$$\begin{aligned} I_\Phi(a,u) &:= \overset{Os}{\iint\limits_{U \times \mathbb{R}^n}} e^{i\Phi(x,\theta)} a(x,\theta) u(x) d(x,\theta) \\ &:= \iint\limits_{U \times \mathbb{R}^n} (L^*)^l e^{i\Phi(x,\theta)} a(x,\theta) u(x) d(x,\theta) \\ &= \lim_{\varepsilon \searrow 0} \iint\limits_{U \times \mathbb{R}^n} e^{i\Phi(x,\theta)} \chi(\varepsilon\theta) a(x,\theta) u(x) d(x,\theta). \end{aligned}$$

We just ensured in 7.3.4 that this is well-defined. For reasons of convenience, we will introduce the following notations as well: In case $m = 0$, we will denote the oscillatory integral by

$$\overset{Os}{\int\limits_{\mathbb{R}^n}} e^{i\Phi(\theta)} a(\theta) d\theta.$$

If $U_x \subset \mathbb{R}^{m_1}$, $U_y \subset \mathbb{R}^{m_2}$ are two open sets, we consider $U_z := U_x \times U_y$ and for $a \in \mathcal{S}^k(U_z \times \mathbb{R}^n, \mathbb{C})$ and a phase function Φ on $U_z \times \mathbb{R}^n$, we will denote the oscillatory integral by

$$\overset{Os}{\iiint\limits_{U_z \times \mathbb{R}^n}} e^{i\Phi(x,y,\theta)} a(x,y,\theta) u(x,y) d(x,y,\theta).$$

The following Lemmata will be needed to calculate certain oscillatory integrals.

7.3.6 Lemma. Let Φ be a phase function on $U \times \mathbb{R}^n$. For any $a \in \mathcal{S}^k_{\rho,\delta}(U \times \mathbb{R}^n, \mathbb{C}^{r' \times r}$ and any $x \in U$, denote by $a_x : \mathbb{R}^n \to \mathbb{C}^{r' \times r}$, $\theta \mapsto a(x,\theta)$.

$$\forall x \in R_\Phi : \overset{Os}{\int\limits_{\mathbb{R}^n}} e^{i\Phi_x(\theta)} a_x(\theta) d\theta = \int_{\mathbb{R}^n} e^{i\Phi(x,\theta)} L^l_\Phi(a_x)(\theta) d\theta,$$

$l - k \min(\rho, 1-\delta) < -n$.

Proof. First we verify that

$$\forall x \in R_\Phi : \forall \theta \in \mathbb{R}^n : L_{\Phi_x}(a_x)(\theta) = L_\Phi(a_x)(\theta).$$

This follows from the simple fact that

$$L_\Phi(a_x)(\theta) = \sum_{\nu=1}^m a_\nu(x,\theta) \partial_{\theta_\nu}(a_x)(\theta) + \sum_{\mu=1}^n b_\nu(x,\theta) \underbrace{\partial_{x_\mu}(a_x)}_{=0}(\theta) + c = \sum_{\nu=1}^m a_\nu(x,\theta) \partial_{\theta_\nu}(a_x)(\theta) + c$$

$L_{\Phi_x}(a_x)(\theta)$.

Therefore we may calculate for any $x \in R_\Phi$

$$\overset{Os}{\int\limits_{\mathbb{R}^n}} e^{i\Phi_x(\theta)} a_x(\theta) d\theta = \int_{\mathbb{R}^n} e^{i\Phi_x(\theta)} L^l_{\Phi_x}(a_x)(\theta) d\theta = \int_{\mathbb{R}^n} e^{i\Phi(x,\theta)} L^l_\Phi(a_x)(\theta) d\theta.$$

$\square$

7.3.7 Lemma. Let Φ be a phase funcion, $a \in \mathcal{S}^k_{\rho,\delta}(U \times \mathbb{R}^n, \mathbb{C}^{r' \times r})$, $u \in \mathscr{C}^\infty_c(U)$. Let $L = L(U \times \mathbb{R}^n)$ be a PDO satisfying (**??**). Additionally suppose that there exists $l \in \mathbb{N}$, $l - k\min(\rho, 1-\delta) < -n$, such that for any $(x,\theta) \in U \times \dot{\mathbb{R}}^n$ at least one of the following two conditions is satisfied:

(i) $((L^*)^l e^{i\Phi}(x,\theta) = e^{i\Phi(x,\theta)}$,

(ii) $a(x,\theta)u(x) = 0$.

Proof. The hypothesis implies that for any $(x,\theta) \in U \times \dot{\mathbb{R}}^n$

$$e^{i\Phi(x,\theta)}a(x,\theta)u(x) = (L^*)^l(e^{i\Phi(x,\theta)})a(x,\theta)u(x), \tag{7.10}$$

regardless which of the two conditions is satisfies for (x,θ). Again choose χ as defined in (7.7). We calculate

$$\begin{aligned}
{}^{Os}\iint_{U\times\mathbb{R}^n} e^{i\Phi(x,\theta)}a(x,\theta)u(x)d(x,\theta) &\overset{7.3.5}{=} \lim_{\varepsilon\searrow 0}\iint_{U\times\mathbb{R}^n} e^{i\Phi(x,\theta)}a(x,\theta)u(x)\chi(\varepsilon\theta)d(x,\theta)\\
&\overset{(7.10)}{=} \lim_{\varepsilon\searrow 0}\iint_{U\times\mathbb{R}^n} (L^*)^l(e^{i\Phi(x,\theta)})a(x,\theta)u(x)\chi(\varepsilon\theta)d(x,\theta)\\
&= \lim_{\varepsilon\searrow 0}\iint_{U\times\mathbb{R}^n} (e^{i\Phi(x,\theta)})L^l(a(x,\theta)u(x)\chi(\varepsilon\theta))d(x,\theta)\\
&= \iint_{U\times\mathbb{R}^n} (e^{i\Phi(x,\theta)})L^l(a(x,\theta)u(x))d(x,\theta),
\end{aligned}$$

where the last equation follows from DCT (!ToDo). □

7.4. Regularity of associated Distributions

We just defined an object $I_\Phi(a, \_)$ that sends a function $u \in \mathscr{D}(U)$ to a complex number. Sound familiar?

7.4.1 Definition. For any fixed phase function Φ and any symbol $a \in \mathcal{S}^k_{\rho,\delta}$, the map $\mathcal{A} := \mathcal{A}_\Phi(a) : \mathscr{D}(U) \to \mathbb{C}$,

$$u \mapsto I_\Phi(a,u)$$

is the *distribution associated to* a *and* Φ.

7.4.2 Lemma. An associated distribution is in fact a distribution, i.e. $\mathcal{A} \in \mathscr{D}'(U)$.

Proof. Linearity in u is obvious. To see continuity, assume $K \Subset U$ is compact and $x \in K$. By the Leibniz rule, the fact that $a \in \mathcal{S}^k_{\rho,\delta}$ and the choice of l

$$\begin{aligned}
|\mathcal{A}(a)(u)(x)| &\leq \iint_{U\times\mathbb{R}^n} |e^{i\Phi(x,\theta)}L^l(a(x,\theta)u(x))|d(x,\theta)\\
&\leq \iint_{U\times\mathbb{R}^n} C(1+|\xi|)^{-n}\|u\|_{\mathscr{C}^l(K)}|d(x,\theta)\\
&\leq C'\|u\|_{\mathscr{C}^l(K)}|K|\iint_{\mathbb{R}^n}(1+|\theta|)^{-n}|d\theta.
\end{aligned}$$

□

7.4.3 Theorem (Singular Support of associated distributions)**.** With the notation from 7.4.1 and 7.2.3:

$$\text{sing-supp}\, \mathcal{A}_\Phi \subset S_\Phi.$$

If there exists a conical neighbourhood $D \supset C_\Phi$, such that $a|_D \equiv 0$. Then $\text{sing-supp}\, \mathcal{A}_\Phi = \emptyset$

Proof. Let $u \in \mathscr{D}(R_\Phi)$.
STEP 1: It is sufficient to define a function $A \in \mathscr{C}^\infty(R_\Phi)$, such that $\mathcal{A}|_{R_\Phi} = \langle A \rangle$. Define $A : R_\Phi \to \mathbb{C}$ by

$$A(x) := \overset{Os}{\int_{\mathbb{R}^n}} e^{i\Phi_x(\theta)} a_x(\theta) d\theta$$

(!ToDo ist smooth wegen parameter dependence) Now let $L = L_\Phi(U \times \mathbb{R}^n)$,

$$L = \sum_{\nu=1}^{m} a_\nu(x,\theta)\partial_{\theta_\nu} + \sum_{\mu=1}^{m} b_\mu(x,\theta)\partial_{x_\mu} + c(\theta), \qquad \tilde{L} := \sum_{\nu=1}^{m} a_\nu(x,\theta)\partial_{\theta_\nu} + c(\theta).$$

In case $m = 0$ the operator constructed in 7.3.1 does not have any x-derivatives. Therefore for any $x \in R_\Phi$, we may chose $L_{\Phi_x} = \tilde{L}$ and calculate

$$A(x) = \overset{Os}{\int_{\mathbb{R}^n}} e^{i\Phi_x(\theta)} a_x(\theta) d\theta = \int_{\mathbb{R}^n} e^{i\Phi_x(\theta)} L^l_{\Phi_x}(a_x)(\theta) d\theta = \int_{\mathbb{R}^n} e^{i\Phi(x,\theta)} \tilde{L}^l(a_x)(\theta) d\theta \tag{7.11}$$

STEP 2: We claim that the operator $\tilde{L}$ satisfies the hypothesis of Lemma 7.3.7: Let $(x,\theta) \in U \times \mathbb{R}^n$. Then there are two possible cases: If $x \in S_\Phi$, then $a(x,\theta)u(x) = 0$, because $u \in \mathscr{C}_c^\infty(R_\Phi)$ by hypothesis. In case $x \in R_\Phi$, we obtain

$$\tilde{L}^*(e^{i\Phi})(x,\theta) = L^t_{\Phi_x}(e^{i\Phi_x})(\theta) = e^{i\Phi_x}(\theta) = e^{i\Phi(x,\theta)}. \tag{7.12}$$

STEP 3: Since $\tilde{L}$ does not have any x_μ-derivarives, we may calculate

$$\begin{aligned}\langle A\rangle(u) &= \int_U A(x)u(x)dx = \int_U \overset{Os}{\int_{\mathbb{R}^n}} e^{i\Phi_x(\theta)} a(x,\theta)d\theta\, u(x)dx \overset{(7.11)}{=} \int_U \int_{\mathbb{R}^n} e^{i\Phi(x,\theta)} \tilde{L}^l(a_x)(\theta) d\theta\, u(x)dx \\ &= \int_{U\times\mathbb{R}^n} e^{i\Phi(x,\theta)} \tilde{L}^l(au)(x,\theta) d(x,\theta) \overset{7.3.7}{=} \overset{Os}{\iint_{U\times\mathbb{R}^n}} e^{i\Phi(x,\theta)} a(x,\theta)u(x) d(x,\theta) = \mathcal{A}(u).\end{aligned} \tag{7.13}$$

STEP 4: The second statement is proven in exactly the same fashion: We define $A : U \to \mathbb{C}$ by the right hand side of (7.11). Now we take any $u \in \mathscr{C}_c^\infty(U)$. For any $x \in U$ there are only two possibilities: Either $(x,\theta) \in D$, then $a(x,\theta)u(x) = 0$. Or $x \in U \setminus \pi(D) \subset R_\Phi$, then $\tilde{L} = L_{\Phi_x}$ and we again obtain (7.12). Calculation (7.13) proves the statement.

□

7.5. Fourier Integral Operators "FIO"

7.5.1 Definition (Fourier Integral Operator)**.** Assume $U_x \subset \mathbb{R}^{m_x}$, $U_y \subset \mathbb{R}^{m_y}$, $m_y, m_x \in \mathbb{N}$, $U_z := U_x \times U_y$. Let $\Phi : U_z \times \mathbb{R}^n \to \mathbb{C}$ be a phase function, $a \in \mathcal{S}^k_{\rho,\delta}(U_z \times \mathbb{R}^n, \mathbb{C}^{r'\times r})$. By 7.4.2, the map $\mathcal{A} : U_z \to \mathbb{C}$,

$$\mathcal{A}(w) := I_\Phi(a, w)$$

defines a distribution $\mathcal{A} \in \mathscr{D}'(U_z)$. By (the easy part of) the Schwartz Kernel Theorem 5.6.4, the map $\mathscr{A} : \mathscr{D}(U_y) \to \mathscr{D}'(U_x)$, defined by

$$\mathscr{A}(u)(v) := \mathcal{A}(v \otimes u) = \overset{Os}{\iiint\limits_{U_x \times U_y \times \mathbb{R}^n}} e^{i\Phi(x,y,z)} a(x,y,\theta) u(y) v(x) d(x,y,\theta), \tag{7.14}$$

is linear and continuous. We call such a map a *Fourier Integral Operator (or just a "FIO") with amplitude a.*

7.5.2 Definition (Operator phase function)**.** Let Φ be a phase function on $U_x \times U_y \times \mathbb{R}^n$. Then Φ is an *operator phase function*, if both of the following conditions are satisfied:

(i) $\forall x \in U_x : \forall y \in U_y : \forall \theta \neq 0 : \nabla_{(y,\theta)} \Phi(x,y,\theta) \neq 0$,

(ii) $\forall x \in U_x : \forall y \in U_y : \forall \theta \neq 0 : \nabla_{(x,\theta)} \Phi(x,y,\theta) \neq 0$.

In that case, for any $y \in U_y$, the function $(x,\theta) \mapsto \Phi(x,y,\theta)$ is a phase function on $U_x \times \mathbb{R}^n$ and for any $x \in U_x$, the function $(y,\theta) \mapsto \Phi(x,y,\theta)$ is a phase function on $U_y \times \mathbb{R}^n$.

7.5.3 Theorem (FIO Regularity and Extension)**.** Let $\mathscr{A}$ be a FIO as in 7.5.1 and let Φ be a phase function.

(i) If Φ satisfies condition 7.5.2,(i), then $\mathscr{A}$ is a map $\mathscr{D}(U_y) \to \langle \mathscr{E}(U_x) \rangle$. The composition $\rangle\_\langle \circ \mathscr{A} : \mathscr{D}(U_y) \to \mathscr{E}(U_x)$ is continuous and sometimes also just denoted as $\mathscr{A} : \mathscr{D}(U_y) \to \mathscr{E}(U_x)$.

(ii) If Φ satisfies condition 7.5.2,(ii), then $\mathscr{A}$ has a continuous extension $\mathscr{A} : \mathscr{E}'(U_y) \to \mathscr{D}'(U_x)$.

Proof.

(i) We have to show that for all $u \in \mathscr{D}(U_x)$ there exists $Au \in \mathscr{E}(U_x)$, such that $\mathscr{A}(u) = \langle Au \rangle$. Define

$$Au(x) := \overset{Os}{\iint\limits_{Y \times \mathbb{R}^n}} e^{i\Phi_x(y,\theta)} a(x,y,\theta) u(y) d(y,\theta). \tag{7.15}$$

By hypothesis, this is an oscillatory integral. (!ToDo smooth parameter dependence). The hypothesis ensures that the operator L_Φ for the oscillatory integral $\mathscr{A}(u)(v)$ does not contain any derivatives in x-directions (c.f. 7.3.1). Therefore, we may calculate for any $v \in \mathscr{D}(U_x)$

$$\begin{aligned}
\mathscr{A}(u)(v) &= \overset{Os}{\iiint\limits_{U_x \times U_y \times \mathbb{R}^n}} e^{i\Phi(x,y,\theta)} (a(x,y,\theta) u(y) v(x)) d(x,y,\theta) \\
&= \iiint\limits_{U_x \times U_y \times \mathbb{R}^n} e^{i\Phi(x,y,\theta)} L^l (a(x,y,\theta) u(y) v(x)) d(x,y,\theta) \\
&= \iiint\limits_{U_x \times U_y \times \mathbb{R}^n} e^{i\Phi(x,y,\theta)} L^l (a(x,y,\theta) u(y)) v(x) d(x,y,\theta) \\
&= \int_{U_x} \iint\limits_{U_y \times \mathbb{R}^n} e^{i\Phi_x(y,\theta)} L^l (a(x,y,\theta) u(y)) d(y,\theta) v(x) dx \\
&= \int_{U_x} \overset{Os}{\iint\limits_{U_y \times \mathbb{R}^n}} e^{i\Phi_x(y,\theta)} a(x,y,\theta) u(y) d(y,\theta) v(x) dx = \langle Au \rangle (v).
\end{aligned}$$

By 5.4.8 the map $\mathscr{A} : \mathscr{D}(U_y) \to \mathscr{E}(U_x)$ is a composition of continuous maps.

(ii) Let $u \in \mathscr{D}(U_y)$, $v \in \mathscr{D}(U_x)$. By hypothesis, for any $y \in U_y$, Φ_y is a Phase function on $U_x \times \mathbb{R}n$. Define

$$\forall y \in U_y : A^t(v)(y) := I_{\Phi_x}(a_y, v) = \overset{Os}{\iint\limits_{U_x \times \mathbb{R}^n}} e^{i\Phi_y(x,\theta)} a_y(x,\theta) v(x) d(x,\theta).$$

Notice that this is exactly (7.15) with the roles of x and y interchanged. Consequently $A^t(v)$ is a smooth function as well.
Now define the extension $\mathscr{A} : \mathscr{E}'(U_y) \to \mathscr{D}'(U_x)$ by

$$\forall u \in \mathscr{E}'(U_y) : \forall v \in \mathscr{D}(U_x) : \mathscr{A}(u)(v) := u(A^t v).$$

In terms of pairings this reads as

$$\langle \mathscr{A} u, v \rangle_{\mathscr{D}'(U_x) \times \mathscr{D}(U_x)} = \langle u, A^t v \rangle_{\mathscr{E}'(U_y) \times \mathscr{E}(U_y)},$$

which explains why we think of A^t as a transpose.
This is an extension, because for any $u \in \mathscr{D}(U)$

$$\begin{aligned}
\mathscr{A}(u)(v) &= \overset{Os}{\iiint\limits_{U_x \times U_y \times \mathbb{R}^n_\theta}} e^{i\Phi(x,y,\theta)} a(x,y,\theta) u(y) v(x) d(x,y,\theta) \\
&= \lim_{\varepsilon \searrow 0} \iiint\limits_{U_x \times U_y \times \mathbb{R}^n_\theta} e^{i\Phi(x,y,\theta)} \chi(\varepsilon\theta) a(x,y,\theta) u(y) v(x) d(x,y,\theta) \\
&= \lim_{\varepsilon \searrow 0} \int_{U_y} \iint\limits_{\times U_x \times \mathbb{R}^n_\theta} e^{i\Phi(x,y,\theta)} \chi(\varepsilon\theta) a(x,y,\theta) v(x) d(x,\theta) u(y) dy \\
&= \int_{U_y} \lim_{\varepsilon \searrow 0} \iint\limits_{\times U_x \times \mathbb{R}^n_\theta} e^{i\Phi(x,y,\theta)} \chi(\varepsilon\theta) a(x,y,\theta) v(x) d(x,\theta) u(y) dy = \langle u \rangle (A^t v).
\end{aligned}$$

(!ToDo Vertauschung von Limesbildung und Integration) To see that $\mathscr{A}$ is continuous assume

$$u_j \xrightarrow[\mathscr{E}'(U_y)]{} 0 .$$

This implies for any $v \in \mathscr{D}(U_x)$

$$\lim_{j \to \infty} \mathscr{A}(u_j) = \lim_{j \to \infty} u_j(A^t v) = 0,$$

by definition of convergence in $\mathscr{E}'(U_y)$. This implies

$$\mathscr{A}(u_j) \xrightarrow[\mathscr{E}'(U_x)]{} 0 .$$

□

7.5.4 Definition (composition of sets)**.** Let X, Y be sets, $S \subset X \times Y$ and $K \subset Y$. Define

$$S \circ K := \{x \in X \mid \exists y \in K : (x,y) \in S\}.$$

7.5.5 Theorem. Let Φ be an operator phase function. For any $u \in \mathscr{E}'(U_y)$

$$\text{sing-supp}(\mathscr{A} u) \subset S_\Phi \circ \text{sing-supp}\, u.$$

Proof.
STEP 1: In a first step we prove that for any $K \Subset U_y$:

$$\forall u \in \mathscr{E}'(U_y) : \operatorname{supp} u \Subset K \Rightarrow \text{sing-supp}(\mathscr{A}u) \subset S_\Phi \circ K.$$

To that end it suffices to check that for any $\tilde{U}_x \overset{\circ}{\subseteq} X \setminus S \circ K$, $\mathscr{A}u$ is smooth. Choose $u_j \in \mathscr{D}_X(U_y)$, such that

$$\langle u_j \rangle \xrightarrow[\mathscr{E}'(U_y)]{} u \ .$$

Now we calculate for any $v \in \mathscr{D}(\tilde{U}_x)$

$$\begin{aligned}\mathscr{A}(u)(v) &= \lim_{j\to\infty} \langle \mathscr{A}u_j, v\rangle_{\mathscr{D}'(U_x)\times\mathscr{D}(U_x)} = \lim_{j\to\infty} \langle \mathscr{A}u_j, v\rangle_{\mathscr{D}'(\tilde{U}_x)\times\mathscr{D}(\tilde{U}_x)} \\ &= \lim_{j\to\infty} \langle \mathcal{A}, v\otimes u_j\rangle_{\mathscr{D}'(\tilde{U}_x\times \operatorname{int} K)\times\mathscr{D}(\tilde{U}_x\times \operatorname{int} K)}\end{aligned}$$

By Theorem 7.4.3, $\text{sing-supp}\,\mathcal{A} \subset S_\Phi$. We will now show that the distribution $\mathcal{A}|_{\tilde{U}_x\times \operatorname{int} K}$ is regular by showing that

$$S_\Phi \cap (\tilde{U}_x \times \operatorname{int} K) = \emptyset.$$

This we do by contradiction: Assume there exists $(x,y) \in S_\Phi \cap (\tilde{U}_x \times \operatorname{int} K)$. By definition $x \in \tilde{U}_x$, $y \in \operatorname{int} K$. Since $(x,y) \in S_\Phi$, this implies $x \in S_\Phi \circ K$ by definition of $\circ$. Therefore $x \in \tilde{U}_x \cap S_\Phi \circ K$. This contradicts the choice of $\tilde{U}_x$.
Consequently, there exists $A \in \mathscr{E}(\tilde{U}_x \times \operatorname{int} K)$, such that $\mathcal{A}|_{K\times \operatorname{int} K} = \langle A\rangle$. This in turn defines a continuous operator $\tilde{\mathcal{A}} : \mathscr{E}'(\operatorname{int} K) \to \mathscr{E}(\tilde{U}_x)$ by 5.6.5, which is just the restriction of $\mathscr{A}$. Thus

$$\tilde{\mathscr{A}}u_j \xrightarrow[\mathscr{E}(\tilde{U}_x)]{} \tilde{\mathscr{A}}u \ ,$$

thus $\mathscr{A}u = \tilde{\mathscr{A}} \in \mathscr{E}(\tilde{U}_x)$.
STEP 2: Since $\text{sing-supp}\,\mathscr{A}u \subset U_y$ is closed, there are ε-neighbourhoods K_ε, such that $\text{sing-supp}\,\mathscr{A}u \subset K_\varepsilon \subset \bar{K}_\varepsilon \subset U_y$. Choose $\chi_\varepsilon \in \mathscr{D}(U_y)$, such that $\chi|_{\bar{K}_\varepsilon} \equiv 1$. We may decompose

$$u = \underbrace{\chi_\varepsilon u}_{=:u_1} + \underbrace{(1-\chi_\varepsilon)u}_{=:u_2}.$$

Now by construction u_2 is regular. By choosing ε small enogh, we see that $\operatorname{supp} u_1 \setminus \text{sing-supp}\, u_1$ is arbitrarily small (!ToDo naja, n bisschen handwaving..)

□

8. YDO: Pseudodifferential Operators

"The corps is mother, the corps is father."

Bester, Ψ-Cop, 2259

8.0.6 Definition (Pseudodifferential Operator)**.** Let $k \in \mathbb{R}$, $\sigma \in \mathcal{S}^k(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{C}^{r' \times r})$. An operator $T_\sigma : \mathscr{S}(\mathbb{C}^r) \to \mathscr{S}(\mathbb{C}^{r'})$, defined by

$$(T_\sigma u)(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \sigma(x,\xi)\mathcal{F}(u)(\xi) d\xi \tag{8.1}$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \sigma(x,\xi) u(y) dy d\xi \tag{8.2}$$

is a *pseudodifferential operator of order* k (or just "is a ΨDO". We denote by $\Psi^k := \Psi^k(\mathbb{R}^n; \mathbb{C}^{r' \times r})$ the space of all these operators. Analogously denote

$$\Psi(\mathbb{R}^n, \mathbb{C}^{r' \times r}) := \bigcup_{k \in \mathbb{R}} \Psi^k(\mathbb{R}^n; \mathbb{C}^{r' \times r}), \qquad \Psi^{-\infty}(\mathbb{R}^n, \mathbb{C}^{r' \times r}) := \bigcap_{k \in \mathbb{R}} \Psi^k(\mathbb{R}^n; \mathbb{C}^{r' \times r}),$$

If $\sigma \in \mathcal{S}^{-k}$, $k > 0$, then T_σ is *smoothing of order* k. A linear map $\tau : \mathscr{S} \to \mathscr{S}$, which extends to a bounded linear operator $\tau : H^s \to H^{s+k}$ for all $s, k \in \mathbb{R}$ is an *infinitely smoothing operator*. (!ToDo YDOs mit symbolen in -infty sind also infinitely smoothing) Two pseudodifferential operators P and P' are *equivalent*, if $P - P'$ is an infinitely smoothing operator. We denote the equivalence class of P by $[P]$.

P is infinitely smooth if and only if the symbol is in $S^{-\infty}$

8.0.7 Theorem. Let $k \in \mathbb{R}$, $p \in \mathcal{S}^k(\mathbb{R}^n, \mathbb{C}^{r' \times r})$ and $u \in \mathscr{S}(\mathbb{C}^r)$. Then the function Pu defined by (8.1) automatically satisfies $Pu \in \mathscr{S}(\mathbb{C}^{r'})$. If p has compact x-support, for any $s \in \mathbb{R}$, this operator has a continuous extension $P : H^s \to H^{s-k}$.

Proof.
Step 1 ($Pu \in \mathscr{S}$):
Step 2 (Extension):

□

8.0.8 Theorem. Any bounded PDO is a ΨDO. More precisely: Let

$$P = \sum_{|\alpha| \leq k} P_\alpha D^\alpha \in \mathrm{Diff}^k(\mathbb{R}^n, \mathbb{C}^r, \mathbb{C}^{r'}), \qquad \forall |\alpha| \leq k : P_\alpha \in \mathscr{C}_b^\infty(\mathbb{R}^n, \mathbb{C}^{r' \times r})$$

be a differential operator with full symbol p. Then $p \in \mathcal{S}^k(\mathbb{C}^{r' \times r})$ and for any $u \in \mathscr{S}(\mathbb{C}^r)$

$$Pu(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi)\mathcal{F}(u)(\xi) d\xi.$$

Proof.

Step 1 ($p \in \mathcal{S}^k$): Since p is a polynomial of degree k, $\partial_\xi^\beta p = 0$, if $|\beta| > k$ (see Lemma A.1.6). For any $|\beta| \le k$ and any $\alpha \in \mathbb{N}^n$ we calculate

$$\begin{aligned}
|D_x^\alpha D_\xi^\beta p(x,\xi)| &\le \sum_{|\gamma|\le k} |\partial_x^\alpha (P_\gamma)(x)||\partial_\xi^\beta \xi^\gamma| \overset{\text{A.1.6}}{\le} \sum_{|\gamma|\le k} |\partial_x^\alpha (P_\gamma)(x)||\beta! \binom{\gamma}{\beta} \xi^{\gamma-\beta}| \\
&\overset{\text{A.2.1}}{\le} \sum_{|\gamma|\le k} \beta! \binom{\gamma}{\beta} \|P_\gamma\|_{\mathscr{C}^{|\alpha|}} |\xi|^{|\gamma|-|\beta|} \\
&\le \max_{|\gamma|\le k} \|P_\gamma\|_{\mathscr{C}^{|\alpha|}} \sum_{|\gamma|\le k} \beta! \binom{\gamma}{\beta} (1+|\xi|)^{k-|\beta|} \le C_{\alpha,\beta}(1+|\xi|)^{k-|\beta|}
\end{aligned}$$

Step 2 (integral representation): By the Fourier Inversion formula (c.f. 4.4.26), we have

$$u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \mathcal{F}(u)(\xi) d\xi$$

Since u is compactly supported, we may interchange differentiation and integration in order to obtain

$$\begin{aligned}
Pu(x) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \sum_{|\alpha|\le k} P_\alpha(x) D_x^\alpha e^{i\langle x,\xi\rangle} \mathcal{F}(u)(\xi) d\xi = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \sum_{|\alpha|\le k} P_\alpha(x) \xi^\alpha e^{i\langle x,\xi\rangle} \mathcal{F}(u)(\xi) d\xi \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi) \mathcal{F}(u)(\xi) d\xi.
\end{aligned}$$

□

8.1. The Symbol Calculus

8.1.1 Definition. For any symbol $\sigma \in S^k(U \times \mathbb{R}^n, \mathbb{C}^{r' \times r})$, $\sigma_\alpha \in S^{|\alpha|}(U \times \mathbb{R}^n, \mathbb{C}^{r' \times r})$, $\alpha \in \mathbb{N}^n$, we say

$$\sigma \sim \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha :\Longleftrightarrow \sigma \sim \sum_{j=0}^{\infty} \sum_{|\alpha|=j} \sigma_\alpha.$$

8.1.2 Theorem (Workhorse Theorem)**.** Let $k \in \mathbb{R}$

$$a = a(x,y,\xi) \in S^k((\mathbb{R}^n \times \mathbb{R}^n) \times \mathbb{R}^n, \mathbb{C}^{r' \times r})$$

be a symbol with compact support in x and y. By definition

$$\forall \alpha, \beta, \gamma \in \mathbb{N}^n : \exists C_{\alpha,\beta,\gamma} > 0 : |D_x^\alpha D_y^\beta D_\xi^\gamma (a)(x,y,\xi)| \le C_{\alpha,\beta,\gamma}(1+|\xi|)^{k-|\gamma|}.$$

Then the operator $K : \mathscr{S}(\mathbb{C}^r) \to \mathscr{S}(\mathbb{C}^{r'})$, defined by

$$K(u)(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a(x,y,\xi) u(y) dy d\xi$$

is a ΨDO, i.e. $K \in \Psi^k(\mathbb{R}^n, \mathbb{C}^{r' \times r})$ and its symbol σ_K has an asymptotic expansion

$$\sigma_K(x,\xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{i^{|\alpha|}}{\alpha!} (D_\xi^\alpha D_y^\alpha a)(x,x,\xi)$$

8.1.3 Remark. In case a does not depend on y, the operator K is just given as

$$\begin{aligned}Ku(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} a(x,\xi) \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} u(y) dy d\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} a(x,\xi) \mathcal{F}(u)(\xi) d\xi\end{aligned}$$

8.1.4 Corollary. In the situation of 8.1.2 assume in addition that there exists an open neighbourhood $U_\Delta \overset{\circ}{\subseteq} \mathbb{R}^n \times \mathbb{R}^n$ of the diagonal $\Delta := \{(x,x) \in \mathbb{R}^n \times \mathbb{R}^n\}$, then $K \in \Psi^{-\infty}$.

Proof. By 8.1.2, $\sigma_K \sim 0$. And L/M 3.4 and Def of equivalence □

8.1.5 Definition (formally adjoint)**.** Let $P \in \Psi^k(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{C}^{r' \times r})$. An operator $P^* : \mathscr{S}(\mathbb{C}^{r'}) \to \mathscr{S}(\mathbb{C}^r)$ is *formally adjoint to* P, if

$$\forall s \in \mathscr{C}_c^\infty(\mathbb{R}^n, \mathbb{C}^r) : \forall t \in \mathscr{C}_c^\infty(\mathbb{R}^n, \mathbb{C}^{r'}) : \langle P(s), t\rangle_{L^2} = \langle s, P^*(t)\rangle_{L^2}$$

8.1.6 Theorem (adjoints)**.** For any $P \in \Psi_K^k - (\mathbb{R}^n \times \mathbb{R}^n, \mathbb{C}^{r' \times r})$ there exists a unique formal adjoint

$$P^* \in \Psi^k(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{C}^{r \times r'})$$

and its symbol has a asymptotic expansion

$$\sigma_{P^*} \sim \sum_{\alpha \in \mathbb{N}^n} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha \sigma_P^*$$

Proof. Let $s \in \mathscr{D}_K(\mathbb{R}^n, \mathbb{C}^r)$, $t \in \mathscr{D}_K(\mathbb{R}^n, \mathbb{C}^{r'})$, let $\psi \in \mathscr{C}_c^\infty(\mathbb{R}^n, \mathbb{R})$ such that

$$\psi|_K \equiv 1.$$

We calculate

$$\begin{aligned}\langle P(s), t\rangle_{L^2} &= \int_{\mathbb{R}^n} \langle P(s)(x), t(x)\rangle_{\mathbb{C}^{r'}} dx \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \langle \sigma_P(x,\xi)\mathcal{F}(s)(\xi), t(x)\rangle_{\mathbb{C}^{r'}} d\xi dx \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \langle \sigma_P(x,\xi)s(y), t(x)\rangle_{\mathbb{C}^{r'}} dy d\xi dx \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \langle s(y), \sigma_P(x,\xi)^* t(x)\rangle_{\mathbb{C}^{r'}} dy d\xi dx \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \langle \psi(y)s(y), \sigma_P(x,\xi)^* t(x)\rangle_{\mathbb{C}^{r'}} dx d\xi dy \\ &= \int_{\mathbb{R}^n} (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle s(y), e^{-i\langle x-y,\xi\rangle} \psi(y)\sigma_P(x,\xi)^* t(x)\rangle_{\mathbb{C}^{r'}} dx d\xi dy \\ &= \langle s, P^*(t)\rangle_{L^2},\end{aligned}$$

where

$$P^*(t)(y) := (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle y-x,\xi\rangle} \psi(y)\sigma_P(x,\xi)^* t(x) dx d\xi \tag{8.3}$$

Now the symbol $\psi(y)\sigma_P(x,\xi)^*$ satisfies the hypothesis of 8.1.2. This implies that P^* is a ΨDO and that its symbol has an asymptotic expansion

$$\begin{aligned}\sigma_{P^*}(x,\xi) &\sim \sum_{\alpha\in\mathbb{N}^n} \frac{i^\alpha}{\alpha!} D_\xi^\alpha D_y^\alpha(\psi(x)\sigma_P(y,\xi)^*)|_{x=y} \\ &= \sum_{\alpha\in\mathbb{N}^n} \frac{i^\alpha}{\alpha!} \psi(x) D_\xi^\alpha D_y^\alpha(\sigma_P(y,\xi)^*)|_{x=y} \\ &= \sum_{\alpha\in\mathbb{N}^n} \frac{i^\alpha}{\alpha!} \psi(x) D_\xi^\alpha D_x^\alpha(\sigma_P(x,\xi)^*) \\ &= \sum_{\alpha\in\mathbb{N}^n} \frac{i^\alpha}{\alpha!} D_\xi^\alpha D_x^\alpha(\psi(x)\sigma_P(x,\xi)^*) \\ &= \sum_{\alpha\in\mathbb{N}^n} \frac{i^\alpha}{\alpha!} D_\xi^\alpha D_x^\alpha(\sigma_P^*)(x,\xi).\end{aligned}$$

□

8.1.7 Remark. Since adjoints are unique, $P^{**} = P$.

8.1.8 Theorem (composition)**.** Let $P \in \Psi^{k_1}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{C}^{r' \times r})$, $Q \in \Psi^{k_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{C}^{r'' \times r'})$. Then $Q \circ P \in \Psi^{k_1+k_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{C}^{r'' \times r})$ and the symbol has a formal development

$$\sigma_{Q\circ P}(x,\xi) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{i^{|\alpha|}}{\alpha!} (D_\xi^\alpha \sigma_Q)(D_x^\alpha \sigma_P)(x,\xi).$$

Proof. By definition for any $s \in \mathscr{C}_c^\infty(\mathbb{R}^n, \mathbb{C}^r)$, $x \in K$

$$(Q\circ P)(s)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \sigma_Q(x,\xi) \mathcal{F}(Ps)(\xi) d\xi, \tag{8.4}$$

so we need a reasonable expression for $\mathcal{F}(Ps)$. Since $P^{**} = P$, we obtain

$$(Ps)(x) = P^{**}(s)(x) \overset{(8.3)}{=} (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \sigma_{P^*}(y,\xi)^* s(y) dy d\xi. \tag{8.5}$$

Define $\tau(y,\xi) := \sigma_{P^*}(y,\xi)^*$. We claim that

$$\mathcal{F}(Ps)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle} \tau(y,\xi) s(y) dy \tag{8.6}$$

To that end, we calculate $\mathcal{F}^{-1}$ of the right hand side:

$$\begin{aligned}&\mathcal{F}^{-1}\left(\xi \mapsto \int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle}\tau(y,\xi)s(y)dy\right)(x) \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle}\tau(y,\xi)s(y)dyd\xi \\ &\overset{(8.5)}{=} Ps(x).\end{aligned}$$

Therefore, we calculate

$$\begin{aligned}(Q\circ P)(s)(x) &\overset{(8.4)}{=} (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \sigma_Q(x,\xi)\mathcal{F}(Ps)(\xi)d\xi \\ &\overset{(8.6)}{=} (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} \underbrace{\sigma_Q(x,\xi)\sigma_{P^*}^*(y,\xi)}_{=:a(x,y,\xi)} s(y) dy d\xi\end{aligned}$$

By 4.5.2(iv)

$$a(x,y,\xi) = \sigma_Q(x,\xi)\sigma^*_{P^*}(y,\xi) \in S^{k_1+k_2}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{C}^{r''\times r}),$$

thus by Theorem 8.1.2

$$Q \circ P \in \Psi^{k_1+k_2}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{C}^{r''\times r})$$

and its symbol has an asymptotic expansion

$$\begin{aligned}
\sigma_{Q\circ P}(x,\xi) &\sim \sum_{\alpha\in\mathbb{N}^n} \frac{i^\alpha}{\alpha!} D^\alpha_\xi D^\alpha_y(\sigma_Q(x,\xi)\tau(y,\xi))|_{x=y} \\
&= \sum_{\alpha\in\mathbb{N}^n} \frac{i^\alpha}{\alpha!} D^\alpha_\xi(\sigma_Q(x,\xi) D^\alpha_y(\tau(y,\xi)))|_{x=y} \\
&= \sum_{\alpha\in\mathbb{N}^n} \frac{i^\alpha}{\alpha!} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta_\xi(\sigma_Q)(x,\xi) D^{\alpha-\beta}_\xi D^{\alpha-\beta+\beta}_x(\tau)(x,\xi) \\
&= \sum_{\alpha\in\mathbb{N}^n} \sum_{\beta\leq\alpha} \frac{i^{|\beta-\alpha|+|\beta|}}{\alpha!} \frac{\alpha!}{\beta!(\alpha-\beta)!} D^\beta_\xi(\sigma_Q)(x,\xi) D^{\alpha-\beta}_\xi D^{\alpha-\beta+\beta}_x(\tau)(x,\xi) \\
&= \sum_{\alpha\in\mathbb{N}^n} \sum_{\substack{\beta+\gamma=\alpha,\\ \beta,\gamma\in\mathbb{N}^n}} \frac{i^{|\gamma|+|\beta|}}{\beta!\gamma!} D^\beta_\xi(\sigma_Q)(x,\xi) D^\gamma_\xi D^\gamma_x D^\beta_x(\tau)(x,\xi) \\
&= \sum_{\beta,\gamma\in\mathbb{N}^n} \frac{i^{|\gamma|+|\beta|}}{\beta!\gamma!} D^\beta_\xi(\sigma_Q)(x,\xi) D^\gamma_\xi D^\gamma_x D^\beta_x(\tau)(x,\xi) \\
&= \sum_{\beta\in\mathbb{N}^n} \frac{i^{|\beta|}}{\beta!} D^\beta_\xi(\sigma_Q)(x,\xi) D^\beta_x \left(\sum_{\gamma\in\mathbb{N}^n} \frac{i^{|\gamma|}}{\gamma!} D^\gamma_\xi D^\gamma_x(\tau)(x,\xi) \right) \\
&\overset{8.1.6}{\sim} \sum_{\beta\in\mathbb{N}^n} \frac{i^{|\beta|}}{\beta!} D^\beta_\xi(\sigma_Q)(x,\xi) D^\beta_x(\sigma_P)(x,\xi).
\end{aligned}$$

asymptotische doppelentwicklungen?

□

8.1.9 Theorem (diffeomorphism invariance)**.** Let $U, V \mathring{\subseteq} \mathbb{R}^n$, $F : U \to V$ be a diffeomorphism. Then for any $K \Subset U$ the map

$$F_* : \Psi^k_K(U \times \mathbb{R}^n, \mathbb{C}^{r'\times r}) \to \Psi^k_{F(K)}(V \times \mathbb{R}^n, \mathbb{C}^{r'\times r}),$$

defined by

$$\forall s \in \mathscr{D}_K(U, \mathbb{C}^r) : F_*(P)(s) := P(s \circ F) \circ F^{-1}.$$

is well-defined.

Proof. The crucial and only part is to show that $F_*(P)$ is again a ΨDO. Let $G := F^{-1}$ and for any $x \in V$, write $\tilde{x} := G(x) \in U$. Before we start, we make the auxilliary calculation

$$\begin{aligned}
\forall x, y \in U : \tilde{x} - \tilde{y} = G(x) - G(y) &= \int_0^1 \partial_t(G(tx + (1-t)y)dt \\
&= \underbrace{\int_0^1 \nabla G(tx + (1-t)y)dt}_{=:H(x,y)}(x-y) = H(x,y)(x-y)
\end{aligned} \tag{8.7}$$

Notice that

$$H(x,x) = \nabla G(x),$$

thus there exists an open neighbourhood $\mathcal{O}$ of the diagonal in $V \times V$ such that

$$H : \mathcal{O} \to GL(\mathbb{R}^n).$$

Choose $\chi \in \mathscr{D}(\mathcal{O})$ such that $\chi \equiv 1$ in a (smaller) neighbourhood of the diagonal. Let

$$\begin{array}{rcl} J : V & \to & \mathbb{R} \\ x & \mapsto & \det(H(x,x)) \neq 0 \end{array}$$

be the Jacobian determinant of G. We calculate for any $s \in \mathscr{C}^\infty(V, \mathbb{C}^r)$

$$\begin{aligned} F_*(P)(s)(x) = P(s \circ F)(\tilde{x}) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle \tilde{x}-\tilde{y}, \xi\rangle} \sigma_P(\tilde{x}, \xi) s(F(\tilde{y})) d\tilde{y} d\xi \\ &\overset{(8.7)}{=} (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle H(x,F(\tilde{y}))(x-F(\tilde{y})), \xi\rangle} \sigma_P(\tilde{x}, \xi) s(F(\tilde{y})) J(F(\tilde{y}))^{-1} J(F(\tilde{y})) d\tilde{y} d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle H(x,y)(x-y), \xi\rangle} \sigma_P(G(x), \xi) s(y) J(y) dy d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y, H(x,y)^t \xi\rangle} \sigma_P(G(x), \xi) s(y) J(y) dy d\xi \end{aligned}$$

Now we multiply the integrand with $\chi + (1-\chi) \equiv 1$. We obtain two integrals, which we analyze seperately: For the integral with $(1-\chi)$ we obtain (up to $(2\pi)^{-n}$)

$$\begin{aligned} &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y, H(x,y)^t \xi\rangle} (1-\chi(x,y)) \sigma_P(G(x), \xi) s(y) J(y) dy d\xi \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y, \tilde{\xi}\rangle} \underbrace{(1-\chi(x,y)) \sigma_P(G(x), (H(x,y)^t)^{-1} \tilde{\xi}) \det(H(x,y))^{-1} J(y)}_{=:a''(x,y,\tilde{\xi})} s(y) dy d\tilde{\xi} \end{aligned}$$

Now a'' is a symbol, which satisfies the Workhorse Theorem 8.1.2 and vanishes in a neighbourhood of the diagonal. Hence by 8.1.4 this defines an infinitely smoothing operator. For the integral with χ we obtain analogously

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y, \tilde{\xi}\rangle} \underbrace{\chi(x,y) \sigma_P(G(x), (H(x,y)^t)^{-1} \tilde{\xi}) \det(H(x,y))^{-1} J(y)}_{=:a'(x,y,\tilde{\xi})} s(y) dy d\tilde{\xi}$$

Again a' satisfies the hypothesis of the workhorse Theorem 8.1.2, hence $F_*(P) = P' + P''$, where P'' is infinitely smoothing and P' is a ΨDO as claimed. □

A. Appendix

A.1. Leibniz Formulae

Before we can start with the analysis of the PDO Algebra itself, we first have to investigate the application of a PDO to a function especially to the product of two functions. We assume the reader to be very familiar with the product rule from basic calulus, i.e. if $f, g \in \mathcal{C}^1(U)$, then

$$\forall x \in U : \partial_i(fg)(x) = (\partial_i f)(x)g(x) + f(x)(\partial_i g(x))$$

This can be generalized considerably and will be done in this section. The following notation conventions will be useful.

A.1.1 Definition. For any $n, k \in \mathbb{N}$ we define

$$\binom{n}{k} := \begin{cases} \frac{n!}{k!(n-k)!}, & k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

In addition, if $\alpha, \beta \in \mathbb{N}^n$, we define

$$\binom{\alpha}{\beta} := \begin{cases} \frac{\alpha!}{\beta!(\alpha-\beta)!}, & \alpha \leq \beta, \\ 0, & \text{otherwise.} \end{cases}$$

A.1.2 Lemma. The binomials satisfy the following law of addition

$$\forall \alpha \leq \beta : \forall 1 \leq i \leq n : \binom{\alpha}{\beta - e_i} + \binom{\alpha}{\beta} = \binom{\alpha + e_i}{\beta}$$

A.1.3 Theorem (Leibniz Rule)**.** Let $U \subset \mathbb{R}^n$ be open, $f, g \in \mathcal{C}^k(U)$, $\alpha \in \mathbb{N}^n$, $|\alpha| = k$. Then

$$\partial^\alpha(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} (\partial^\beta f)(\partial^\gamma g)$$

As a didactial motivation we will prove the very important special case $n = 1$ seperately. Logically it is not needed in the proof of the general case and thus may be skipped.

Proof. [for $n = 1$] In that case, the statement is

$$(f \cdot g)^{(k)} = \sum_{\nu=0}^{k} \binom{k}{\nu} f^{(\nu)} \cdot g^{(k-\nu)}$$

We will prove this via induction over k. For $k = 1$ this is the ordinary product rule:

$$(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$$

For the induction step $k \to k+1$ consider

$$\begin{aligned}
(f \cdot g)^{(k+1)} &= \left((f \cdot g)^{(k)}\right)' = \left(\sum_{\nu=0}^{k} \binom{k}{\nu} f^{(\nu)} \cdot g^{(k-\nu)}\right)' \\
&= \sum_{\nu=0}^{k} \binom{k}{\nu} f^{(\nu+1)} \cdot g^{(k-\nu)} + f^{(\nu)} \cdot g^{(k+1-\nu)} \\
&\overset{(1)}{=} \sum_{\nu=1}^{k+1} \binom{k}{\nu-1} f^{(\nu)} \cdot g^{(k-(\nu-1))} + \sum_{\nu=0}^{k} \binom{k}{\nu} f^{(\nu)} \cdot g^{(k+1-\nu)} \\
&\overset{(2)}{=} \sum_{\nu=1}^{k} \left(\binom{k}{\nu-1} + \binom{k}{\nu}\right) f^{(\nu)} \cdot g^{(k-\nu+1)} + f^{(k+1)}g + fg^{(k+1)} \\
&\overset{(3)}{=} \sum_{\nu=1}^{k} \binom{k+1}{\nu} f^{(\nu)} \cdot g^{(k-\nu+1)} + \binom{k+1}{k+1} f^{(k+1)}g + \binom{k+1}{0} fg^{(k+1)} \\
&= \sum_{\nu=0}^{k+1} \binom{k+1}{\nu} f^{(\nu)} \cdot g^{(k+1-\nu)}
\end{aligned}$$

(1): Here we splitted up the sum and shifted the index of the first one up by one.
(2): Here we separated the summands $\nu = k+1$ in the first sum and $\nu = 0$ in the second sum from the rest and combined the two remaining sums. (3): This uses the addition theorem for binomials, c.f. LemBinomAdd, and the fact that

$$\binom{k+1}{k+1} = 1 = \binom{k+1}{0}.$$

□

Proof. [General Case] We will prove this statment as well by induction over $k = |\alpha|$. If $k = 1$, the statement is just the ordinary product rule. So by induction assume the statement is valid for k. Any multi-index $\tilde{\alpha}$ with $|\tilde{\alpha}| = k+1$ can be written in the form

$\tilde{\alpha} = \alpha + e_j$, where $|\alpha| = k$ and $1 \leq j \leq n$. Using induction hypothesis we calculate:

$$\begin{aligned}
\partial^{\tilde{\alpha}}(fg) =& \partial^{e_j}\partial^{\alpha}(fg) = \sum_{0\leq\beta\leq\alpha} \binom{\alpha}{\beta} \partial_j\Big(\partial^{\beta} f \, \partial^{\alpha-\beta} g\Big) \\
=& \sum_{0\leq\beta\leq\alpha} \binom{\alpha}{\beta} \partial^{\beta+e_j} f \, \partial^{\alpha-\beta} g + \sum_{0\leq\beta\leq\alpha} \binom{\alpha}{\beta} \partial^{\beta} f \, \partial^{\alpha+e_j-\beta} g \\
\overset{(1)}{=}& \sum_{e_j\leq\beta\leq\alpha+e_j} \binom{\alpha}{\beta-e_j} \partial^{\beta} f \, \partial^{\alpha-(\beta-e_j)} g + \sum_{0\leq\beta\leq\alpha} \binom{\alpha}{\beta} \partial^{\beta} f \, \partial^{\alpha+e_j-\beta} g \\
\overset{(2)}{=}& \binom{\alpha}{\alpha} \partial^{\alpha+e_j} f \, g + \sum_{\substack{0\leq\beta\leq\alpha \\ \beta_j=0}} \binom{\alpha}{\beta} \partial^{\beta} f \, \partial^{\alpha+e_j-\beta} g \\
&+ \sum_{e_j\leq\beta\leq\alpha} \binom{\alpha}{\beta-e_j} \partial^{\beta} f \, \partial^{\alpha+e_j-\beta)} g + \sum_{e_j\leq\beta\leq\alpha} \binom{\alpha}{\beta} \partial^{\beta} f \, \partial^{\alpha+e_j-\beta} g \\
\overset{(3)}{=}& \binom{\alpha+e_j}{\alpha+e_j} \partial^{\alpha+e_j} f \, g + \sum_{\substack{0\leq\beta\leq\alpha \\ \beta_j=0}} \binom{\alpha+e_j}{\beta} \partial^{\beta} f \, \partial^{\alpha+e_j-\beta} g \\
&+ \sum_{e_j\leq\beta\leq\alpha} \binom{\alpha+e_j}{\beta} \partial^{\beta} f \, \partial^{\alpha+e_j-\beta)} g \\
=& \sum_{0\leq\beta\leq\alpha+e_j} \binom{\alpha+e_j}{\beta} \partial^{\beta} f \, \partial^{\alpha+e_j-\beta} g \\
=& \sum_{0\leq\beta\leq\tilde{\alpha}} \binom{\tilde{\alpha}}{\beta} \partial^{\beta} f \, \partial^{\tilde{\alpha}-\beta} g.
\end{aligned}$$

(1): Here we "shifted" the first sum by e_j.
(2): Here we separated the summand $\beta = \alpha + e_j$ from the first sum and all the summands with $\beta_i = 0$ from the second sum.
(3): Here we used the addition law LemBinomAdd on the last two sums and the fact that $\beta_j = 0$ implies

$$\binom{\alpha_j}{\beta_j} = 1 = \binom{\alpha_j+1}{\beta_j},$$

thus

$$\binom{\alpha}{\beta} = \prod_{\substack{\nu=1, \\ \nu\neq j}}^{n} \binom{\alpha_\nu}{\beta_\nu} \cdot \binom{\alpha_j}{\beta_j} = \prod_{\substack{\nu=1, \\ \nu\neq j}}^{n} \binom{\alpha_\nu}{\beta_\nu} \cdot \binom{\alpha_j+1}{\beta_j} = \binom{\alpha+e_j}{\beta}.$$

begin crap

$$\begin{aligned}
\partial^{\tilde{\alpha}}(fg) = \partial_j \partial^\alpha (fg) = \partial_j \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g \\
&= \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} (\partial_j \partial^\beta f)(\partial^{\alpha-\beta} g) + (\partial^\beta f)(\partial_j \partial^{\alpha-\beta} g) \\
&\overset{(1)}{=} \sum_{i \neq j: 0 \leq \beta_i \leq \alpha_i, 0 \leq \beta_j \leq \alpha_j} \prod_{i=1, i\neq j}^{n} \binom{\alpha_i}{\beta_i} \binom{\alpha_j}{\beta_j} (\partial_j \partial^\beta f)(\partial^{\alpha-\beta} g) + \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g) \\
&\overset{(2)}{=} \sum_{i \neq j: 0 \leq \beta_i \leq \alpha_i, 1 \leq \beta_j \leq \alpha_j} \prod_{i=1, i\neq j}^{n} \binom{\alpha_i}{\beta_i} \binom{\alpha_j - 1}{\beta_j - 1} (\partial^\beta f)(\partial^{\alpha-\beta} g) + \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g) \\
&\overset{(3)}{=} \sum_{0 \leq \beta \leq \alpha} \prod_{i=1, i\neq j}^{n} \binom{\alpha_i}{\beta_i} \binom{\alpha_j - 1}{\beta_j - 1} (\partial^\beta f)(\partial^{\alpha-\beta} g) + \sum_{0 \leq \beta \leq \alpha} \prod_{i=1, i\neq j}^{n} \binom{\alpha_i}{\beta_i} \binom{\alpha_j - 1}{\beta_j} (\partial^\beta f)(\partial^{\alpha-\beta} g) \\
&= \sum_{0 \leq \beta \leq \alpha} \left(\binom{\alpha_j - 1}{\beta_j - 1} + \binom{\alpha_j - 1}{\beta_j} \right) \prod_{i=1, i\neq j}^{n} \binom{\alpha_i}{\beta_i} (\partial^\beta f)(\partial^{\alpha-\beta g)} \\
&\overset{(4)}{=} \sum_{0 \leq \beta \leq \alpha} \binom{\alpha_j}{\beta_j} \prod_{i=1, i\neq j}^{n} \binom{\alpha_i}{\beta_i} (\partial^\beta f)(\partial^{\alpha-\beta} g) \\
&= \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g)
\end{aligned}$$

Where we have used the following facts:

(1) In the first sum, we just wrote down the index set and the expression more complicated. In the second sum we only added summands with multi-indices β, such that

$$\binom{\tilde{\alpha}}{\beta} = \prod_{1 \leq i \neq j \leq n} \binom{\alpha_i}{\beta_i} \cdot \binom{\alpha_j}{\alpha_j + 1}$$

(2) In the first sum this is an index shift

$$(0 \leq \beta_j \leq \tilde{\alpha}_j = \alpha_j - 1) \mapsto (1 \leq \beta_j \leq \alpha_j)$$

and the plugging in of the definition of $\tilde{\alpha}$.

(3) In the first sum, we just added summands where $\beta_j = 0$ and thus

$$\binom{\alpha_j - 1}{\beta_j - 1} = 0$$

In the second sum we plugged in the definitions.

(4) This is the addition law for binomial coefficients.

□

end crap

A.1.4 Remark. This statement is also valid for D^α instead of ∂^α (by just multiplying the equation with $(-i)^\alpha$).

A.1.5 Corollary (Matrix valued Leibniz Formula)**.** Let $U \subset \mathbb{R}^n$ be open and $F \in \mathcal{C}^k(U, \mathbb{C}^{s\times r})$, $G \in \mathcal{C}^k(U, \mathbb{C}^{t\times s})$ and $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq k$. Then

$$\partial^\alpha(FG)(x) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} (\partial^\beta F)(x)(\partial^{\alpha-\beta}G)(x).$$

Proof. Notice that $FG \in \mathscr{C}^k(U, \mathbb{C}^{r\times t})$. Thus we calculate for any $1 \leq i \leq r$, $1 \leq j \leq t$

$$\begin{aligned}\partial^\alpha(FG)^i_j &= \sum_{\nu=1}^{s} \partial^\alpha(F^i_\nu G^\nu_j) \overset{\text{A.1.3}}{=} \sum_{\nu=1}^{s}\sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta(F^i_\nu)\partial^{\alpha-\beta}(G^\nu_j) \\ &= \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \sum_{\nu=1}^{s} \partial^\beta(F^i_\nu)\partial^{\alpha-\beta}(G^\nu_j) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} (\partial^\beta(F)\partial^{\alpha-\beta}(G))^i_j.\end{aligned}$$

□

One specific product is of particular importance.

A.1.6 Lemma. Let $\alpha, \delta \in \mathbb{N}^n$ be any two multi-indices. Then for all $\xi \in \mathbb{R}^n$

$$\partial^\delta_\xi \xi^\alpha = \begin{cases} \delta!\binom{\alpha}{\delta}\xi^{\alpha-\delta}, & \delta \leq \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

Proof.
Step 1$n = 1$: Let $k \in \mathbb{N}$ and $f_k : \mathbb{R} \to \mathbb{R}$, $t \mapsto t^k$. Then for any $l \leq k$

$$f_k^{(l)}(t) = k(k-1)\dots(k-l+1)t^{k-l} = \frac{k!}{(k-l)!}t^{k-l} = \binom{k}{l} l! t^{k-l}.$$

On the other hand, if $l > k$, we obtain $f_k^{(l)}(t) = 0$. Consequently

$$\forall k, l \in \mathbb{N} : \forall t \in \mathbb{R} : f_k^{(l)}(t) = (t^k)^{(l)} = \begin{cases} \binom{k}{l} l! x^{k-l}, & l \leq k, \\ 0, & \text{otherwise} \end{cases} \tag{A.1}$$

Step 2general case: We calculate

$$\partial^\delta(\xi^\alpha) = \partial_1^{\delta_1}\dots\partial_n^{\delta_n}(\xi_1^{\alpha_1}\cdot\dots\cdot\xi_n^{\alpha_n}) = \prod_{i=1}^{n} \partial_i^{\delta_i}(\xi_i^{\alpha_i})$$

Now in case $\delta \leq \alpha$ it follows from (A.1) that

$$\partial^\delta(\xi^\alpha) = \prod_{i=1}^{n} \binom{\alpha_i}{\delta_i} \delta_i! \xi_i^{\alpha_i-\delta_i} = \binom{\alpha}{\delta} \delta! \xi^{\alpha-\delta}.$$

If $\delta \leq \alpha$ does not hold, it follows also from (A.1) that $\partial^\delta(\xi^\alpha) = 0$ in this case.

□

Proof. [old] By induction over n. We always write $\partial := \partial_\xi$.

Step 1 (induction start $n = 1$): In this case $\alpha, \delta \in \mathbb{N}_0$. In case $\delta \leq \alpha$

$$\partial^\delta \xi^\alpha = \xi^{\alpha-\delta} \prod_{i=0}^{\delta+1} \alpha - i = \xi^{\alpha-\delta} \frac{\alpha!}{(\alpha-\delta)!} = \delta! \binom{\alpha}{\delta} \xi^{\alpha-\delta}.$$

This identity implies that in case $\delta > \alpha$.

$$\partial^\delta(\xi^\alpha) = \partial^{\delta-\alpha}(\partial^\alpha(\xi^\alpha)) = 0.$$

Step 2 (induction step $n \to n+1$): Suppose the formula is valid for multi-indices of length n and let $\alpha, \delta \in \mathbb{N}^{n+1}$. Define

$$\tilde{\alpha} := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n, \qquad \tilde{\delta} := (\delta_1, \ldots, \delta_n) \in \mathbb{N}^n.$$

We calculate

$$\partial^\delta \xi^\alpha = \partial^{\delta_{n+1}}(\partial^{\tilde{\delta}}(\xi^{\tilde{\alpha}} \xi_{n+1}^{\alpha_{n+1}})) = \partial^{\delta_{n+1}}(\xi_{n+1}^{\alpha_{n+1}} \cdot \partial^{\tilde{\delta}}(\xi^{\tilde{\alpha}})) = \partial^{\delta_{n+1}}(\xi_{n+1}^{\alpha_{n+1}}) \cdot \partial^{\tilde{\delta}}(\xi^{\tilde{\alpha}}) \tag{A.2}$$

Now if $\delta \leq \alpha$ does not hold, one of these factors equals zero by induction hypothesis. In case $\delta \leq \alpha$ does hold, we may continue this equation using the induction hypothesis by

$$(A.2) = \frac{1}{\delta_{n+1}!} \binom{\alpha_{n+1}}{\delta_{n+1}} \xi_{n+1}^{\alpha_{n+1}-\delta_{n+1}} \frac{1}{\tilde{\delta}!} \binom{\tilde{\alpha}}{\tilde{\delta}} \xi^{\tilde{\alpha}-\tilde{\delta}} = \frac{1}{\delta!} \binom{\alpha}{\delta} \xi^{\alpha-\delta}.$$

□

This allows us to generalize Leibniz formula for PDOs.

A.1.7 Theorem (Leibniz Formula for PDO). Let $U \subset \mathbb{R}^n$ be open $F \in \mathcal{C}^k(U, \mathbb{C}^{r \times s})$, $g \in \mathcal{C}^k(U, \mathbb{C}^r)$ open and

$$P(x, D) = \sum_{|\alpha| \leq k} p_\alpha(x) D^\alpha \in \operatorname{Diff}^k_{\mathbb{C}}(U, s, t)$$

be a PDO with symbol $p(x, \xi)$. Then

$$P(D)(Fg) = \sum_{|\mu| \leq k} \frac{1}{\mu!} P^{(\mu)}(D)(F) D^\mu g$$

where $P^{(\mu)}(D) \in \operatorname{Diff}^{k-|\mu|}_{\mathbb{C}}(U, s, t)$ is the PDO with symbol

$$p^{(\mu)}(x, \xi) = \partial_\xi^\mu p(x, \xi)$$

Proof. By the Leibniz formula **??** above

$$P(x, D)(Fg) = \sum_{|\alpha| \leq k} p_\alpha(x) D^\alpha(Fg) = \sum_{|\alpha| \leq k} p_\alpha(x) \sum_{\mu \leq \alpha} \binom{\alpha}{\mu} D^{\alpha-\mu}(F) D^\mu(g)$$

Furthermore

$$p^{(\mu)}(x, \xi) = \sum_{|\alpha| \leq k} p_\alpha(x) \partial_\xi^\mu \xi^\alpha \overset{A.1.6}{=} \sum_{|\alpha| \leq k} p_\alpha(x) \mu! \binom{\alpha}{\mu} \xi^{\alpha-\mu}$$

Consequently

$$P(x,D)(Fg) = \sum_{|\alpha|\le k}\sum_{\mu\le\alpha} p_\alpha(x)\binom{\alpha}{\mu}D^{\alpha-\mu}(F)D^\mu g \overset{\text{A.1.6}}{\sum}_{|\mu|\le k}\frac{1}{\mu!}\sum_{|\alpha|\le k} p_\alpha(x)\mu!\binom{\alpha}{\mu}D^{\alpha-\mu}(F)D^\mu g$$
$$= \sum_{|\mu|\le k}\frac{1}{\mu!}P^{(\mu)}(D)(F)D^\mu g$$

□

We would like to generalize generalize these results to products with finitely many factors.

A.1.8 Theorem (Binomial Theorem)**.** For any $x, y \in \mathbb{C}$, $n \in \mathbb{N}$

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$

In this context $0^0 := 1$.

A.1.9 Theorem (Multinomial Theorem)**.** Let $z_1,\dots,z_k \in \mathbb{C}^n$, $\alpha \in \mathbb{N}^n$. Then

$$\left(\sum_{i=1}^{k} z_i\right)^\alpha = \sum_{B\in(\mathbb{N}^n)^k, |B|=\alpha}\binom{\alpha}{B}Z^B$$

where $B = (B_1,\dots,B_k)$ is a tuple of multi-indices, $Z^B := z_1^{B_1}\dots z_k^{B_k}$ and

$$|B| := \sum_{i=1}^{k} B_i \qquad \binom{\alpha}{B} := \frac{\alpha!}{B_1!\dots B_k!}$$

Proof. We use induction over k. For $k=1$, the statement is clear since both side equal z_1^α. For $k=2$

$$\left(\sum_{i=1}^{k} z_i\right)^\alpha = \prod_{i=1}^{n}(z_1^i + z_2^i)^{\alpha_i} \overset{\text{A.1.8}}{\prod}{}^{n}_{i=1}\sum_{\gamma_i=1}^{\alpha_i}\binom{\alpha_i}{\gamma_i}(z_1^i)^{\gamma_i}(z_2^i)^{\alpha_i-\gamma_i}$$
$$= \sum_{\gamma_1=1}^{\alpha_1}\dots\sum_{\gamma_n=1}^{\alpha_n}\prod_{i=1}^{n}\binom{\alpha_i}{\gamma_i}z_1^{\gamma_i}z_2^{\alpha_i-\gamma_i} = \sum_{\gamma\le\alpha}\frac{\alpha!}{\gamma!(\alpha-\gamma)!}z_1^\gamma z_2^{\alpha-\gamma} = \sum_{B\in(\mathbb{N}^n)^2, |B|=\alpha}\binom{\alpha}{B}Z^B$$

For the induction step assume $k \ge 3$ and that the statement holds for k. Define $y := \sum_{i=1}^{k} z_k$, $Y = (z_1,\dots,z_k) \in (\mathbb{C}^n)^k$ Using the induction start for $k=1,2$ and the induction hypothesis, we obtain:

$$\left(\sum_{i=1}^{k+1} z_i\right)^\alpha = \left(\sum_{i=1}^{k} z_i + z_{k+1}\right)^\alpha = (y+z_{k+1})^\alpha = \sum_{\gamma+\delta=\alpha}\binom{\alpha}{(\gamma,\delta)}(y,z_{k+1})^{(\gamma,\delta)}$$
$$= \sum_{\gamma+\delta=\alpha}\frac{\alpha!}{\gamma!\delta!}z_{k+1}^\delta\left(\sum_{i=1}^{k} z_i\right)^\gamma = \sum_{\gamma+\delta=\alpha}\frac{\alpha!}{\gamma!\delta!}z_{k+1}^\delta\left(\sum_{C\in(\mathbb{N}^n)^k, |C|=\gamma}\binom{\gamma}{C}Y^C\right)$$
$$= \sum_{\gamma+\delta=\alpha}\sum_{C\in(\mathbb{N}^n)^k, |C|=\gamma}\frac{\alpha!}{\gamma!\delta!}\frac{\gamma!}{C_1!\dots C_k!}z_1^{C_1}\dots z_k^{C_k}z_{k+1}^\delta = \sum_{B\in(\mathbb{N}^n)^{k+1}, |B|=\alpha}\binom{\alpha}{B}Z^B$$

□

A.1.10 Theorem (Leibniz rule for multiple factors)**.** Let $f_1, \ldots, f_k \in \mathcal{C}^\infty(U, \mathbb{C})$, $U \subset \mathbb{R}^n$, $\alpha \in \mathbb{N}^n$. Then

$$\partial^\alpha \left(\prod_{i=1}^k f_i \right) = \sum_{B \in (\mathbb{N}^n)^k, |B| = \alpha} \binom{\alpha}{B} \partial^B F$$

where $F = (f_1, \ldots, f_k) : U \to \mathbb{C}^k$, $B = (B_1, \ldots, B_k)$ is a tuple of multi-indices, $\partial^B F := (\partial^{B_1} f_1) \ldots (\partial^{B_k} f_k)$ and

$$|B| := \sum_{i=1}^k B_i \qquad\qquad \binom{\alpha}{B} := \frac{\alpha!}{B_1! \ldots B_k!}$$

Proof. We use induction over k. For $k = 1$, the statement is clear since both side equal $\partial^\alpha f_1$. For $k = 2$ this has already been proven as A.1.3. For the induction step $k \to (k+1)$ consider:

$$\partial^\alpha \left(\prod_{i=1}^{k+1} f_i \right) = \partial^\alpha \left(\prod_{i=1}^{k} f_i f_{k+1} \right) = \sum_{\gamma+\delta=\alpha} \binom{\alpha}{(\gamma,\delta)} \partial^\gamma \left(\prod_{i=1}^{k} f_i f_{k+1} \right) f_{k+1}^\delta$$

$$= \sum_{\gamma+\delta=\alpha} \frac{\alpha!}{\gamma!\delta!} z_{k+1}^\delta \left(\sum_{C \in (\mathbb{N}^n)^k, |C|=\gamma} \binom{\gamma}{C} \partial^C (f_1, \ldots, f_k) \right)$$

$$= \sum_{\gamma+\delta=\alpha} \sum_{C \in (\mathbb{N}^n)^k, |C|=\gamma} \frac{\alpha!}{\gamma!\delta!} \frac{\gamma!}{C_1! \ldots C_k!} (\partial^{C_1} f_1) \ldots (\partial^{C_k} f_k)(\partial_{k+1}^\delta f) = \sum_{B \in (\mathbb{N}^n)^{k+1}, |B|=\alpha} \binom{\alpha}{B} \partial^B F$$

□

A.2. Auxilliary Lemmata

A.2.1 Lemma. The Euclidean norm $| \ | = \| \ \|_2$ on $\mathbb{R}^n$ satisfies

$$\forall \alpha \in \mathbb{N}^n : \forall x \in \mathbb{R}^n : |x^\alpha| \leq |x|^\alpha = |x|^{|\alpha|}.$$

Proof. The last equality holds by definition. The first inequality is proven by stupid induction over $|\alpha| = k$. If $k = 1$ there exists $1 \leq j \leq n$ such that $\alpha = e_j$. Therefore

$$|x|^{|\alpha|} = \sqrt{\sum_{k=1}^n x_k^2} \geq \sqrt{x_j^2} = |x^\alpha|.$$

For the induction step $k \to k+1$ notice decompose an α satisfying $|\alpha| = k+1$ into $\beta + e_j$ where $|\beta| = 1$ and calculate

$$|x^\alpha| = |x^\beta||x_j| \leq |x|^{|\beta|}|x_j| \leq |x|^{|\alpha|}.$$

□

A.2.2 Lemma. For any $k \in \mathbb{N}$ there exist constants $\{c_\alpha > 0 \mid |\alpha| \leq 2k\}$ such that

$$\forall x \in \mathbb{R}^n : |x|^k \leq \sum_{|\alpha| \leq 2k} c_\alpha |x^\alpha|.$$

Proof. We use induction over k.

STEP 1 ($k = 0$): Clearly

$$|x|^0 = 1 =: c_0|x^0|$$

does the job.

STEP 2 ($k \to k+1$): Let

$$p_k(x) = \sum_{|\alpha| \leq 2k} c_\alpha x^\alpha$$

Using the induction hypothesis, we calculate for any $|x| \geq 1$

$$|x| \sum_{|\alpha| \leq 2k} c_\alpha |x^\alpha| \leq |x|^2 \sum_{|\alpha| \leq 2k} c_\alpha |x^\alpha| = \sum_{j=1}^{n} x_j^2 \sum_{|\alpha| \leq 2k} c_\alpha |x^\alpha| = \sum_{j=1}^{n} \sum_{|\alpha| \leq 2k} c_\alpha |x^{\alpha+2e_j}|$$

Since for any $|x| \leq 1$, $|x|^{k+1} \leq 1 = x^0$ as well, the estimate

$$\forall x \in \mathbb{R}^n : |x|^{k+1} = |x||x|^k \leq \sum_{j=1}^{n} \sum_{|\alpha| \leq 2k} c_\alpha |x^{\alpha+2e_j}| + 1$$

does the job.

□

A.2.3 Lemma.

$$\forall x, y \in \mathbb{R}_{\geq 0} : \forall k \in \mathbb{N} : (x+y)^k \leq 2^k (x^k + y^k)$$

Proof. By common sense

$$(x+y)^k \leq (2 \max(x,y))^k = 2^k \max(x,y)^k \leq 2^k (x^k + y^k).$$

□

A.2.4 Lemma. Let $s \in \mathbb{R}$ and define $f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto |x|^s$ (remember, that $|x| := \|x\|_2$) and let $1 \leq p < \infty$. Then

$$f \in L^p(B_1(0)) \Leftrightarrow -s < \frac{n}{p}, \qquad f \in L^p(\mathbb{R}^n \setminus B_1(0)) \Leftrightarrow -s > \frac{n}{p}.$$

A.2.5 Lemma (Peetre Inequality)**.** For any $\xi \in \mathbb{R}^n$ let $\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}$ and let $|\_|$ be the Euclidean norm.

$$\forall \xi, \eta \in \mathbb{R}^n : \forall s \in \mathbb{R} : \langle \xi \rangle^s \leq 2^{|s|} \langle \xi - \eta \rangle^{|s|} \langle \eta \rangle^s$$

Proof. [see Abels] In a first step, we calculate

$$\langle \xi \rangle^2 = (1 + |\xi|^2) \leq (1 + |\xi|)^2 \leq (1 + |\xi|)^2 + (1 - |\xi|)^2 = 2(1 + |\xi|^2),$$

thus we obtain

$$\langle \xi \rangle \leq (1 + |\xi|) \leq \sqrt{2} \langle \xi \rangle \leq 2 \langle \xi \rangle \tag{A.3}$$

Let's assume $s \geq 0$. By the triangle inequality

$$(1 + |\xi|) \leq (1 + |\xi - \eta| + |\eta|) \leq (1 + |\xi - \eta|)(1 + |\eta|) \tag{A.4}$$

thus

$$\langle\xi\rangle^s \overset{(A.3)}{\leq} (1+|\xi|)^s \overset{(A.4)}{\leq} (1+|\xi-\eta|)^s(1+|\eta|)^s \overset{(A.3)}{\leq} 2^s\langle\xi-\eta\rangle^s\langle\eta\rangle^s$$

This implies the claim for $s \geq 0$. In case $s < 0$, interchange the roles of ξ and η in the previous inequality and apply this to $-s$. We obtain

$$\begin{aligned} &\langle\eta\rangle^{-s} \leq 2^{-s}\langle\eta-\xi\rangle^{-s}\langle\xi\rangle^{-s} \\ \Longrightarrow &\langle\xi\rangle^{s} \leq 2^{-s}\langle\eta-\xi\rangle^{-s}\langle\eta\rangle^{s} = 2^{|s|}\langle\eta-\xi\rangle^{|s|}\langle\eta\rangle^{s}. \end{aligned}$$

□

A.2.6 Lemma. For any k-times differentiable function $h : U \subset \mathbb{R}^n \to \mathbb{R}^m$ define

$$\|h\|_{\mathcal{C}^k} := \sup_{x\in U} \max_{\alpha\in\mathbb{N}_0^n:\, |\alpha|\leq k} \max_{1\leq i\leq m} |\partial^\alpha h^i|(x).$$

Now let $F \in \mathcal{C}_b^k(U \subset \mathbb{R}^n, V \subset \mathbb{R}^m)$, $k \geq 1$, and $g \in \mathcal{C}_b^k(V, \mathbb{R})$. There exists $C > 0$, such that

$$\|g \circ F\|_{\mathcal{C}^k(U)} \leq C_k \|F\|_{\mathcal{C}^k(U)} \|g\|_{\mathcal{C}^k(V)}. \tag{A.5}$$

Proof. This can be proven by induction over k using the chain rule and the Leibniz rule: For $k = 1$, this follows from

$$|\partial_j(g \circ F)| \leq \sum_{i=1}^n |\partial_i F^j\, \partial_i g| \leq n\|F\|_{\mathcal{C}^1(U)}\|g\|_{\mathcal{C}^1(V)}$$

For the induction step, we just notice that for any $\alpha \in \mathbb{N}^n$, such that $|\alpha| = k+1$ there exist $\beta \in \mathbb{N}^n$ and $1 \leq i \leq j$, such that $|\beta| = k$ and $\alpha = \beta + e_j$. Therefore

$$|\partial^\alpha(g\circ F)| = |\partial^\beta\partial_j(g\circ F)| \leq \sum_{i=1}^n |\partial^\beta(\partial_i F^j \partial_i g)| \leq \sum_{i=1}^n \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} |\partial^\gamma\partial_i F^j\, \partial^{\beta-\gamma}\partial_i g)| \leq C\|F\|_{\mathcal{C}^{k+1}}\|g\|_{\mathcal{C}^{k+1}(V)}.$$

□

A.2.7 Lemma. The map $\alpha : \mathbb{R}^n \to \mathrm{B}^n$, $x \mapsto \frac{x}{\sqrt{1+|x|^2}}$, is a diffeomorphism with inverse $\alpha^{-1} : \mathrm{B}^n \to \mathbb{R}^n$.

A.3. Linear Algebra

A.3.1 Theorem (Adjoints)**.** Assume that

(i) (V, g), (W, h) are finite-dimensional hermitian vector spaces over $\mathbb{C}$,

(ii) $B = (b_1, \ldots, b_n)$ is a basis of V, $C = (c_1, \ldots, c_m)$ is a basis of W,

(iii) $G = (g_{ij}) \in GL(n, \mathbb{C})$ is the coordinate matrix of G with respect to B,

(iv) $H = (h_{ij}) \in GL(m, \mathbb{C})$ is the coordinate matrix of H with respect to C,

(v) $f : V \to W$ is a $\mathbb{C}$-linear map,

(vi) $M = c_C^B(f)$ is the coordinate matrix of f with respect to B and C,

(vii) $N = c_V^C(f^*)$ is the coordinate matrix of f^* with respect to C and B.

Then

$$N = G^{-1} M^* H,$$

where $M^* := \bar{M}^t$.

Proof. By definition, for any $1 \leq i \leq n$, $1 \leq j \leq m$, we claculate on the one hand

$$\begin{aligned} h(f(b_i), c_j) &= h\Big(\sum_{k=1}^m M_{ki} c_k\Big) = \sum_{k=1}^m M_{ki} h(c_k, c_j) = \sum_{k=1}^m M_{ki} H_{kj} \\ &= \sum_{k=1}^m M^t_{ik} H_{kj} = (M^t H)_{ij} \end{aligned}$$

and on the other hand

$$\begin{aligned} g(b_i, f^*(c_j)) &= g\Big(b_i, \sum_{l=1}^n N_{lj} b_l\Big) = \sum_{l=1}^n \bar{N}_{lj} g(b_i, b_l) = \sum_{l=1}^n \bar{N}_{lj} G_{il} \\ &= \sum_{l=1}^n G_{il} \bar{N}_{lj} = (G\bar{N})_{ij}. \end{aligned}$$

By definition $h(f(b_i), c_j) = g(b_i, f^*(c_j))$ and therefore, we obtain

$$M^t H = G\bar{N} \implies G^{-1} M^t H = \bar{N} \implies N = \bar{G}^{-1} M^* \bar{H}.$$

□

A.4. Connections and Vector bundles

Let $E \to M$ be a smooth vector bundle over $\mathbb{K}$. In this section we discuss the concecpt of a connection on a vector bundle. There are slightly different definitions of a connection, which are all very common in the literature and are all equivalent. In order to see this and to give some better intuition and understanding, we remind some easy facts from linear algebra.

A.4.1 Definition. Let V, W be $\mathbb{K}$ vector spaces. For any $v \in V$ the map $B_v : V' \times W \to W$, $(v', w) \mapsto v'(v)w$, is bilinear. By the universal property of the tensor product, there exists a unique linear map $\beta_v : V' \otimes W \to W$, such that $\beta_v \circ \otimes = B_v$. The bilinear map $\langle \_, \_ \rangle_W : V \times (V' \otimes W) \to W$, defined by $(v, x) \mapsto \beta_v(x)$, is the W-pairing of V. It satisfies

$$\forall v \in V : \forall v' \in V' \forall w \in W : \langle v, v' \otimes w \rangle_W = v'(v) w.$$

A.4.2 Definition (connection)**.** An $\mathbb{K}$-linear map $\Gamma(E) \to \Gamma(T^*M \otimes E)$, which satisfies

$$\forall f \in \mathscr{C}^\infty(M) : \forall \sigma \in \Gamma(E) : D(f\sigma) = df \otimes \sigma + f D\sigma$$

is a *connection* on E.

A.4.3 Definition (covariant derivative)**.** A $\mathbb{K}$-bilinear map $\nabla : \Gamma(E) \times \Gamma(M) \to \Gamma(E)$, $(X, \sigma) \mapsto \nabla_X \sigma$ satisfying

(i) $\forall f \in \mathscr{C}^\infty(M) : \forall \sigma \in \Gamma(E) : \nabla_{fX} \sigma = f \nabla_X \sigma$

(ii) $\forall f \in \mathscr{C}^\infty(M) : \forall \sigma \in \Gamma(E) : \nabla_X(f\sigma) = X(f)\sigma + f\nabla_X\sigma$

is a *covariant derivative on* M.

There is an intimate relation between connections and covariant derivatives. The words are almost used as synonoms in the literature.

A.4.4 Lemma. There is a bijection $\varphi : \{\text{connections on } E\} \to \{\text{covariant derivatives on } E\}$. For any connection D, define the covariant derivative $\nabla^D := \varphi(D)$ to be the following: For any $X \in \Gamma(M)$ define

$$\nabla^D_X(\sigma) := \langle X, D\sigma\rangle_E,$$

where $\langle X, D\sigma\rangle_E$ is the (pointwise defined) E-pairing between TM and T^*M.

Proof.

STEP 1 (∇^D_X is a covariant derivative): By construction ∇^D_X it is $\mathbb{K}$-binlinear and $\mathscr{C}^\infty(M)$-linear in X. By the connection property, we obtain

$$\nabla^D_X(f\sigma) = \langle X, D(f\sigma)\rangle_E = \langle X, df\otimes\sigma + fD\sigma\rangle_E = df(X)\sigma + \langle X, fD\sigma\rangle_E = X(f)\sigma + f\nabla^D_X(\sigma)$$

□

A.4.5 Lemma (Affine structure)**.** Let ∇^0 and ∇^1 be two linear connections on $E \to M$, where E is vector bundle over $\mathbb{K}$. The difference

$$\forall X \in \mathcal{T}(M) : \forall s \in -(E) : A(X,s) := \nabla^1_X s - \nabla^0_X s$$

is a tensor field $A \in \Gamma(T^*M \otimes E)$, i.e. a map $\mathcal{T}(M) \times \Gamma(E) \to \Gamma(E)$, that is multilinear over $\mathscr{C}^\infty(M)$. It is called the *difference tensor*. The set $\nabla(E)$ of all connections on E is given by

$$\nabla(E) = \{\nabla^0 + A \mid A \in \Gamma(T^*M \otimes E)\}$$

and is therefore an affine space.

Proof. The linearity over $\mathbb{K}$ in X and s is clear. For any $f \in \mathcal{C}^\infty(M)$, we calculate

$$\begin{aligned} A(fX,s) &= \nabla^1_{fX}s - \nabla^0_{fX}s = f\nabla^1_X s - f\nabla^0_X s = fA(X,s) \\ A(X,fs) &= \nabla^1_X(fs) - \nabla^0_X(fs) = X(f)s + f\nabla^1_X s - X(f)s - f\nabla^0_X s = fA(X,s), \end{aligned}$$

thus $A \in T^2_1(E)$ is a tensor field. Conversely if $A \in \mathcal{T}^2_1(E)$ is an arbitrary tensor field, the map

$$\nabla^1 := \nabla^0 + A$$

is a connection: Clearly it is linear over $\mathbb{R}$ in both arguments, linear over $\mathcal{C}^\infty(M)$ in X by definition and the Leibniz rule follows immediately from

$$\nabla^1(X,fs) = \nabla^0_X(fs) + A(X,fs) = X(f)s + f\nabla^0_X s + fA(X,s) = X(f)s + \nabla^1_X s.$$

□

A.4.6 Lemma. Let ∇ be a linear connection on M and let $\{E_i\}$ be a local frame on some open neighbourhood $U \subset M$, and let $\{E^i\}$ be its dual coframe.

(i) There is a uniquely determined matrix of 1-forms $\omega_i^j \in \mathcal{T}^*(U)$, called *connection 1-forms* for this frame such that

$$\forall X \in \mathcal{T}(U) : \nabla_X E_i = \omega_i^j(X)E_j$$

(ii) Let τ be the torsion tensor from Problem 4-2 and $\{\tau^i\}$ be the *torsion 2-forms* defined by

$$\forall X, Y \in \mathcal{T}(U) : \tau(X,Y) = \tau^j(X,Y)E_j$$

Then *Cartan's first structure euqation*

$$dE^j = E^i \wedge \omega_i^j + \tau^j$$

holds.

Proof.

(i) Uniqueness follows easiliy from

$$\nabla_X E_i = \nabla_{X^k E_k} E_i = X^k \nabla_{E_k} E_i = X^k \Gamma_{ki}^j E_j$$

Since E_j is a local frame, we have no choice but to define

$$\omega_i^j := \Gamma_{ki}^j E^k$$

Clearly $\omega_i^j \in \mathcal{T}^*(U)$, which shows existence.

(ii) Let's expand the various terms. For the differential, we obtain

$$dE^j(X,Y) \overset{[?,\ 12.17]}{X} (Y^j) - Y(X^j) - E^j([X,Y])$$

For the torsion, we obtain

$$\begin{aligned}
&\tau^j(X,Y) = E^j(\tau(X,Y)) = E^j(\nabla_X Y) - E^j(\nabla_Y X) - E^j([X,Y]) \\
&= E^j((X(Y^k) + X^i Y^l \Gamma_{il}^k)E_k) - E^j((Y(X^k) + Y^i X^l \Gamma_{il}^k)E_k) - E^j([X,Y]) \\
&= X(Y^j) + X^i Y^l \Gamma_{il}^j - Y(X^j) - Y^i X^l \Gamma_{il}^j - E^j([X,Y]) \\
&= X(Y^j) - Y(X^j) - E^j([X,Y]) - X^i Y^k \Gamma_{ik}^j + Y^i X^k \Gamma_{ik}^j
\end{aligned}$$

The wedge can be written as

$$\begin{aligned}
&(E^i \wedge \omega_i^j)(X,Y) = \frac{(1+1)!}{1! \cdot 1! \cdot 2}(E^i \otimes \omega_i^j - \omega_i^j \otimes E^i)(X,Y) = X^i \omega_i^j(Y) - \omega_i^j(X) Y^i \\
&= X^i Y^k \Gamma_{ki}^j - X^k Y^i \Gamma_{ki}^j = X^k Y^i \Gamma_{ik}^j - X^i Y^k \Gamma_{ik}^j
\end{aligned}$$

These three equations together imply

$$(dE^j - E^i \wedge \omega_i^j - \tau^j)(X,Y) = 0$$

□

List of Symbols

$\mathscr{S}$ the Schwartz space, page 61

s_λ scaling, page 68

sing-supp singular support, page 96

S_Φ , page 124

supp f support of f, page 51

τ_y translation, page 68

U $U \subset \mathbb{R}^n$ usually is an open set

ξ_φ^α power of the pullback of a cotangent vector, page 19

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