Differential Geometry II

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SS 2010

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Contents

1 Distance Functions and the Riccati Equation

Let M be a Riemannian n-manifold.

1.1 Definition (distance function)**.** Let $U \subset M$ be open. A smooth function $f : U \to \mathbb{R}$ satisfying

$$\|\operatorname{grad} f\| \equiv 1$$

is a *local distance function.*
The non-empty level sets $M_r := f^{-1}(r)$ are hypersurfaces, i.e. $(n-1)$-dimensional submanifolds of M.

1.2 Example. Let $p \in M$ and $\varepsilon > 0$, such that $\exp_p : B_\varepsilon(0_p) \to B_\varepsilon(p)$ is a diffeomorphism. Let $U := B_\varepsilon(p)$ and $f : U \setminus \{p\} \to \mathbb{R}$, $f(q) := d(p,q)$. Then f is a local distance function: By construction U is open and f is smooth. In order to show, that the condition on the gradient holds, choose normal coordinates near p and consider the radial distance function r near p (c.f. [2, (5.9)]) and the radial unit vector field ∂_r. By definition $f(q) = d(p,q) = r(q)$, which together with the Gauss Lemma implies

$$\operatorname{grad} f|_q = \operatorname{grad} r|_q \overset{[2,\ 6.9]}{=} \partial_r|_q \overset{[2,\ 5.11e)]}{=} \dot{c}_q(f(q)),$$

where c_q is the unique minimizing unit speed geodesic from p to q.

From now on let $f : U \to \mathbb{R}$ be a local distance function.

1.3 Lemma. Let $c : [a,b] \to U \subset M$ be a piecewise smooth curve, define

$$p := c(a) \qquad q := c(b) \qquad f(p) =: r_0 \qquad f(q) =: r_1$$

and suppose $r_0 \leq r_1$. Then

$$L(c) \geq r_1 - r_0,$$

where equalitiy holds if and only if c is (up to reparametrization) the solution curve c_p of

$$\dot{c}_p = \operatorname{grad} f \circ c_p \qquad c_p(0) = p.$$

In particular c_p is a geodesic and shortest path in U from $M_{r_0} = f^{-1}(r_0)$ to $M_{r_1} = f^{-1}(r_1)$.

Proof. We calculate

$$\begin{aligned} L(c) &= \int_a^b \|\dot{c}(t)\| dt \overset{\|\operatorname{grad} f\| = 1}{=} \int_a^b \|\dot{c}(t)\| \|\operatorname{grad} f|_{c(t)}\| dt \geq \int_a^b |\langle \dot{c}(t), \operatorname{grad} f|_{c(t)} \rangle| dt \\ &\geq \left| \int_a^b \langle \dot{c}(t), \operatorname{grad} f|_{c(t)} \rangle dt \right| \geq \int_a^b df|_{c(t)}(\dot{c}(t)) dt \\ &= \int_a^b (f \circ c)'(t) dt = f(c(b)) - f(c(a)) = f(q) - f(p) = r_1 - r_0. \end{aligned} \tag{1.1}$$

Analogously for c_p

$$L(c_p) = \int_a^b \|\dot{c}_p(t)\| dt = \int_a^b \|\operatorname{grad} f(c(t))\| dt = \int_a^b (f \circ c)'(t) dt = r_1 - r_0.$$

In particular c_p is a shortest path in U by construction, thus a geodesic in U, thus a geodesic in M. Conversely, if c is not c_p (up to reparametrization), there exists $t_0 \in I$, such that $\dot{c}(t_0) \neq \operatorname{grad} f|_{c(t)}$. This implies strict inequality in the Cauchy-Schwarz-Inequality in (1.1). $\square$

We are interested in the second derivatives of f.

1.4 Definition (Hessian)**.** Let $f \in \mathcal{C}^\infty(M)$. For any two smooth vector fields X, Y define

$$\begin{aligned}\operatorname{Hess} f(X,Y) &:= \nabla^2 f(X,Y) = Y(X(f)) - (\nabla_Y X)(f) = Y(\langle \operatorname{grad} f, X\rangle) - (\nabla_Y X)(f)\\ &= \nabla_Y(\langle \operatorname{grad} f, X\rangle) - \langle \operatorname{grad} f, \nabla_Y X\rangle = \langle \nabla_Y \operatorname{grad} f, X\rangle\end{aligned}$$

(c.f. [2, Exc. 4.5]). This defines a symmetric tensor field [1] , i.e.

$$\operatorname{Hess} f(X,Y) = \langle \nabla_X \operatorname{grad} f, Y\rangle$$

as well.

1.5 Lemma. Let $f : U \to \mathbb{R}$ be a distance function, $X, Y \in \mathcal{T}(U)$, $r \in \mathbb{R}$ and $p \in M_r := f^{-1}(r) \cap U$.

(i) Then $\operatorname{Lin}(\operatorname{grad} f|_p) = (T_pM_r)^\perp = N_pM_r$ and for any $X, Y \in T_pM_r$:

$$\operatorname{Hess} f(X,Y) = \langle \nabla_X Y, -\operatorname{grad} f\rangle = \langle II(X,Y), -\operatorname{grad} f\rangle = h(X,Y),$$

i.e. $\operatorname{Hess} f$ is the scalar second fundamental form h of M_r with respect to the unit normal field $-\operatorname{grad} f$.

(ii) If $X \in T_pM$, $Y \in N_pM_r$, then $\operatorname{Hess} f(X,Y) = 0$.

(iii) Moreover the restriction of $U \in T^1_1(U)$

$$U(X) := \nabla_X \operatorname{grad} f$$

onto $T_pM_r \to T_pM_r$ is the shape operator (”*Weingartenabbildung*”) of M_r with respect to $-\operatorname{grad} f$ (c.f. [2, p.140]).

Proof. We consider M_r as a submanifold of M.
STEP 1 (Characterization of the tangential space): For any arbitrary curve $\gamma : I \to M_r$ through p we have $r = f \circ \gamma$, consequently

$$0 = df|_{\gamma(t)}(\dot\gamma(0)) = \langle \operatorname{grad} f|_p, \dot\gamma(0)\rangle \Rightarrow \operatorname{grad} f|_p \perp \dot\gamma(0),$$

thus $\operatorname{Lin}(\operatorname{grad} f|_p) \subset (T_pM_r)^\perp$. Equality holds for dimensional reasons.

STEP 2 (Analysis in normal directions): Since f is a distance function, we have $\|\operatorname{grad} f\| = 1$. This implies

$$0 = \nabla_X\langle \operatorname{grad} f, \operatorname{grad} f\rangle = 2\langle \nabla_X \operatorname{grad} f, \operatorname{grad} f\rangle,$$

thus $\nabla_X \operatorname{grad} f \perp \operatorname{grad} f$. By step 1 we obtain $\nabla_X \operatorname{grad} f \in \mathcal{T}(M_r)$.
For $Y := \operatorname{grad} f$, we get

$$Y(f) = df(Y) = \langle \operatorname{grad} f, \operatorname{grad} f\rangle = 1 \Rightarrow X(Y(f)) = 0$$

and

$$(\nabla_X Y)(f) = df(\nabla_X Y) = \langle \operatorname{grad} f, \nabla_X \operatorname{grad} f\rangle = 0.$$

Consequently

$$\operatorname{Hess} f(X,Y) = X(Y(f)) - \nabla_X Y(f) = 0.$$

[1] Since g is symmetric and ∇ is torsion free:

$$\operatorname{Hess} f(Y,X) - \operatorname{Hess} f(X,Y) = X(Yf) - \nabla_X Y(f) - Y(Xf) - \nabla_Y X(f) = [X,Y](f) - \nabla_X Y(f) - \nabla_Y X(f) = 0.$$

STEP 3 (Analysis in tangential directions): If $Y \in \mathcal{T}(M_r)$, we obtain

$$\begin{aligned}\operatorname{Hess} f(X,Y) &= \langle \nabla_X \operatorname{grad} f, Y \rangle = \nabla_X(\underbrace{\langle Y, \operatorname{grad} f \rangle}_{=0}) - \langle \nabla_X Y, \operatorname{grad} f \rangle \\ &= \langle \nabla_X Y, -\operatorname{grad} f \rangle = \langle II(X,Y), -\operatorname{grad} f \rangle.\end{aligned}$$

This implies the statements concerning the shape operator by definition (c.f. [2, (8.3)]). □

At this point we assume the reader to be familiar with covariant differentiation of vector and tensor fields on manifolds and along curves (otherwise see for example LRM). Since there is a canonical isomorphism $\operatorname{End}(V) \to T^1_1(V)$ between the endomorphisms of a vector space and its tensors of type $(1,1)$, we also obtain a diffeomorphism $\operatorname{End}(M) \to \mathcal{T}^1_1(M)$ between the endomorphism fields and the tensor fields of type $(1,1)$ on M (c.f. [2, 2.1]). Using this identification we can also introduce covariant differentiation of endomorphism fields in a very general setting. We have to use the fact, that a linear connection on a manifold induces connections on all tensor bundles in a canonical manner (see [2, 4.6]). If you are unfamiliar with the concept of covariant differentiation of endomorphism fields, you may also want to consult the appendix first.

1.6 Definition (Riccati equation and solution)**.** Let $c : I \to M$ be a unit speed geodesic and $U \in \operatorname{End}(c^\perp)$ be a symmetric field of endomorphisms along c. If $R := R_v := R(\_, \dot{c})(\dot{c}) \in \operatorname{End}(c^\perp)$, $v = \dot{c}(0)$, is the curvature endomorphism along c, we call

$$U' + U^2 + R = 0$$

the *Riccati equation*. We say $U \in \operatorname{End}(c^\perp)$ is a *solution of the Riccati equation*, if

$$\forall X \in \mathcal{T}(c^\perp) : U'(X) + U^2(X) + R(X) = 0.$$

1.7 Definition (Jacobi equation and solution)**.** Let $c : I \to M$ be a unit speed geodesic. We call

$$J'' + R \circ J = 0$$

the *Jacobi equation*. We say $J \in \operatorname{End}(c^\perp)$ is a *solution of the Jacobi equation*, if

$$\forall X \in \mathcal{T}(c^\perp) : J''(X) + R(J(X)) = 0.$$

1.8 Theorem (Riccati equation)**.** Let $f : U \to \mathbb{R}$ be a local distance function, $r \in \mathbb{R}$, such that $M_r := f^{-1}(r) \cap U \neq \emptyset$ and let $U \in \operatorname{End}(U)$ be the shape operator of M_r as in Lemma 1.5. Let $c : I \to M_r$ be a unit speed solution curve of $\dot{c} = \operatorname{grad} f \circ c$, $f \circ c = \mathrm{id}$ [2] and restrict U to $U \in \operatorname{End}(c^\perp)$. Then U is a solution of the Riccati equation, i.e.

$$U' + U^2 + R = 0.$$

We remark that U, U' and R are symmetric operators on $\mathcal{T}(c^\perp)$.

Proof. Since this is a local question it suffices to check this near an $r_0 \in I$. let $\sigma :]-\varepsilon, \varepsilon[\to M_{r_0}$ be any smooth curve satisfying $\sigma(0) = c(r_0)$. Let c_s be the solution curve of

$$\dot{c}_s = \operatorname{grad} f \circ c_s \qquad c_s(r_0) = \sigma(s)$$

[2]This is always possible since

$$(f \circ c)'(t) = df|_{c(t)}(\dot{c}(t)) = \langle \operatorname{grad} f_{c(t)}, \operatorname{grad} f_{c(t)} \rangle = 1.$$

Then $H :]-\varepsilon,\varepsilon[\times]r_0-\delta,r_0+\delta[$, $(s,r)\mapsto H(s,r):=c_s(r)$, is a smooth variation. According to 1.3 the variation is a variation through geodesics and consequently the variation field J is a Jacobi field (c.f. [2, 10.2]). By definition of U

$$\begin{aligned}D_rJ(r) &= D_r\partial_sH(0,r) \overset{[2,\,6.3]}{=} D_s\partial_rH(0,r) = D_s\operatorname{grad} f|_{H(s,r)}|_{s=0}\\ &= \nabla_{\partial_sH(0,r)}\operatorname{grad} f|_{H(0,r)} = U_{c(r)}(\partial_sH(0,r)) = U_r(J(r)).\end{aligned}$$

Inserting this in the product rule, we obtain (using the Jacobi equation)

$$U'_r(J(r)) = D_r(UJ)(r) - U_r(D_rJ(r)) = (D_r^2J)(r) - U_r^2(J(r)) = -R(J(r),\dot c(r))\dot c(r) - U_r^2(J(r)).$$

By choosing σ accordingly we can create $(n-1)$ normal Jacobi fields in that way which are all linearly independent on a small neighbourhood near r_0. Therefore U is a solution of the Riccati equation.
By [2, p.140] the shape operator U is symmetric.
By the symmetries of the curvature tensor (c.f. [2, 7.4]), we obtain

$$\langle R(X,\dot c)(\dot c),Y\rangle = Rm(X,\dot c,\dot c,Y) = Rm(\dot c,Y,X,\dot c) = Rm(Y,\dot c,\dot c,X) = \langle R(Y,\dot c)(\dot c),X\rangle = \langle X,R(Y,\dot c)(\dot c)\rangle$$

and therefore R is symmetric. Thus $U' = -U^2 - R$ is symmetric as well. □

In order to solve the Riccati equation, we do not necessarily need a local distance function and the corresponding level sets.

1.9 Theorem (Riccati and Jacobi equation)**.** Let $c : I \to M$ be a unit speed geodesic, $E_1 = \dot c$, and let $E_2,\dots,E_n$ be a parallel ONB along c. Let $t_0 \in I$ and let $X_2,\dots,X_n$ be any basis of $(\dot c(t_0))^\perp$. For any $2 \le i \le n$ let J_i be the Jacobi field (existence is guaranteed by [2, 10.4]) along c satisfying

$$J_i(t_0) = X_i \qquad\qquad D_tJ_i(t_0) = U_0(X_i),$$

where U_0 is a given symmetric endomorphism on $\dot c(t_0)^\perp$. Define a tensor $J_t : (\dot c(t))^\perp \to (\dot c(t))^\perp$ along c by

$$J_t\Big(\sum_{i=2}^n \alpha_iE_i\Big) := \sum_{i=2}^n \alpha_iJ_i.$$

The endomorphism $J = J_t$ solves the Jacobi equation $J'' + RJ = 0$, is invertible for any t near t_0 and

$$U_t := J'_t \circ J_t^{-1}$$

is a symmetric solution of the Riccati equation.
Conversely, if U_t is a symmetric solution of the Riccati equation, a field J satisfying $J'_t = U_tJ_t$ is a solution of the Jacobi equation $J''_t + R_tJ_t = 0$.

Proof. By construction $J_i(t_0) = X_i$ and the X_i are a basis of $(\dot c(t))^\perp$. So J_{t_0} is invertible, which implies that it is invertible in a small neighbourhood of t_0 by smoothness. There the Jacobi equation (c.f. [2, (10.2)]) implies, that for any $2 \le i \le n$

$$\begin{aligned}&J''_t(E_i) + (R_t\circ J_t)(E_i) = D_t(J'_t(E_i)) - J_t(D_tE_i) + R_t(J_i)\\ &= D_t^2(J_i) + R_t(J_i) = 0\end{aligned}$$

or just

$$J''_t + R_tJ_t = 0.$$

on $\dot c^\perp$. For t near t_0 define

$$U_t := J'_t \circ J_t^{-1} : \mathcal{T}(c)^\perp \to \mathcal{T}(c)^\perp.$$

Using A.3, we obtain for any $Y \in \mathcal{T}(c^{\perp})$

$$
\begin{aligned}
&D_t U(Y) = D_t(U(Y)) - U(D_t(Y)) \\
&= D_t^2 J(J^{-1}(Y)) + D_t J(D_t(J^{-1}(Y))) + D_t J(J^{-1}(D_t Y)) - D_t J(J^{-1}((D_t(Y)))) \\
&= -R(J((J^{-1}(Y))) + D_t J(D_t(J^{-1}(J(J^{-1}(Y)))) \\
&= -R(Y) + D_t J(-J^{-1}(D_t J)(J^{-1}(Y))) \\
&= -R(Y) - U^2(Y) = 0.
\end{aligned}
$$

Conversely let U be a symmetric solution of the Riccati equation, i.e.

$$D_t U + U^2 + R = 0$$

and additionally let

$$J' = U \circ J.$$

Now it suffices to check the Jacobi equation for J on a parallel ONB. Therefore if E is parallel

$$
\begin{aligned}
&J''(E) + R(J(E)) = (J(E))'' + R(J(E)) = (J'(E))' + R(J(E)) = (U(J(E)))' + R(J(E)) \\
&= U'(J(E)) + U(J'(E)) + R(J(E)) = -U^2(J(E)) - R(J(E)) + U^2(J(E)) + R(J(E)) = 0.
\end{aligned}
$$

□

2 Comparison Theory for the Riccati Equation

In the last section we discussed the Riccati equation as an equation of field of endomorphisms on $\mathcal{T}(c^\perp)$ along c. Now we discuss the corresponding one dimensional ODE of the same type.

2.1 Definition. Let $\kappa \in \mathbb{R}$ and let $u : I \subset \mathbb{R}$ be differentiable. Then u

(i) is a *solution of the Riccati inequality*, if

$$u' \leq -u^2 - \kappa.$$

(ii) is a *solution of the Riccati equation*, if

$$u' = -u^2 - \kappa.$$

(iii) is a *solution of the Jacobi equation*, if

$$u'' + \kappa u = 0.$$

Notice that we already encountered the Jacobi equation as an equation of fields of endomorphisms in the last chapter. The study of this equation as an equation of vector fields is a classic topic in differential geometry (c.f. [2, 10]).

2.2 Lemma. Let $c : I \to M$ be a unit speed geodesic, U be a symmetric solution of the Riccati equation along c, E be a parallel vector field along c such that $E \perp \dot{c}$ and $\|E\| = 1$. Define $u : I \to \mathbb{R}$, $u := \langle U(E), E\rangle$, and let $K(E(t) \wedge \dot{c}(t))$ denote the sectional curvature of the plane determined by $E(t)$ and $\dot{c}(t)$. Assume $K(E(t) \wedge \dot{c}(t)) \geq \kappa$ for some $\kappa \in \mathbb{R}$. Then u is a solution of the Riccati inequality

$$u' \leq -u^2 - \kappa.$$

Proof. By definition

$$\begin{aligned} u' &= \langle (U(E))', E\rangle + \langle U(E), E'\rangle = \langle U_t'(E) + U_t(E'), E\rangle = \langle -U^2(E) - R_t(E), E\rangle \\ &= -\langle U(E), U(E)\rangle - \langle R(E, \dot{c})\dot{c}, E\rangle = -\langle U(E), U(E)\rangle - K(E \wedge \dot{c}(t)). \end{aligned}$$

By the theorem of Cauchy/Schwarz

$$u^2 = \langle U(E), E\rangle^2 \leq \|U(E)\|^2 \|E\|^2 = \langle U(E), U(E)\rangle.$$

This implies the statement. □

2.3 Lemma. Let $c : I \to M$ be a unit speed geodesic, U be a symmetric solution of the Riccati equation and J be a field of isomorphisms satisfying $J' = J \circ U$ (in particular one may choose U and J as in Theorem 1.9). Define $u : I \to \mathbb{R}$ by

$$u := \frac{1}{n-1}(\ln(\det(J)))'.$$

Then $u = \frac{1}{n-1}\operatorname{tr}(U)$ and

$$u' \leq -u^2 - \frac{1}{n-1}\operatorname{Ric}(\dot{c}, \dot{c}).$$

Consequently, if there exists a constant $\kappa \in \mathbb{R}$, such that $\operatorname{Ric}(\dot{c}, \dot{c}) \geq (n-1)\kappa$, then u is a solution of the Riccati inequality.

Proof. Let $E_1, \dots, E_n$ be a parallel ON frame along c, $t_0 \in I$ and let j^i_k be the matrix of J w.r.t. this frame. Then $\det J := \det(j^i_k)$ does not depend on this choice of frame. The rules of differentiation for the determinant imply (since J is invertible):

$$(n-1)u = \frac{1}{\det(J)} \det'|_J(J') = \frac{1}{\det(J)} \det(J) \operatorname{tr}(J^{-1}J') = \operatorname{tr}(U).$$

Furthermoore since U solves the Riccati equation

$$\begin{aligned} u' &= \frac{1}{n-1} \operatorname{tr}(U)' = \frac{1}{n-1} \operatorname{tr}(U') = -\frac{1}{n-1} \operatorname{tr}(U^2) - \frac{1}{n-1} \operatorname{tr}(R_t) \\ &\overset{(1)}{\leq} -\frac{1}{(n-1)^2} \operatorname{tr}(U)^2 - \frac{1}{n-1} \operatorname{Ric}(\dot{c}, \dot{c}) = -u^2 - \frac{1}{n-1} \operatorname{Ric}(\dot{c}, \dot{c}) \leq -u^2 - \kappa. \end{aligned}$$

(1): By definition $\operatorname{tr}(R) = \operatorname{Ric}(\dot{c}, \dot{c})$ (c.f. 1.6). The estimate for the trace is done in the next Lemma 2.4. □

2.4 Lemma. Let $U \in \mathbb{R}^{n \times n}$ be symmetric. Then

$$\operatorname{tr}(U)^2 \leq \operatorname{tr}(U^2) n,$$

where equalitiy holds if and only if there exists $\lambda \in \mathbb{R}$ such that $U = \lambda E$.

Proof. Define the scalar product

$$\langle A, B \rangle = \operatorname{tr}(AB^t) = \sum_{i=1}^n (AB^t)^i_i = \sum_{i,j=1}^n A^i_j B^i_j$$

on $\mathbb{R}^{n \times n}$. Using the Cauchy/Schwarz inequality, we calculate

$$\operatorname{tr}(U)^2 = \operatorname{tr}(UE^t)^2 = \langle U, E \rangle^2 \leq \|U\|^2 \|E\|^2 = \langle U, U \rangle \langle E, E \rangle = \operatorname{tr}(UU^t) \operatorname{tr}(EE^t) = \operatorname{tr}(U^2) n.$$

□

2.5 Lemma (Jacobi and Riccati equation)**.** Let $\kappa \in \mathbb{R}$. If $j : I \subset \mathbb{R} \to \mathbb{R}$ solves the Jacobi equation

$$j'' + \kappa j = 0$$

and $\forall t \in I : j(t) \neq 0$, then $u := j'/j$ solves the Riccati equation

$$u' = -u^2 - \kappa.$$

Conversely if u solves the Riccati equation, then any solution j of

$$j' = uj$$

solves the Jacobi equation

$$j'' + \kappa j = 0.$$

Proof. We calculate

$$u' = \left(\frac{j'}{j} \right)' = \frac{j''j - j'j'}{j^2} = \frac{-\kappa j^2 - j'j'}{j^2} = -\kappa \frac{j^2}{j^2} - u^2 = -u^2 - \kappa$$

and conversely

$$j'' + \kappa j = (uj)' + \kappa j = u'j + uj' + \kappa j = -u^2 j - \kappa j + u^2 j + \kappa j = 0.$$

□

2.6 Definition. We denote by sn_κ the unique solution of the Jacobi equation satisfying

$$\operatorname{sn}_\kappa(0) = 0 \qquad\qquad \operatorname{sn}'_\kappa(0) = 1$$

und by cs_κ the unique solution of the Jacobi equation satisfying

$$\operatorname{cs}_\kappa(0) = 1 \qquad\qquad \operatorname{cs}'_\kappa(0) = 0.$$

For our comparison theory we require specific solutions of the Jacobi equation, which will be useful throughout the script. We collect some easy facts about them.

2.7 Lemma (Properties of sn_κ and cs_κ)**.** For any $\kappa \in \mathbb{R}$, the following hold.

(i) The solutions are explicitely given by $\operatorname{sn}_\kappa, \operatorname{cs}_\kappa : \mathbb{R} \to \mathbb{R}$

$$\operatorname{sn}_\kappa(t) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} t) & , \kappa > 0 \\ t & , \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa} t) & , \kappa < 0 \end{cases} \qquad \operatorname{cs}_\kappa(t) = \begin{cases} \cos(\sqrt{\kappa} t) & , \kappa > 0 \\ 1 & , \kappa = 0 \\ \cosh(\sqrt{-\kappa} t) & , \kappa < 0 \end{cases}$$

(ii) Defining

$$R_\kappa := \begin{cases} \frac{\pi}{\sqrt{\kappa}} & , \kappa > 0 \\ \infty & , \kappa \leq 0 \end{cases} \qquad L_\kappa := \begin{cases} \frac{\pi}{2\sqrt{\kappa}} & , \kappa > 0 \\ \infty & , \kappa \leq 0 \end{cases},$$

we obtain

$$\forall t \in]0, R_\kappa[: \operatorname{sn}_\kappa(t) > 0, \qquad \forall t \in]0, L_\kappa[: \operatorname{cs}_\kappa(t) > 0.$$

(iii) We obtain the symmetries

$$\forall t \in \mathbb{R} : \operatorname{sn}_\kappa(-t) = -\operatorname{sn}_\kappa(t), \qquad \forall t \in \mathbb{R} : \operatorname{cs}_\kappa(-t) = \operatorname{cs}_\kappa(t).$$

(iv) These functions satisfy

$$\operatorname{sn}'_\kappa = \operatorname{cs}_\kappa, \qquad \operatorname{cs}'_\kappa = -\kappa \operatorname{sn}_\kappa.$$

(v) In particular

$$\operatorname{ct}_\kappa := \frac{\operatorname{cs}_\kappa}{\operatorname{sn}_\kappa}$$

solves the Riccati equation on $]0, R_\kappa[$ and satisfies

$$\lim_{t \searrow 0} \operatorname{ct}_\kappa(t) = +\infty.$$

Proof.

(i) Just verify, that these functions solve the desired initial value problem.

(ii) This is a direct consequence of (i) and the zeros of all the functions occuring there.

(iii) Follows from 1 and the symmetries of all the functions occuring there.

(iv) This is a consequence of the uniqueness of initial value problems: The function $\operatorname{sn}_\kappa'$ satisfies

$$(\operatorname{sn}_\kappa')'' + \kappa \operatorname{sn}_\kappa' = (\operatorname{sn}_\kappa'')' + \kappa \operatorname{sn}_\kappa' = -\kappa \operatorname{sn}_\kappa' + \kappa \operatorname{sn}_\kappa' = 0,$$

$$\operatorname{sn}_\kappa'(0) = 1 = \operatorname{cs}_\kappa(0), \qquad \operatorname{sn}_\kappa''(0) = -\kappa \operatorname{sn}_\kappa(0) = 0 = \operatorname{cs}_\kappa'(0)$$

Thus $\operatorname{sn}_\kappa'$ is a solution of the Jacobi equation with the same initial values as cs_κ. Analogously, we verify:

$$(\operatorname{cs}_\kappa')'' + \kappa \operatorname{cs}_\kappa' = (\operatorname{cs}_\kappa'')' + \kappa \operatorname{cs}_\kappa' = -\kappa \operatorname{cs}_\kappa' + \kappa \operatorname{cs}_\kappa' = 0,$$

$$\operatorname{cs}_\kappa'(0) = 0 = -\kappa \operatorname{sn}_\kappa(0), \qquad \operatorname{cs}_\kappa''(0) = -\kappa \operatorname{cs}_\kappa(0) = -\kappa = -\kappa \operatorname{sn}_\kappa'(0).$$

(v) Follows from (ii), (iv) and Lemma 2.5.

□

2.8 Lemma. Let $\kappa \in \mathbb{R}$ and let $u, v : [a,b] \to \mathbb{R}$ be solutions of the Riccati inequality rsp. the Riccati equality, i.e.

$$u' \leq -u^2 - \kappa, \qquad v' = -v^2 - \kappa.$$

(i) The function $v - u$ is monotonously increasing. In particular, if $v(a) \geq u(a)$, then

$$\forall t \in [a,b] : v(t) \geq u(t).$$

(ii) If $v(a) \geq u(a)$ and additionally $\exists t_0 \in [a,b]$: $v(t_0) = u(t_0)$, we obtain

$$v|_{[a,t_0]} \equiv u|_{[a,t_0]}.$$

Proof. Let F be any anti-derivative of $u + v$. We calculate

$$\begin{aligned}((v-u)e^F)' &= (v'-u')e^F + (v-u)(e^F)' = (v' - u' + (v-u)(v+u))e^F \\ &= (v' - u' + v^2 - u^2)e^F \geq (-v^2 - \kappa + u^2 + \kappa + v^2 - u^2)e^F = 0.\end{aligned}$$

□

2.9 Lemma. Let $u :]0,b[\to \mathbb{R}$ solve the Riccati inequality $u' \leq -u^2 - \kappa$ and assume

$$\lim_{t \searrow 0} u(t) = +\infty.$$

This implies

$$\forall t \in]0,b[: u(t) \leq \operatorname{ct}_\kappa(t)$$

and if there exists $t_0 \in]0,b[$, such that $u(t_0) = \operatorname{ct}_\kappa(t_0)$, then

$$u|_{]0,t_0]} = \operatorname{ct}_\kappa|_{]0,t_0]}.$$

Proof. Assume to the contrary that there exists t_0, such that $u(t_0) > \operatorname{ct}_\kappa(t_0)$. Then $u(t_0) > \operatorname{ct}_\kappa(t_0 - \varepsilon)$ for a sufficiently small $\varepsilon > 0$, thus $u(t) > \operatorname{ct}_\kappa(t - \varepsilon)$ on $]\varepsilon, t_0]$ by Lemma 2.8. This contradicts

$$u(\varepsilon) = \lim_{t \searrow \varepsilon} u(t) < \infty,$$

thus we obtain the first statement.
In case $u(t_0) = \operatorname{ct}_\kappa(t_0)$, we obtain $u(t) = \operatorname{ct}_\kappa(t)$ for all $t \in]0, t_0]$ by Lemma 2.8. □

3 Cut Locus and Conjugate Locus

In this chapter M is a complete Riemannian manifold.

3.1 Definition. For reasons of convenience we will employ the notation

$$\forall p \in M : \mathbb{S}_p M := \{v \in T_p M \mid \|v\| = 1\}$$

and

$$\mathbb{S}M := \bigcup_{p \in M} \mathbb{S}_p M.$$

3.2 Definition. Let $c : \mathbb{R} \to M$ be a unit speed geodesic, $v = \dot{c}(0)$ and $p = c(0)$. We define

$$\begin{aligned} t_0 &:= t_0(c) := t_0(v) := \sup\{t > 0 \mid t = d(c(t), p)\} \in]0, \infty] \\ t_1 &:= t_1(c) := t_1(v) := \inf\{t > 0 \mid c(t) \text{ is conjugate to } p\} \in]0, \infty], \end{aligned}$$

where we employ the convention $\inf \emptyset = \infty$. If $t_1 < \infty$, there exists a Jacobi field $J \neq 0$ along c such that $J(0) = J(t_1) = 0$.

3.3 Lemma. With the notation above, we always obtain

$$t_1 \geq t_0.$$

More explicitely: If there exists a Jacobi field $J \neq 0$ along c such that $J(0) = J(t) = 0$ for some $t > 0$, then $t \geq t_0$ (see also [2, 10.15]).

Proof. Let $s > t$. For such a J we define

$$X(\tau) := \begin{cases} J(\tau) & 0 \leq \tau \leq t \\ 0 & t \leq \tau \leq s \end{cases}.$$

Then X is a piecewise smooth vector field along $c|_{[0,s]}$ satisfying $X(0) = X(s) = 0$. Since $J \neq 0$ and $J(t) = 0$, we get $J'(t) \neq 0$ (since otherwise $J \equiv 0$). Choose a smooth vector field Y along c satisfying $Y(0) = Y(s) = 0$ and $Y(t) = -J'(t)$ and define $X_\varepsilon := X + \varepsilon Y$. Then X_ε is a piecewise smooth vector field along $c|[0, s]$, which satisfies $X_\varepsilon(0) = X_\varepsilon(s) = 0$ and which is broken at t [3]. We want to calculate the index form $I(X_\varepsilon, X_\varepsilon)$ along $c|[0, s]$ and consider

$$\begin{aligned} I(X, Y) &= -\int_0^s \langle X''(u) + R(X(u), \dot{c}(u))(\dot{c}(u)), Y\rangle du - \sum_{\text{break points } t_i} \langle \Delta_{t_i} X', Y(t_i)\rangle \\ &= -\int_0^t \underbrace{\langle X''(u) + R(X(u), \dot{c}(u))(\dot{c}(u)), Y\rangle}_{=0 \text{ ,since } X \text{ satisfies the Jacobi equation here}} du - \int_t^s \underbrace{\langle X''(u) + R(X(u), \dot{c}(u))(\dot{c}(u), Y)}_{=0 \text{ ,since here } X=0}\rangle du \\ &\quad - \langle X'(t \swarrow) - X'(\nearrow t), Y(t)\rangle \\ &= -\langle J'(t), J'(t)\rangle = -\|J'(t)\|^2. \end{aligned}$$

Analogously

$$I(X, X) = -\langle X'(t \swarrow) - X'(\nearrow t), X(t)\rangle = \langle J'(t), X(t)\rangle = 0.$$

[3] Notice, that the derivative satisfies

$$X'_\varepsilon(\nearrow t) = X'(\nearrow t) + \varepsilon Y'(t) = J'(t) + \varepsilon Y'(t) \neq \varepsilon Y'(t) = X'_\varepsilon(t \swarrow).$$

Consequently the index form satisfies

$$I(X_\varepsilon, X_\varepsilon) = I(X,X) + 2\varepsilon I(X,Y) + \varepsilon^2 I(Y,Y) = -2\varepsilon\|J'(t)\|^2 + \varepsilon^2 I(Y,Y) < 0$$

for small $\varepsilon > 0$. This implies, that the unit speed geodesic $c|[0,s]$ is not minimizing (c.f. [2, 10.13])), which implies $s > t_0$. □

3.4 Definition (Cut locus and conjugate locus)**.** Let $c : \mathbb{R} \to M$, $c(0) = p$ be a unit speed geodesic and let $t_0(c) < \infty$. Then $c(t_0(c))$ is the *cut point* of p along c. The set $C(p)$ of all cut points of p is the *cut locus* of p. We call

$$C_T(p) := \{t_0(v)v \mid v \in \mathbb{S}_pM, t_0(v) < \infty\} \subset T_pM$$

the *tangential cut locus near* p. By definition $\exp_p(C_T(p)) = C(p)$.
If $t_1(c) < \infty$, we call $c(t_1(c))$ the *first conjugate point of* p *along* c. The set of all first conjugate points along c is called the *first conjugate locus of* p.

3.5 Remark. If M is compact, then $t_0 \leq \operatorname{diam} M$ (e.g. on $\mathbb{S}^n$). But $t_1(c) = \infty$ is possible, even for all unit speed geodesics c in M (e.g. on flat Tori).

3.6 Lemma. Let $c : \mathbb{R} \to M$ be a unit speed geodesic, let $t_0 := t_0(c) < \infty$, $t_1 := t_1(c)$, $c(0) =: p$, and let $q := c(t_0)$ be a cut point of p along c. Then there are two possibilities, which are not mutually exclusive [4]:

(i) q is conjugate to p along c, thus $t_1 = t_0$.

(ii) There is a second unit speed geodesic $\bar{c} : \mathbb{R} \to M$ satisfying $\bar{c}(0) = p$ and $\bar{c}(t_0) = q$.

Proof. Assume, that q is not conjugate to p along c. Let $v \in \mathbb{S}_pM$ such that $t_0 = t_0(c) = t_0(v)$. Then $q = \exp_p(t_0 v)$ and there exists a neighbourhood $U \subset T_pM$ of $t_0 v$ and a neighbourhood $V \subset M$ of q, such that $\exp_p : U \to V$ is a smooth diffeomorphism ([2, 10.11]).
Certainly there exists a sequence (t_n) in $\mathbb{R}$ such that $t_n \searrow t_0$ and $c(t_n) \in V$. Thus we have

$$\exp_p(t_n v) = c_{t_n v}(1) = c_v(t_n) = c(t_n).$$

Since $t_n > t_0$, the definition of t_0 implies $d(c(t_n), p) < t_n$. Consequently the curve c is not minimizing the distance between p and $c(t_n)$. By the theorem of Hopf-Rinow there exists a minimizing geodesic between p and $c(t_n)$, which implies

$$\exists w_n \in T_pM : \exp_p(w_n) = \exp_p(t_n v) = c(t_n).$$

This implies $\|w_n\| = d(c(t_n), p) < t_n$. In particular (w_n) is bounded. We claim that $w_n \notin U$, $w_n \in U$ would imply

$$\exp(w_n) = \exp(t_n v) \Rightarrow w_n = t_n v,$$

by the invertibility of $\exp_p : U \to V$. But $w_n \neq t_n v$ by construction, so $w_n \notin U$. Since in addition $t_n v \to t_0 v$, this shows, that $t_0 v$ cannot be an accumulation point of the sequence (w_n).
But since (w_n) is bounded, there exists at least one accumulation point $w \notin U$ (we may assume that (w_n) converges). This point satisfies

$$\|w\| = \lim_{n\to\infty} ||w_n|| \leq \lim_{n\to\infty} t_n = t_0$$

and

$$\exp_p(w) = \lim_{n\to\infty} \exp(w_n) = \lim_{n\to\infty} c(t_n) = c(t_0) = q.$$

Write $w = t'\bar{v}$, where $\|\bar{v}\| = 1$ and $t' := \|w\|$. We claim that $\bar{c} := c_{\bar{v}}$ is our desired geodesic: Clearly $\bar{c}(0) = p$, $\bar{c}(t') = q$ and $\bar{v} \neq v$. We already obvserved $t' \leq t_0$. On the other hand $t_0 = d(p,q) \leq L(\bar{c}|_{[0,t']}) = t'$, which alltogether implies $t' = t_0$. □

[4] For example consider antipodal points $p, q \in \mathbb{S}^n$.

3.7 Remark. If $0 < t < t_0$ none of the possibilites enlisted in Lemma 3.6 above can hold. The first is a direct contradiction to Lemma 3.3. For the second, suppose there exists such a second unit speed geodesic $\bar{c}$. Then for any $t < s < t_0$ the curve $\hat{c}$ defined by

$$\hat{c}(\tau) := \begin{cases} \bar{c}(\tau) & 0 \leq \tau \leq t \\ c(\tau) & t \leq \tau \leq s \end{cases}$$

is a curve from p to $c(s)$ satisfying

$$L(\hat{c}) = L(\bar{c}|[0,t]) + L(c|[t,s]) = t + s - t = s = L(c[0,s]) = d(p, c(s))$$

by definition of t_0. Thus the curve $\hat{c}$ is a minimizing curve from p to $c(s)$. Since $\bar{c} \neq c$, but $\bar{c}(t) = c(t)$, this implies $\dot{\bar{c}}(t) \neq \dot{c}(t)$ and $\hat{c}$ is broken in t. This is a contradiction, since a broken geodesic can never be minimizing (c.f. [2, 6.6]).

3.8 Corollary. Let $p, q \in M$ be arbitrary. Then

$$q \in C(p) \Longleftrightarrow p \in C(q).$$

Proof. Let $c : \mathbb{R} \to M$ be a unit speed geodesic and $c(0) = p$. Let $t_0 := t_0(c)$ and let $q := c(t_0)$ be the cut point of p along c. We claim, that p is the cut point of q along $\tilde{c} : \mathbb{R} \to M$, $s \mapsto c(t_0 - s)$. Since distance is symmetric, certainly

$$t_0 = t_0(c) = d(p, c(t_0)) = d(c(t_0), p) = d(q, \tilde{c}(t_0)).$$

So by definition $\tilde{t}_0 := t_0(\tilde{c}) \geq t_0(c) = t_0$. For the other estimate consider the two possibilites of Lemma 3.6
Case 1: If p is cojugate to q along c, then $t_0(c) = t_1(c)$ and there exists a Jacobi field $J \neq 0$ along c, such that $J(0) = J(t_0) = 0$. Consequently $\tilde{J}(s) := J(t_0 - s)$ is a Jacobi field $\neq 0$ along $\tilde{c}$ satisfying

$$\tilde{J}(0) = J(t_0) = 0 = J(0) = \tilde{J}(t_0).$$

Thus q is conjugate to p along $\tilde{c}$ and $\tilde{t}_0 \leq t_1(\tilde{c}) = t_1(c) = t_0(c)$.
Case 2: Assume p is not conjugate to q along c and assume $\tilde{t}_0 > t_0$. Then by Lemma 3.6 there exists a second geodesic $\bar{c} : \mathbb{R} \to M$ satisfying $\bar{c}(0) = p$ and $\bar{c}(t_0) = q$. Consequently $s \mapsto \bar{c}(t_0 - s)$ is a second geodesic from q to p. But since $\tilde{t}_0 > t_0$, this geodesic must not exist by Remark 3.7 applied to $\tilde{c}$!

□

3.9 Lemma. The map $t_0 : \mathbb{S}M \to]0, \infty]$ is continuous.

Proof. Consider a sequence $v_i \in \mathbb{S}M$, $v_i \to v$, and define

$$t^* := \limsup_{i \to \infty} t_0(v_i) \qquad\qquad t_* := \liminf_{i \to \infty} t_0(v_i).$$

Obviously we have to show that $t^* = t_0(v) = t_*$.
Step 1: For any $t < t^*$ there exists a subsequence (v_{i_k}) such that $t_0(v_{i_k}) > t$. Denoting by c_k the geodesic satisfying $\dot{c}_k(0) = v_{i_k}$, we obtain

$$d(c_k(t), c_k(0)) = t.$$

Since $v_{i_k} \to v$ as $k \to \infty$ and since $\exp_p$ and d are continuous, this implies $d(c(t), c(0)) = t$, where c is the geodesic satisfying $\dot{c}(0) = v$. Consequently $\forall t < t^* : t_0(v) \geq t$ and thus $t_0(v) \geq t^*$.

In the next step we will show, that $t_0(v) \leq t_*$. Since $t_* \leq t^*$ anyway, this implies alltogether

$$t^* \leq t_0(v) \leq t_* \leq t^*$$

and thus the claim.
STEP 2: Now let (v_{i_k}) be a subsequence such that

$$t_* = \liminf_{i\to\infty} t_0(v_i) = \lim_{k\to\infty} t_0(v_{i_k}).$$

Suppose $t_* < \infty$ (otherwise there is nothing to show) and let $c(t_*)$ not be conjugate to $c(0)$. [5] The inverse function theorem on manifolds ([3, 7.10]) implies the existence of an open neighbourhood U of t_*v in TM, such that $\exp|_{U\cap T_qM}$ is invertible for all $q \in \pi(U)$, i.e. near $p := c(0)$, where $\pi : TM \to M$ is the canonical projection.
Since $t_0(v_{i_k}) \to t_*$ by construction, we obtain for all large k, that $t_0(v_{i_k})v_{i_k} \in U$. Since exp is invertible on U, $p_k := \pi(v_{i_k})$ is not conjugate to $q_k := c_k(t_0(v_{i_k})) = \exp(t_0(v_{i_k})v_{i_k})$, where c_k is the unit speed geodesic between p_k and q_k. Applying Lemma 3.6 to every k, the second item is in power and thus there exist vectors $\bar{v}_{i_k} \neq v_{i_k}$ in the same tangent space as v_{i_k}, such that

$$\exp(t_0(v_{i_k})v_{i_k}) = \exp(t_0(v_{i_k})\bar{v}_{i_k}).$$

By construction of U we obtain $t_0(v_{i_k})v_{i_k} \in U$, but $t_0(v_{i_k})\bar{v}_{i_k} \notin U$ for all large k (again since exp is bijective on U). Since the geodesics from 3.6 also have unit speed, we obtain $\|\bar{v}_{i_k}\| = 1$. In particular the sequence $\bar{v}_{i_k}$ is bounded and thus has an accumulation point $\bar{v}$. This point satisfies $\bar{v} \neq v$, but $\bar{c}(0) = c(0)$ and $\bar{c}(t_*) = c(t_*)$. Alltogether we have constructed a second unit speed geodesic between $c(0)$ and $c(t_*)$. This implies $t_0(v) \leq t_*$, since $t_* < t_0(v)$ contradicts Remark 3.7.

□

3.10 Lemma. Let

$$D_T(p) := \{tv \mid v \in \mathbb{S}_pM, 0 \leq t < t_0(v)\}.$$

Then $D_T(p)$ is star-shaped with respect to 0_p, open in T_pM, $\partial D_T(p) = C_T(p)$ and

$$\forall w \in D_T(p) : \exp_p(w) \notin C(p).$$

Proof. The definitions and Lemma 3.9 imply the first statements.
Suppose there exists $w \in D_T(p)$ such that $\exp_p(w) \in C(p)$. Then there is a $\bar{w} \in C_T(p)$ satisfying

$$\exp_p(\bar{w}) = \exp_p(w).$$

Define

$$v := \frac{w}{\|w\|} \qquad \bar{v} := \frac{\bar{w}}{\|\bar{w}\|}.$$

By definition of t_0 the geodesics c und $\bar{c}$ determined by v resp. $\bar{v}$ are minimizing till $\|w\| < t_0(v)$ resp. $\|\bar{w}\| = t_0(\bar{v})$. By hypothesis $c(\|w\|) = \bar{c}(\|\bar{w}\|)$ and consequently

$$\|w\| = d(p, c(\|w\|)) = d(p, c(\|\bar{w}\|)) = \|\bar{w}\|.$$

Thus $t_0(\bar{v}) = \|\bar{w}\| = \|w\| < t_0(v)$ and in particular $v \neq \bar{v}$. Alltogether we have found two geodesics from p to $c(w)$, which by Remark 3.7 implies $\|w\| \geq t_0(v)$. Contradiction! □

[5] Otherwise $t_0 \leq t_1 \leq t_*$ by Lemma 3.3 anyway.

3.11 Theorem. For any $p \in M$ the set $M \setminus C(p)$ is open and $\exp_p : D_T(p) \to M \setminus C(p)$ is a diffeomorphism. The function $f : M \to \mathbb{R}$, $q \mapsto \frac{1}{2}d(p,q)^2$, is smooth on $M \setminus C(p)$ and for any $q \in M \setminus C(p)$ the following holds.

(i) Denoting by c_q the geodesic satisfying $\dot{c}_q(0) = \exp_p|_{D_T(p)}^{-1}(q)$, we obtain $\operatorname{grad} f(q) = \dot{c}_q(1)$.

(ii) Denoting by J the Jacobi field along c_q satisfying $J(0) = 0$, $J(1) = X$, $X \in T_pM$, we obtain $\nabla_X \operatorname{grad} f = J'(1)$.

Proof.
STEP 1 ($\exp_p$ is a diffeomorphism): Since M is complete by hypothesis, for any $q \in M$ there exists a minimizing geodesic form p to q. So $\exp_p : D_T(p) \to M \setminus C(p)$ is surjective. Remark 3.7 implies the injectivity. Since conjugate points do not occur before t_0 by Lemma 3.3, $\exp_p$ has maximal rank on $D_T(p)$ (c.f. [2, 10.11]) and thus is a diffeomorphism.
STEP 2: We will now prove statement (i). Let $q \in M \setminus C(p)$ and $X \in T_qM$. Choose $\varepsilon > 0$ and a smooth curve $\sigma :]-\varepsilon, \varepsilon[\to M \setminus C(p)$, such that $\sigma(0) = q$, $\dot{\sigma}(0) = X$. Define $w :]-\varepsilon, \varepsilon[\to D_T(p) \subset T_pM$, $w(s) := (\exp_p|_{D_T(p)})^{-1}(\sigma(s))$, and $H :]-\varepsilon, \varepsilon[\times [0,1] \to M$ by

$$H(s,t) := \exp_p(tw(s)).$$

So $t \mapsto H(s,t)$ is the radial geodesic from p to $\sigma(s)$ which implies

$$f(\sigma(s)) = \frac{1}{2}d(p,\sigma(s))^2 = \frac{1}{2}\left(\int_0^1 \|\partial_t H(s,t)\| dt\right)^2 = \frac{1}{2}\int_0^1 \langle \partial_t H(s,t), \partial_t H(s,t)\rangle dt.$$

In the last step we are just for once allowed to interchange the square with the integral, because $t \mapsto H(s,t)$ has constant speed. Furthermore

$$\begin{aligned}
\langle \operatorname{grad} f(q), X\rangle &= df_q(X) = X_q(f) = \dot{\sigma}(0)(f) = \partial_s(f \circ \sigma)(0) \\
&= \partial_s \frac{1}{2}\int_0^1 \langle \partial_t H(s,t), \partial_t H(s,t)\rangle dt|_{s=0} = \int_0^1 \langle D_s\partial_t H(s,t), \partial_t H(s,t)\rangle dt|_{s=0} \\
&= \int_0^1 \langle D_t \partial_s H, \partial_t H\rangle(0,t) dt \overset{(1)}{=} \langle \partial_s H, \partial_t H\rangle(0,t)|_{(0,0)}^{(0,1)} \\
&= \langle \partial_s H(0,1), \partial_t H(0,1)\rangle - \langle \partial_s H(0,0), \partial_t H(0,0) \overset{(2)}{=} \langle X, \dot{c}_q(1)\rangle
\end{aligned}$$

(1): We have
$$\partial_t \langle \partial_s H, \partial_t H\rangle = \langle D_t \partial_s H, \partial_t H\rangle + \langle \partial_s H, D_t \partial_t H\rangle$$
and $D_t \partial_t H(0,\_) = 0$ since $H(0,\_)$ is a geodesic from p to $\sigma(0) = q$.
(2): By construction

$$\begin{aligned}
&H(s,1) = \exp_p(1 \cdot w(s)) = \sigma(s) \Rightarrow \partial_s H(0,1) = \dot{\sigma}(0) = X \\
&H(0,t) = \exp_p(tw(0)) \Rightarrow \partial_t H(0,1) = \partial_t \exp_p(tw(0))|_{t=1} = \partial_t \exp_p(t\exp_p|_{D_T(p)}^{-1}(q))|_{t=1} = \dot{c}_q(1) \\
&H(s,0) = \exp_p(0w(s)) = p \Rightarrow \partial_s H(0,0) = 0
\end{aligned}$$

STEP 3: To prove the second statement, we remark that

$$\nabla_X \operatorname{grad} f|_q = \nabla_{\dot{\sigma}(0)} \dot{c}_q(1) = D_s \partial_t H(0,1) = D_t \partial_s H(0,1).$$

But $\partial_s H(0,\_)$ is (according to Step 2, (2)) the Jacobi field J as in (ii).

□

3.12 Corollary. $M \setminus C(p)$ is diffeomorphic to the slice

$$D^n := \{x \in \mathbb{R}^n \mid \|x\| < 1\},$$

where $n = \dim M$.

Proof. There is only to show that $D_T(p)$ is diffeomorphic to $\mathbb{D}^n$. We leave this as an exercise. $\square$

3.13 Theorem. If M is compact, then $C(p)$ is a strong deformation retraction of $M \setminus \{p\}$, i.e. there is a continuous map $H : (M \setminus \{p\}) \times [0,1] \to M \setminus \{p\}$ satisfying $H(q,0) = q$ for all $q \in M \setminus \{p\}$, $H(q,s) = q$ for all $q \in C(p)$ and $s \in [0,1]$ and $H(q,1) \in C(p)$ for all $q \in M$.

Proof. Exercise. $\square$

3.14 Remark. The function $d(p, \_) = \sqrt{(}2f)$ is smooth on M without p and $C(p)$ and there its gradient has unit length (c.f. chapter 1).

4 Injectivity Radius and Curvature Bounds from above

4.1 Definition (Injectivity radius)**.** For any $p \in M$ we call [6]

$$i(p) := \min\{t_0(v) \mid v \in \mathbb{S}_pM\} = \min\{\|w\| \mid w \in C_T(p)\} = \min\{d(p,q) \mid q \in C(p)\} \in]0,\infty]$$

the *injectivity radius of* M *in* p. We call

$$i(M) := \inf\{i(p) \mid p \in M\} = \inf\{t_0(v) \mid v \in \mathbb{S}M\}$$

the *injectivity radius of* M.

By Remark 3.7 and Lemma 3.6 the number $i(p)$ is the maximal radius $r > 0$, such that

$$\exp_p : B_r(0_p) \to B_r(p)$$

is a diffeomorphism. Notice that the shape of $D_T(p)$ may be very different from $B_{i(p)}(0_p)$.

4.2 Lemma. The injectivity radius is a continuous function $i : M \to]0,\infty]$. In particular for any compact manifold M

$$0 < i(M) \leq \operatorname{diam} M.$$

Proof. By Lemma 3.9 the function $t_0 : \mathbb{S}M \to]0,\infty]$ is continuous. Let $p \in M$, U be an open neighbourhood near p, such that there exists a smooth ON frame $E_1,\ldots,E_n$ over U. This defines a continuous function $f : U \times \mathbb{S}^{n-1} \to]0,\infty]$ by

$$(q,\xi) \mapsto t_0\Big(\sum_{i=1}^n \xi^i E_i|_q\Big)$$

and

$$i(q) = \min\{f(q,\xi) | \xi \in \mathbb{S}^{n-1}\}.$$

Thus i is a composition of continous functions and hence continuous. This also implies the second statment. $\square$

4.3 Theorem. Let $q \in C(p)$ such that $d(p,q) = i(M)$. Then one of the following is true:

(i) q is conjugate to p along a minimizing geodesic.

(ii) There are precisely two unit speed minimizing geodesics c, $\bar{c}$ from p to q and

$$\dot{c}(t_0) = -\dot{\bar{c}}(t_0),$$

where $t_0 = d(p,q) = i(p)$.

Proof. Assume the first statement does not hold. By Lemma 3.6 there exist two unit speed geodesics from p to q. The second statement follows, provided that we can show $\dot{c}(t_0) = -\dot{\bar{c}}(t_0)$ for any two such unit speed minimizing geodesics c and $\bar{c}$.

Step 1: Suppose to the contrary that $v_1 := \dot{c}(t_0) \neq -\dot{\bar{c}}(t_0) =: -v_2$. Then there exists a vector $w \in T_pM$ such that

$$\langle w, \dot{c}(t_0)\rangle, \langle w, \dot{\bar{c}}(t_0)\rangle < 0.$$

This can be seen as follows: By hypothesis $\|v_1\| = \|v_2\| = 1$ and consequently the Cauchy/Schwarz inequality implies

$$|\langle v_1, v_2\rangle| \leq \|v_1\|\|v_2\| = 1.$$

[6] remind that t_0 is continuous by 3.9

Since equality holds if and only if v_1, v_2 are linear dependent, which in our case holds if and only if $v_1 = \pm v_2$, strict inequality holds. Define $w := -(v_1 + v_2) \neq 0$. Then

$$\langle v_1, w\rangle = -\langle v_1, v_1\rangle - \langle v_1, v_2\rangle = -1 - \langle v_1, v_2\rangle < 0$$

and similar $\langle v_2, w\rangle < 0$.
STEP 2: Now let $\sigma :]-\varepsilon, \varepsilon[\to M$ be a smooth curve satisfying $\sigma(0) = q$, $\dot\sigma(0) = w$. Since q is not conjugate to p along c and $\bar c$, $\exp_p$ is locally invertible in $t_0\dot c(0)$ and $t_0\dot{\bar c}(t_0)$. Thus there exists $0 < \delta \leq \varepsilon$ and smooth curves $v, \bar v :]-\delta, \delta[\to T_pM$, such that

$$\forall s \in]-\delta, \delta[: \exp_p(v(s)) = \exp_p(\bar v(s)) = \sigma(s).$$

By construction $v(0) = t_0\dot c(0)$ and $\bar v(0) = t_0\dot{\bar c}(0)$. We define geodesic variations $H, \bar H :]-\delta, \delta[\times [0,1] \to M$ by

$$H(s,t) := \exp_p(tv(s)) \qquad \bar H(s,t) := \exp_p(t\bar v(s)).$$

By definition

$$H(0,t) = \exp_p(tv(0)) = \exp_p(tt_0\dot c(0)) = c_{tt_0\dot c(0)}(1) = c_{\dot c(0)}(tt_0) = c(tt_0)$$

and similar $\bar H(0,t) = \bar c(tt_0)$.
STEP 3: We claim, that the first variation formula implies

$$L(H(s,\_)) < L(H(0,\_)) \qquad L(\bar H(s,\_) < L(\bar H(0,\_))$$

for any $s > 0$ sufficiently small. Since H is a variation of the geodesic $c(tt_0)$ and since the variation is constant at the left, all terms in the variation formula vanish except

$$\begin{aligned}\partial_s L(H(s,\_))|_{s=0} &= \langle \partial_s H(s,1)|_{s=0}, \partial_t^-(c(tt_0))|_{t=1}\rangle = \langle \partial_s(\exp_p(v(s)))|_{s=0}, \dot c(t_0)t_0\rangle \\ &= t_0\langle \dot\sigma(0), \dot c(t_0)\rangle = t_0\langle w, \dot c(t_0)\rangle < 0.\end{aligned}$$

Here the last inequality holds by construction of w. Thus $s \mapsto L(H(s,\_))$ is strictly decreasing in a neighbourhood of 0. Analogously this also holds for $\bar H$.
Furthermore

$$L(H(0,\_)) = \int_0^1 \|\partial_t H(0,t)\|dt = \int_0^1 \|\partial_t(c(tt_0))\|dt = \int_0^1 \|\dot c(tt_0)t_0\|dt = t_0$$

and

$$L(H(s,\_)) = \int_0^1 \|\partial_t \exp_p(tv(s))\|dt = \int_0^1 \|\partial_t c_{tv(s)}(1)\|dt = \int_0^1 \|\partial_t c_{v(s)}(t)\|dt = \|v(s)\|$$

and by the same token $L(\bar H(s,\_)) = \|\bar v(s)\|$.
Alltogether this implies

$$\|v(s)\| = L(H(s,\_)) < L(H(0,\_)) = t_0 = i(p)$$

and similar for $\|\bar v(s)\|$. But $\exp_p(v(s)) = \exp_p(\bar v(s))$, so $\exp_p$ is not injective on the ball with radius $i(p)$. Contradiction! $\square$

4.4 Corollary. Let $p \neq q \in M$ such that $d(p,q) = \mathrm{diam}(M)$. Then one of the following is true:

(i) q is conjugate to p along a minimizing geodesic;

(ii) There exists a closed unit speed geodesic c through p and q satisfying $c(0) = c(2t_0) = p$, $c(t_0) = q$. We have $\dot{c}(2t_0) = \dot{c}(0)$ by definition.

These statements motivate to look for bounds for the first conjugate locus along geodesics from below. We will show, that curvature bounds from above yield those estimates.

4.5 Lemma. Let $c : [0, b] \to M$ be a unit speed geodesic, let $\kappa : [0, b] \to \mathbb{R}$ be continuous and for all $t \in [0, b]$ and all tangential 2-planes $\sigma \in T_{c(t)}M$ satisfying $\dot{c}(t) \in \sigma$ let

$$K(\sigma) \leq \kappa(t).$$

Let $0 \neq J \in \mathcal{T}(c^{\perp})$ be a Jacobi field and let $f : [0, b] \to \mathbb{R}$ satisfy

$$f'' + \kappa f = 0 \qquad f(0) = \|J(0)\| \qquad f'(0) = \|J\|'(0) := \lim_{t \searrow 0} \|J\|'(t).$$

If $f > 0$ on $]0, a]$, $a \leq b$, then for any $t \in]0, a]$

(i) $\frac{\|J\|'}{\|J\|}(t) = \frac{\langle J, J' \rangle}{\langle J, J \rangle}(t) \geq \frac{f'}{f}(t)$.

(ii) For any $0 < s < t$: $\frac{\|J\|}{f}(s) \leq \frac{\|J\|}{f}(t)$.

(iii) $\|J(t)\| \geq f(t)$, so there are no conjugate points on $c|_{[0,a]}$.

Proof. First assume, that J has no zeros $]0, t[$.
Step 1: Then we may calculate

$$\begin{aligned} \|J\|'' &= \left(\frac{\langle J', J \rangle}{\|J\|} \right)' = \frac{1}{\|J\|^2} \left(\langle J'', J \rangle \|J\| + \langle J', J' \rangle \|J\| - \frac{\langle J', J \rangle^2}{\|J\|} \right) \\ &= \frac{1}{\|J\|^3} \left(-Rm(J, \dot{c}, \dot{c}, J)\|J\|^2 + \underbrace{\|J'\|^2\|J\|^2 - \langle J', J \rangle^2}_{\geq 0 \text{ ,C.S.}} \right) \geq -K(J \wedge \dot{c})\|J\| \geq -\kappa\|J\|. \end{aligned}$$

Thus we obtain on $]0, t[$

$$(\|J\|'f - \|J\|f')' = \|J\|''f + \|J\|'f' - \|J\|'f' - \|J\|f'' = \|J\|''f + \kappa\|J\|f \geq 0.$$

So $\|J\|'f - \|J\|f'$ is monotonously increasing on $]0, t[$ and $\|J\|'(0)f(0) - \|J(0)\|f'(0) = 0$. We have shown

$$\|J\|'f - \|J\|f' \geq 0$$

on $[0, t]$.
Step 2: This implies

$$\left(\frac{\|J\|}{f} \right)' = \frac{\|J\|'f - f'\|J\|}{f^2} \geq 0,$$

and therefore $\frac{\|J\|}{f}$ is monotonously increasing on $[0, t]$.
Step 3: Since $J \neq 0$ by hypothesis one of the two following cases must hold.
Case 1: If $J(0) \neq 0$, there is a small neighbourhood $[0, \varepsilon]$, $\varepsilon > 0$, on which J has no zeros. Consequently all steps above hold on $[0, \varepsilon]$ and since

$$\frac{\|J\|(0)}{f(0)} = \frac{\|J(0)\|}{\|J(0)\|} = 1.$$

Step 2 implies $\|J\| \geq f$ on $[0, \varepsilon]$. But now this inequality holds on $]0, t[$ for all t such that J has no zeros on $]0, t[$. Meaning if $\|J\|(t) = 0$, this implies $f(t) \leq 0$ and therefore $t > a$, which cannot happen by hypothesis. Alltogether this implies that J has no zeros on $]0, a]$ and all the steps above imply the statement.

CASE 2: If $J(0) = 0$, this implies $J'(0) \neq 0$, since we are assuming $J \neq 0$. By Corollary A.12 we may write $J(t) = tX(t)$ where $J'(0) = X(0) \neq 0$. Therefore

$$\begin{aligned}\|J\|'(0) &= \lim_{t \searrow 0} \|J\|'(t) = \lim_{t \searrow 0} \frac{\langle J'(t), J(t)\rangle}{\|J(t)\|} = \lim_{t \searrow 0} \frac{\langle X(t) + tX'(t), tX(t)\rangle}{\|tX(t)\|} \\ &= \lim_{t \searrow 0} \frac{\langle X(t), X(t)\rangle + t\langle X(t), X'(t)\rangle}{\|X(t)\|} = \|X(0)\| = \|J'(0)\|. \end{aligned} \tag{4.1}$$

Thus we may use l'Hôpital's rule to calculate

$$\lim_{s \searrow 0} \frac{\|J\|(s)}{f(s)} = \lim_{s \searrow 0} \frac{\|J\|'(s)}{f'(s)} = \frac{\|J\|'(0)}{f'(0)} = 1.$$

Now we may argue as in the first case.

□

4.6 Remark. If equality holds in (i) or (iii) for some $t > 0$ or in (ii) for some pair $0 < s < t$, then $J = fE$ on $[0, t]$, where E is parallel along c satisfying $E \perp \dot{c}$, $\|E\| = 1$ and $K(E \wedge \dot{c}) = \kappa$.

The case $\kappa \equiv \text{const}$ is of particular importance.

4.7 Theorem. Employing the same notation and hypothesis of the preceeding Theorem 4.5, we obtain for any $\kappa \equiv \text{const}$ (c.f. 2.6 and 2.7):

(i) Rauch's comparison theorem for the curvature bounded from above: If $J(0) = 0$, $J'(0) \perp \dot{c}(0)$ and $\|J'(0)\| = 1$, then
$$\forall t \in]0, R_\kappa[: \|J(t)\| \geq \text{sn}_\kappa(t).$$

(ii) Berger's comparison theorem for the curvature bounded from above: If $J(0) \perp \dot{c}(0)$, $\|J(0)\| = 1$ and $J'(0) = 0$, then
$$\forall t \in]0, L_\kappa[: \|J(t)\| \geq \text{cs}_\kappa(t).$$

Proof. This is a direct consequence of Theorem 4.5 and Lemma 2.7 since $f = \text{sn}_\kappa$ resp. $f = \text{cs}_\kappa$. □

4.8 Corollary. Let all the sectional curvatures K of M satisfy $K \leq \kappa \equiv \text{const}$ and let $p \in M$. Then for any $w, X \in T_pM$, $0 < \|w\| < R_\kappa$, we obtain

$$X \perp w \Rightarrow \|(\exp_p)_*|_w(X)\| \geq \frac{\text{sn}_\kappa(\|w\|)}{\|w\|}\|X\|.$$

Proof. If $X = 0$, the statement is trivial. If $X \neq 0$, we apply Lemma A.8 to $Y := \frac{X}{\|X\|}$ and obtain

$$(\exp_p)_*|_w(X) = \frac{1}{\|w\|} J(\|w\|)\|X\|,$$

where J is the Jacobi field through p satisfying $J(0) = 0$ and $J'(0) = Y$. So the conclusion follows from comparison theorem 4.7,(i), since $\|J'(0)\| = 1$. □

4.9 Remark. In case $K = \kappa$ the Jacobi fields may be computed explicitly (c.f. [2, 10.8]) and one may check that in that case equality holds.

This corollary shows, that manifolds with $K \leq \kappa$ are "larger" in a certain sense, than manifolds with $K \equiv \kappa$. We will make this intuition more precise in the following theorem.

4.10 Theorem (Volume comparison)**.** Let M_0 be a complete Riemannian manifold having constant sectional curvature κ and let M be complete having sectional curvature $K \leq \kappa$. Let $p_0 \in M_0$, $p \in M$ and $\varepsilon \in]0, R_\kappa[$ sufficiently small such that $\exp_{p_0} : B_\varepsilon(0_{p_0}) \to B_\varepsilon(p_0) \subset M_0$ and $\exp_p : B_\varepsilon(0_p) \to B_\varepsilon(p)$ are both diffeomorphisms. Let $I : T_{p_0}M_0 \to T_pM$ be a linear isometry and let

$$F := \exp_p \circ I \circ \exp_{p_0}|_{B_\varepsilon(p_0)}^{-1} : B_\varepsilon(p_0) \to B_\varepsilon(p).$$

Let $q \in B_\varepsilon(p_0)$ and $X \in T_qM_0$. Then

$$\|F_*X\| \geq \|X\|$$

In particular if $\varepsilon < i(p)$, then $\mathrm{vol}(B_\varepsilon(p_0)) \leq \mathrm{vol}(B_\varepsilon(p))$ and equality implies, that F is an isometry. In that case $K \equiv \kappa$ on $B_\varepsilon(p)$.

If we compare volumes, we assume, that ε is sufficiently small, such that M_0 and M are oriented on these small balls and that I preserves orientation. These hypothesis simplify working with integration.

Proof.
Step 1 (Representation by Jacobi fields): By construction F is a diffeomorphism, so

$$\exists w \in B_\varepsilon(0_{p_0}) \subset T_{p_0}M_0 : q = \exp_{p_0}(w) \qquad \text{and} \qquad \exists Y \in T_{p_0}M_0 : X = (\exp_{p_0})_*|_w(Y).$$

We denote $w = \|w\|v$ mit $\|v\| = 1$. Then (c.f. A.8)

$$X = (\exp_{p_0})_*|_w(Y) = \frac{1}{\|w\|}J_0(\|w\|),$$

where J_0 is the Jacobi field along c_v satisfying $J_0(0) = 0$ and $J_0'(0) = Y$. By the same token

$$(\exp_p)_*|_{I(w)}(IY) = \frac{1}{\|w\|}J(\|w\|),$$

where J is the Jacobi field along c_{Iv} in M satisfying $J(0) = 0$ and $J'(0) = IY$. Alltogether

$$F_*|_q(X) = (\exp_p)_* \circ I \circ (\exp_{p_0}|_{B_\varepsilon(p_0)}^{-1})_*((\exp_{p_0})_*|_wY) = (\exp_p)_*|_{I(w)}(IY) = \frac{1}{\|w\|}J(\|w\|).$$

Step 2 (Estimate on the norm): If $w = 0$, we have $F_*|_q = I$ and the conclusion follows.
Step 2.1: If $w \neq 0$ and $Y = \lambda w \in \mathrm{Lin}_\mathbb{R}(w) \subset T_{p_0}M_0$, Gauss' Lemma (c.f. A.10) implies

$$\begin{aligned}\|F_*|_q(X)\| = \|{\exp_p}_*|_{I(w)}(I(Y))\| = |\lambda|\|{\exp_p}_*|_{I(w)}(I(w))\| &\overset{A.10}{=} |\lambda|\|I(w)\| = |\lambda|\|w\| \\ &\overset{A.10}{=} |\lambda|\|{\exp_{p_0}}_*|_w(w)\| = \|{\exp_{p_0}}_*|_w(\lambda w)\| = \|X\| \geq \|X\|.\end{aligned}$$

Step 2.2: Next we consider the case $Y \perp w$, $w, Y \neq 0$ (if $Y = 0 \Leftrightarrow X = 0$ again the statement is trivial). Let E be a unit length parallel vector field along c_v satisfying $E(0) = \frac{Y}{\|Y\|}$. Then

$$J_0 = \mathrm{sn}_\kappa \|Y\|E,$$

since M_0 has constant sectional curvature κ. So $\mathrm{sn}_\kappa \|Y\|E$ is a Jacobi field (c.f. [2, 10.8]) satisfying the initial conditions

$$\mathrm{sn}_\kappa(0)\|Y\|E(0) = 0 = J_0(0) \qquad D_t(\mathrm{sn}_\kappa \|Y\|E)(0) = \dot{\mathrm{sn}}_\kappa(0)\|Y\|E(0) = Y = D_tJ_0(0)$$

by construction. Thus

$$\|X\| = \|{\exp_{p_0}}_*|_w(Y)\| = \frac{1}{\|w\|}\|J_0(\|w\|)\| = \frac{1}{\|w\|}\|\,\mathrm{sn}_\kappa(\|w\|)\|Y\|E(\|w\|)\| = \frac{\mathrm{sn}_\kappa(\|w\|)}{\|w\|}\|Y\|.$$

Alltogether we obtain (with J as obove), that

$$\|F_*|_q(X)\| = \frac{1}{\|w\|}\|J(\|w\|)\| = \|(\exp_p)_*|_{I(w)}(I(Y))\| \overset{4.8}{\geq} \frac{\mathrm{sn}_\kappa(\|w\|)}{\|w\|}\|Y\| = \|X\|.$$

Step 2.3: In the general case we may orthogonally decompose Y by

$$Y = Y^T + Y^\perp \in \mathrm{Lin}_{\mathbb{R}}(w) \oplus w^\perp,$$

which corresponds to an orthogonal decomposition of X (again by Gauss' Lemma A.10)

$$X = X^T + X^\perp := \exp_{p_0 *}|_w(Y^T) + \exp_{p_0 *}|_w(Y^\perp).$$

Alltogether this implies

$$\|F_*|_q(X)\|^2 = \|F_*|_q(X^T) + F_*|_q(X^\perp)\|^2 = \|F_*|_q(X)^T + F_*|_q(X)^\perp\|^2 = \|X^T\|^2 + \|X^\perp\|^2 = \|X\|^2.$$

Step 3: Provided $\varepsilon < i(p)$, the maps $\exp_{p_0}|_{B_\varepsilon(0_{p_0})}^{-1} = x_0$ and $\exp_p \circ I|_{B_\varepsilon(0_{p_0})}^{-1} = x$ are charts on $U_0 = B_\varepsilon(p_0)$ rsp. $U = B_\varepsilon(p)$. With respect to these charts and the first part

$$\det_g(F(q)) \geq \det_{g_0}(q),$$

where g_0 and g are the fundamental forms of the metrics with respect to these coordinates.

□

4.11 Theorem. Let $p \in M$, $\kappa \in \mathbb{R}$, $0 < r < R < R_\kappa$, $U \subset (M \setminus (C(p) \cup \{p\})) \cap B_R(p)$ and $K \leq \kappa$ on U. Let $S_r(p) := \{q \in M \mid d(p,q) = r\}$ and $q \in S_r(p) \cap U$. We denote by c_q the unit speed minimizing geodesic from p to q and by h the scalar second fundamental from of S_r w.r.t. $-\dot{c}_q(r)$. Then

$$\forall X \in T_qS_r(p) : h(X,X) \geq \mathrm{ct}_\kappa(r)\|X\|^2.$$

Proof. Let $q = \exp_p(rv)$ and $X = (\exp_p)_*|_{rv}(Y)$ such that $v, Y \in T_pM$, $\|v\| = 1$ and assume $\|Y\| = 1$ (otherwise one has to replace Y by $\frac{Y}{\|Y\|}$ in the following). By Gauss' Lemma A.10 $Y \perp v$. Define a geodesic variation $H : \mathbb{R} \times [0,r] \to M$ of c_q by

$$H(s,t) := \exp_p(t(\cos(s)v + \sin(s)Y))$$

Then $\partial_t H(s,r)$ is the outward pointing unit normal to $H(s,r) \in S_r(p)$. In particular $H(0,r) = q$, $\partial_t H(0,t)|_{t=r} = \dot{c}_q(r)$ and $\partial_s H|_{s=0} =: J$ is a variation field of a variation through geodesics. Consequently J is a Jacobi field along $H(0, \_) = c_v$ satisfying $J(0) = 0$.

$$\begin{aligned}
&D_tJ(0) = D_t\partial_s \exp_p(t(\cos(s)v + \sin(s)Y))|_{s=0}|_{t=0} = D_s\partial_t \exp_p(t(\cos(s)v + \sin(s)Y))|_{t=0}|_{s=0} \\
&\quad = D_s(\cos(s)v + \sin(s)Y)|_{s=0} = Y.
\end{aligned}$$

Thus $\|J(0)\| = 0$ and $\|D_tJ(0)\| = \|Y\| = 1$. We may apply Lemma 4.5,(i) to $f = \mathrm{sn}_\kappa$ and obtain using Lemma 2.7, that

$$\frac{\langle J(r), D_tJ(r)\rangle}{\langle J(r), J(r)\rangle} \geq \frac{f'(r)}{f(r)} = \mathrm{ct}_\kappa(r).$$

Notice that

$$J(r) = \partial_s H(s,r)|_{s=0} = \partial_s(\exp_p(r(\cos(s)v + \sin(s)Y)) = \exp_{p_*}|_{rv}(Y) = X$$

and $\dot{c}_q(r) = \partial_t H(0,t)|_{t=r}$. Alltogether we obtain

$$\begin{aligned}
h(X,X) = -\langle -\nabla_X \dot{c}_q(r), X\rangle &= \langle D_s\partial_t H(s,t)|_{t=r}|_{s=0}, J(r)\rangle = \langle D_t\partial_s H(s,r)|_{s=0}|_{t=r}, J(r)\rangle \\
&= \langle J'(r), J(r)\rangle \geq \mathrm{ct}_\kappa(r)\|J(r)\|^2.
\end{aligned}$$

□

4.12 Corollary. We assume the same hyothesis as in Theorem 4.11. Then the geodesic spheres $S_r(p)$ are level sets of the distance function $f = d_p = d(p, \_)$ and we obtain

$$\operatorname{Hess} d_p(X, X) = h(X, X) \geq \operatorname{ct}_\kappa(r)\|X\|^2,$$

where $r := d_p(q)$ and $X \perp \dot{c}_q(r)$. Furthermore

$$\forall X \in T_qM : \operatorname{Hess} d_p(X, \dot{c}_q(r)) = 0.$$

Proof. By Example 1.2 f is a distance function and we obtain $\operatorname{grad} f|_q = \dot{c}_q(r)$. Thus the claim follows from Theorem 1.5 and 4.11. □

Calculating $\operatorname{Hess} d_p$ using this case differentiation is a bit inconvenient. We will apply "Karcher's Trick", a modification of d_p, in order to obtain a uniform estimate.

4.13 Lemma. For any $\kappa \in \mathbb{R}$ we define $m_\kappa : \mathbb{R} \to \mathbb{R}$

$$m_\kappa(r) := \int_0^r \operatorname{sn}_\kappa(t) dt.$$

(i) We have the explicit formulae

$$m_\kappa(r) = \begin{cases} \frac{1}{\kappa}(1 - \operatorname{cs}_\kappa(r)) & , \kappa \neq 0 \\ \frac{1}{2}r^2 & , \kappa = 0 \end{cases}$$

and

$$m'_\kappa = \operatorname{sn}_\kappa \qquad\qquad m''_\kappa = \operatorname{cs}_\kappa .$$

(ii) The function m_κ is monotonously increasing on $[0, R_\kappa]$.

(iii) We obtain the identity

$$\operatorname{cs}_\kappa + \kappa m_\kappa = 1.$$

Proof.

(i) If $\kappa \neq 0$, we have $\operatorname{sn}_\kappa = -\frac{1}{\kappa} \operatorname{cs}'_\kappa$ by Lemma 2.7 and thus the statement follows from the fundamental theorem of calculus. The case $\kappa = 0$ follows similiarly; here $\operatorname{sn}_\kappa(t) = t$. By differentiating again, we obtain the other equalities.

(ii) This is a direct consequence of Lemma 2.7 since $\operatorname{sn}_\kappa > 0$ on $[0, R_\kappa]$.

(iii) If $\kappa \neq 0$, we obtain according to (i)

$$\operatorname{cs}_\kappa + \kappa m_\kappa = \operatorname{cs}_\kappa + 1 - \operatorname{cs}_\kappa = 1.$$

In case $\kappa = 0$ by Lemma 2.7

$$\operatorname{cs}_\kappa + \kappa m_\kappa = \operatorname{cs}_\kappa = 1.$$

□

4.14 Lemma (Karcher's Trick)**.** Let $\kappa \in \mathbb{R}$, $c : I \to U \subset M$ be a unit speed geodesic in U, $p \in M$ and $d_p : M \to \mathbb{R}$, $q \mapsto d(p, q)$. Define $r, e : I \to \mathbb{R}$ and $l : M \to \mathbb{R}$ by

$$r := d_p \circ c \qquad\qquad e := m_\kappa \circ r \qquad\qquad l := m_\kappa \circ d_p.$$

Then

(i) We always obtain the identity

$$e'' + \kappa e = (\mathrm{cs}_\kappa \circ r)\langle \operatorname{grad} d_p|_c, \dot{c}\rangle^2 + (\mathrm{sn}_\kappa \circ r) \operatorname{Hess} d_p(\dot{c}, \dot{c})$$

(ii) In case we additionally assume the hypothesis of Theorem 4.11, we obtain the estimate

$$e'' + \kappa e \geq 1.$$

(iii) We obtain the uniform estimate

$$\forall q \in M : \forall X \in T_q M : \operatorname{Hess} l(X, X) \geq (1 - \kappa l)\|X\|^2.$$

Proof.

(i) Differentiation yields

$$e'' = ((m_\kappa' \circ r) \cdot r')' = (m_\kappa'' \circ r) r'^2 + (m_\kappa' \circ r) \cdot r'' \overset{\text{(i)}}{=} (\mathrm{cs}_\kappa \circ r) r'^2 + (\mathrm{sn}_\kappa \circ r) \cdot r''.$$

By definition

$$r'(t) = (d_p \circ c)'(t) = \langle \operatorname{grad} d_p|_{c(t)}, \dot{c}(t)\rangle$$

and

$$\begin{aligned} r''(t) &= \langle \operatorname{grad} d_p|_{c(t)}, \dot{c}(t)\rangle' = \langle D_t \operatorname{grad} d_p|_{c(t)}, \dot{c}(t)\rangle = \langle \nabla_{\dot{c}(t)} \operatorname{grad} d_p|_{c(t)}, \dot{c}(t)\rangle \\ &\overset{1.4}{=} \operatorname{Hess} d_p(\dot{c}(t), \dot{c}(t)). \end{aligned}$$

(ii) In that case we may continue by

$$r''(t) = \operatorname{Hess} d_p(\dot{c}(t), \dot{c}(t)) \overset{4.12}{\geq} (\mathrm{ct}_\kappa \circ r)(t)\|\dot{c}(t)^\perp\|^2,$$

where $\dot{c}^\perp$ is the component of $\dot{c}$ perpendicular to $\operatorname{grad} d_p$ and thus tangential to the corresponding geodesic sphere near p. Denoting by $\dot{c}^T$ the component tangential to $\operatorname{grad} d_p$, we obtain alltogether

$$\begin{aligned} e'' &\geq (\mathrm{cs}_\kappa \circ r)\langle \operatorname{grad} d_p|_c, \dot{c}\rangle^2 + (\mathrm{sn}_\kappa \circ r) \cdot (\mathrm{ct}_\kappa \circ r)\|\dot{c}^\perp\|^2 = (\mathrm{cs}_\kappa \circ r)(\langle \operatorname{grad} d_p|_c, \dot{c}\rangle^2 + \|\dot{c}^\perp\|^2) \\ &= (\mathrm{cs}_\kappa \circ r)(\|\dot{c}^T\|^2 + \|\dot{c}^\perp\|^2) = (\mathrm{cs}_\kappa \circ r)\|\dot{c}\|^2 = (\mathrm{cs}_\kappa \circ r). \end{aligned}$$

Thus the claim follows by

$$e'' + \kappa e \geq \mathrm{cs}_\kappa \circ r + \kappa \cdot m_\kappa \circ r \overset{4.13,\text{(iii)}}{=} 1.$$

(iii) The inequality is invariant under scaling of X, so we may assume that $\|X\| = 1$. Let $c :]-\varepsilon, \varepsilon[\to M$ be a unit speed geodesic satisfying $c(0) = q$, $\dot{c}(0) = X$. By definition (where the r is taken w.r.t. this c):

$$l \circ c = m_\kappa \circ d_p \circ c = m_\kappa \circ r = e.$$

So by definition

$$e'(t) = dl|_{c(t)}(\dot{c}(t)) = \langle \operatorname{grad} l|_{c(t)}, \dot{c}(t)\rangle$$

and

$$e''(0) = \langle \operatorname{grad} l|_{c(t)}, \dot{c}(t)\rangle'(0) = \langle D_t \operatorname{grad} l|_{c(t)}(0), X\rangle = \langle \nabla_X \operatorname{grad} l|_q, X\rangle = \operatorname{Hess} l(X, X).$$

Thus by (ii)

$$\operatorname{Hess} l(X, X) = e''(0) \geq 1 - \kappa e(0) = 1 - \kappa l(q).$$

$\square$

4.1 Inverse Theorem of Toponogov

4.15 Definition (geodesic triangle)**.** A *geodesic triangle* $\Delta = (c_1, c_2, c_3)$ consists of three geodesic segments $c_1 : [a_1, b_1] \to M$, $c_2 : [a_2, b_2] \to M$, $c_3 : [a_3, b_3] \to M$, such that

$$c_1(a_1) = c_2(a_2), \qquad c_1(b_1) = c_3(a_3), \qquad c_2(b_2) = c_3(b_3).$$

If Δ is a geodesic triangle in M and $\bar{\Delta}$ is a geodesic triangle in $\bar{M}$, then $\bar{\Delta}$ is a *comparison triangle*, if

$$\forall i = 1, 2, 3 : L(c_i) = L(\bar{c}_i).$$

Two triangles in M are *congruent* , if there exists an isometry of M such that one triangle is mapped to the second.

4.16 Definition (model spaces)**.** Endow $\mathbb{R}^n$ with the euclidian metric. For any $R > 0$ let

$$\mathbb{S}^n_R := \{x \in \mathbb{R}^{n+1} \mid \|x\| = R\}$$

be the sphere of radius R endowed with the restriction of the Euclidean metric of $\mathbb{R}^{n+1}$. Furthermore denote by

$$H^n_R := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | - x_0^2 + x_1^2 + \ldots + x_n^2 = -R^2\}$$

the hyperbolic space of radius R endowed with the restriction of the Minkowski metric of $\mathbb{R}^{n+1}$.
For any $\kappa \in \mathbb{R}$ the Riemannian manifold

$$M^n_\kappa := \begin{cases} \mathbb{S}^n(\frac{1}{\sqrt{\kappa}}) := \mathbb{S}^n_{\frac{1}{\sqrt{\kappa}}} \subset \mathbb{R}^{n+1} & , \kappa > 0 \\ \mathbb{R}^n & , \kappa = 0 \\ H^n(\kappa) := H^n_{\frac{1}{\sqrt{-\kappa}}} & , \kappa < 0 \end{cases}$$

is the *n-dimensional model space with constant curvature κ.* (They are unique in a sense elaborated later, c.f. 11.15.)

4.17 Lemma. Let $\Delta = (c_1, c_2, c_3)$ be a triangle in M having side lengths $l_i := L(c_i)$ and let $\kappa \in \mathbb{R}$. Let

$$l_1 + l_2 + l_3 < R_\kappa, \qquad l_i + l_j \geq l_k,$$

where (i, j, k) runs through all permutations of $(1, 2, 3)$. Then there exists a comparison trianlge $\bar{\Delta}$ in M^2_κ for Δ, which is unique up to congruence.

4.18 Theorem ("inverse Toponogov's Theorem")**.** Let $p \in M$, $\kappa \in \mathbb{R}$, $K \leq \kappa$ and $R < i(p), R_\kappa$. Let $\Delta = (c_1, c_2, c)$ be a geodesic triangle in $B_R(p)$ consisting of unit speed geodesics c_1, c_2 and c, such that $c_1(0) = c_2(0) = p$, $c_1(l_1) = c(0) =: q_1$, $c_2(l_2) = c(l) =: q_2$ and $l_1 + l_2 + l < R_\kappa$.
Then

$$l_1 + l_2 \geq l, \qquad l_1 + l \geq l_2, \qquad l_2 + l \geq l_1,$$

and the comparison triangle $\bar{\Delta}$ in M^2_κ satisfies

(i) $d(p, c(t)) \leq d(\bar{p}, \bar{c}(t))$,

(ii) $\alpha_i \leq \bar{\alpha}_i$, $i = 1, 2$, where α_i rsp. $\bar{\alpha}_i$ is the angle in Δ rsp. $\bar{\Delta}$ at q_i rsp. $\bar{q}_i$.

Proof. Since c_1, c_2 are both geodesics in $B_R(p)$ starting at p, we obtain $l_1, l_2 < R \leq i(p)$. Consequently c_1,c_2 are minimizing in M, which implies $l_1 \leq l + l_2$, $l_2 \leq l + l_1$.

Case 1: First we will prove the claims (i) und (ii) under the additional hypothesis

$$l \leq l_1 + l_2.$$

By Lemma 4.17 there exists a comparison triangle $\bar{\Delta} = (\bar{c}_1, \bar{c}_2, \bar{c})$ in M^2_κ which is unique up to congruence. Let $\bar{e} = m_\kappa \circ d(\bar{p}, \bar{c})$ be the modified distance function around $\bar{p} = \bar{c}_i(0)$, $i = 1, 2$, along $\bar{c}$ in M^2_κ and again $e = m_\kappa \circ d(p, c)$. We obtain:

$$e(0) = m_\kappa(l_1) = m_\kappa(\bar{l}_1) = \bar{e}(0) \qquad\qquad e(l) = m_\kappa(l_2) = m_\kappa(\bar{l}_2) = \bar{e}(l).$$

Consequently $f := \bar{e} - e$ satisfies $f(0) = f(l) = 0$ and by Karcher's trick 4.14, we obtain [7]

$$\ddot{f} + \kappa f = \underbrace{(\ddot{\bar{e}} + \kappa \bar{e})}_{=1} - \underbrace{(\ddot{e} - \kappa e)}_{\geq 1} \leq 0.$$

This implies $f \geq 0$ by Lemma 4.21. Since m_κ is monotonous, we have shown (i).
Claim (ii) is a consequence of the first energy variation formula: Denote by $\gamma_t : [0,1] \to B_R(p)$ the radial geodesic from p to $c(t)$ and by $\bar{\gamma}_t$ the corresponding geodesic in M^2_κ. Then for any $t \in [0,1]$

$$E(\gamma_t) = \frac{1}{2} \int_0^1 \|\dot{\gamma}_t(s)\|^2 ds = \frac{1}{2} L(\gamma_t)^2 = \frac{1}{2} d(p, \gamma(t))^2 \overset{(i)}{\leq} \frac{1}{2} d(\bar{p}, \bar{\gamma}(t))^2 = E(\bar{\gamma}(t))$$

and $E(\gamma_0) = E(\bar{\gamma}_0)$. Thus $\partial_t E(\gamma_0)|_{t=0} \leq \partial_t E(\bar{\gamma}_0)|_{t=0}$ as well. By the first variation formula (admitting non-proper variations)

$$\partial_t E(\gamma_0)|_{t=0} = \langle \partial_t \gamma_t(1)|_{t=0}, \partial_t^- c_1(l_1) \rangle = \langle \partial_t^+ c(0), \partial_t^- c_1(l_1) \rangle = \alpha_1$$

and analogously for α_2 and $\bar{\alpha}_i$, $i = 1, 2$.
Case 2: In case $l \geq l_1 + l_2$ define $l' := l_1 + l_2$, $c' := c|_{[0,l']}$ and let c_2' be the minimizing radial geodesic from p to $c(l')$. The comparison triangle for $\Delta' = (c_1, c_2', c')$ in M^2_κ is degenerate, because $l' = l_1 + l_2 < R_\kappa$. By (i) the triangle Δ' is itself degenerate and thus $l = l'$. We have reduced this case to the first case.

□

4.19 Remark (omitting the proof)**.** Denote by $|\Delta|$ the region bounded by Δ. Equality in (i) for some $t \in]0, l[$ or equality in (ii) for some i implies, that there exists a totally geodesic isometric immersion $F : M^2_\kappa \supset |\bar{\Delta}| \to B_R(p)$ such that $F \circ \bar{c} = c$ and $F \circ \bar{c}_i = c_i$, $i = 1, 2$.

4.20 Corollary. Let M be simpliy connected having sectional curvature $K \leq 0$. The sum of all angles in a geodesic triangle Δ is less or equal to π. Equality implies, that there is a totally geodesic isometric imersion $F : |\bar{\Delta}| \subset \mathbb{R}^2 \to M$ such that $F(|\bar{\Delta}|) = |\Delta|$.

Proof. By the theorem of Hadamard-Cartan M is diffeomorphic to $\mathbb{R}^n$. Thus $i(p) = \infty$ for any $p \in M$. So the first part follows from 4.18 and the second from Remark 4.19 (which we have not proven here). Furthermore we are using the face, that the sum of the interior angles of a triangle in M^2_κ is always $\leq \pi$ provided $\kappa \leq 0$ (this is a consequence of the Gauss-Bonnet Theorem). □

4.21 Lemma. Let $\kappa \in \mathbb{R}$ and $f : [0, l] \to \mathbb{R}$ be a solution of

$$\ddot{f} + \kappa f \leq 0$$

satisfying $f(0) = f(l) = 0$, $l < R_\kappa$. Then $f \geq 0$. If

[7] Actually the identity $\ddot{\bar{e}} + \kappa \bar{e} = 1$ is not really proven there. One may either check by hand, that if $K = \kappa$ in 4.14 one obtains not only $\ddot{\bar{e}} + \kappa \bar{e} \geq 1$, but $= 1$. Alternatively one may anticipatory use Lemma 7.2, which states that in case $K \geq \kappa$ one obtains $\ddot{\bar{e}} + \kappa \bar{e} \leq 1$, which alltogether shows, that if $\kappa \leq K \leq \kappa$ we obtain $\ddot{\bar{e}} + \kappa \bar{e} = 1$.

(i) $\exists t \in]0,l[\colon f(t) = 0$ or

(ii) $f'(0) = 0$ or

(iii) $f'(l) = 0$,

then $f \equiv 0$.

Proof.

STEP 1: Choose $\varepsilon > 0$ such that $l < R_{\kappa+\varepsilon}$. There exists a positive solution g of

$$\ddot{g} + (\kappa + \varepsilon)g = 0$$

on $[0,l]$, e.g. $g = s_{\kappa+\varepsilon}(t+\delta)$ for sufficiently small $\delta > 0$. We define $h := \frac{f}{g} : [0,l] \to \mathbb{R}$, remark that of course $f = gh$, and so the hypothesis implies

$$0 \geq \ddot{f} + \kappa f = (\dot{g}h + g\dot{h})' + \kappa gh = \ddot{g}h + 2\dot{g}\dot{h} + \kappa gh = (\ddot{g} + \kappa g)h + 2\dot{g}\dot{h} + g\ddot{h} = -\varepsilon gh + 2\dot{g}\dot{h} + g\ddot{h}.$$

Suppose there exists $t \in]0,l[$ such that $f(t) < 0$. Then $h(t) < 0$ as well (since $g \geq 0$), so h has a negative minimum at some $t_0 \in]0,l[$ (since [0,l] is compact and $f(0) = f(l) = 0$). This implies $h(t_0) < 0$, $\dot{h}(t_0) = 0$ and $\ddot{h}(t_0) \geq 0$, which contradicts the estimate above. This proves the first statement.

STEP 2: We discuss the various cases

STEP 2.1 ($f'(0) = 0$): By what we have proven so far $f \geq 0$. Assume there exists $t_0 \in]0,l[$, such that $f(t_0) > 0$. Let k be a solution of

$$\ddot{k} + \kappa k = 0$$

satisfying $k(0) = 0$, $k(t_0) = f(t_0)$, i.e. $k = \frac{f(t_0)}{\mathrm{sn}_\kappa(t_0)} \mathrm{sn}_\kappa$. Then

$$(f-k)'' + \kappa(f-k) = \ddot{f} + \kappa f - (\ddot{k} + \kappa k) \leq 0$$

and thus $f - k \geq 0$ on $[0,t_0]$ by what we have proven so far. Thus $\dot{f}(0) \geq \dot{k}(0) = \frac{f(t_0)}{s_\kappa(t_0)} > 0$ as well. This contradicts our choice of t_0.

STEP 2.2 ($f'(l) = 0$): We proceed in a similar fashion: Assume there exists $t_0 \in]0,l[$, such that $f(t_0) > 0$. Again let k be the solution of $\ddot{k} + \kappa k = 0$, but now satisfying $k(t_0) = f(t_0)$ and $k(l) = 0$, i.e. $k(t) = \frac{f(t_0)}{\mathrm{sn}_\kappa(l-t_0)} \mathrm{sn}_\kappa(l-t)$. In a similar fashion $k - f \geq 0$ on $[t_0,l]$. This implies $f - k \leq 0$ on $[t_0,l]$ and $(f-k)(l) = 0$. Thus $\dot{f}(l) \leq \dot{k}(l) = -\frac{f(t_0)}{\mathrm{sn}_\kappa(l-t_0)} \mathrm{sn}'_\kappa(0) < 0$. Contradiction!

STEP 2.3 ($f(t) = 0$): If $f(t) = 0$, then $\dot{f}(t) = 0$ as well since f has a minimum at t. By what we have proven so far, we obtain on the one hand $f|_{[0,t]} \equiv 0$ and on the other hand $f|_{[t,l]} \equiv 0$, thus $f \equiv 0$.

□

5 Growth of Fundamental Group and Volume

In this section M is a compact connected Riemannian manifold and $\pi : \tilde{M} \to M$ is a universal Riemannian covering (,which implies that both are complete).

5.1 Definition. Let G be a finitely generated group and let $S \subset G$ be a finite generating system. For any $g \in G$, $g \neq e$, define

$$\|g\|_S := \min\{m \geq 0 \mid \exists s_1, \ldots, s_m \in S : g = s_1^{\pm 1} \ldots s_m^{\pm 1}\} \in \mathbb{N}.$$

For the neutral element $e \in G$ we will employ the convention $\|e\|_S := 0$.
For any $R \geq 0$ define

$$N_S(R) := \#\big\{g \in G \,\big|\, \|g\|_S \leq R\big\}.$$

5.2 Lemma. Let G be a finitely generated group and let $S, T \subset G$ be two finite generating systems. Then there exists a constant $k \geq 1$ such that

$$\forall g \in G : \frac{1}{k}\|g\|_T \leq \|g\|_S \leq k\|g\|_T.$$

Furthermore for any $R \geq 0$

$$N_T\Big(\frac{R}{k}\Big) \leq N_S(R) \leq N_T(kR).$$

Proof. First of all we remark, that for any $g, h \in G$ we always have

$$\|gh\|_S \leq \|g\|_S + \|h\|_S$$

(in general equality does not hold). Define

$$k := \max\big\{\max\{\|s\|_T \,|\, s \in S\}, \max\{\|t\|_S \,|\, t \in T\}\big\}.$$

Let $\|g\|_S = m$ and

$$g = s_1^{\pm 1} \ldots s_m^{\pm 1}$$

be a representation of g as a product of (maybe inverse) elements of S. Now any s_i, $1 \leq i \leq m$ can be written as a product of at most k (maybe inverse) elements of T. Therefore

$$\|g\|_T = \|s_1^{\pm 1} \ldots s_m^{\pm 1}\|_T \leq \|s_1^{\pm 1}\|_T \ldots \|s_m^{\pm 1}\|_T \leq km = k\|g\|_S.$$

The other inequality is obtained by interchanging the roles of S and T.
This implies the second statement via

$$\begin{aligned} \|g\|_T \leq \frac{R}{k} &\Longrightarrow \|g\|_S \leq k\|g\|_T \leq k\frac{R}{k} = R \\ \|g\|_S \leq R &\Longrightarrow \|g\|_T \leq k\|g\|_S \leq kR. \end{aligned}$$

$\square$

5.3 Definition (Growth of a function)**.** Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a function. We say f has

(i) *exponential growth*, if

$$\liminf_{R \to \infty} \frac{1}{R} \ln(f(R)) > 0,$$

(ii) *at least polynomial growth of degree n*, if

$$\liminf_{R \to \infty} \frac{f(R)}{R^n} > 0,$$

(iii) *at most polynomial growth of degree* n, if

$$\limsup_{R\to\infty} \frac{f(R)}{R^n} < \infty.$$

5.4 Definition (Growth of a group)**.** Let G be a finitely generated group and let $S \subset G$ be a finite generating systems. Then G has *exponential growth* rsp. *at least polynomial growth of degree* n rsp. *at most polynomial growth of degree* n, if the function $f := N_S$ has these growth properties.

5.5 Remark. By Lemma 5.2 these properties are independent of the choice of generators and thus growth is a well-defined property of the group. By the way, it is shown in [5, 1-7], that $\frac{1}{R}\ln(N_S(R))$ converges as $R \to \infty$.

Our aim in this section is to establish a relationship between the growth of the fundamental group of M and the growth of the volume in M. Before we start, we remind you of the following result from Differential Geometry I:

5.6 Lemma. Let $p \in M$ and $r \leq i(p)$. Then for any $\tilde{p}, \tilde{q} \in \pi^{-1}(p)$

$$B_r(\tilde{p}) \cap B_r(\tilde{q}) = \emptyset$$

and $\pi : B_r(\tilde{p}) \to \tilde{B}_r(p)$ is an isometric diffeomorphism.

In addition to that, we remind the following classical result from Topology

5.7 Theorem. Let $\pi : \hat{X} \to X$ be a covering, $x \in X$, $\hat{x} \in \pi^{-1}(x)$ and let Δ_π be the group of covering transformations of. Then the map $\Psi : \mathcal{N}(\pi_\#(\pi_1(\hat{X}, \hat{x})) \to \Delta_\pi$, sending a homotopy class $[c] \in \mathcal{N}(\pi_\#(\pi_1(\hat{X}, \hat{x})) \subset \pi_1(X, x)$ to the unique covering transformation $D \in \Delta_\pi$, which sends the point $\hat{x}$ to $[c].\hat{x}$, is well-defined and surjective with kernel $\pi_\#(\pi_1(\hat{X}, \hat{x}))$.

Here $\mathcal{N}$ is the normalisator and $[c].\hat{x}$ is the monodromy action. You can find more about this theorem in [4, 11.30].

In our case $\pi : \tilde{M} \to M$, the base space $\tilde{M}$ is simply connected and therefore for any fixed $p \in M$, $\tilde{p} \in \pi^{-1}(p)$ the map $\Psi : \pi_1(M, p) \to G$, $G := \Delta_\pi$, is an isomorphism. Its inverse is given by $\Phi : G \to \pi_1(M, p)$, which is defined as follows: For any $g \in G$ we obtain a point $\tilde{q} := g(\tilde{p}) \in \pi^{-1}(p)$. Since $\tilde{M}$ is simply connected, there exists a path $\tilde{c}$ from $\tilde{p}$ to $\tilde{q}$ which is unique up to homotopy. In this situation $\Phi(g) = [\pi \circ \tilde{c}]$.
Provided $\tilde{c}$ is piecewise smooth, $L(\tilde{c}) = L(\pi \circ c)$ since π is a local isometry.

5.8 Lemma. In our situation $\pi : \tilde{M} \to M$ we denote for any $g \in G$ and $\tilde{p} \in \tilde{M}$

$$\|g\|_{\tilde{p}} := d(\tilde{p}, g(\tilde{p})).$$

Then for any $g, h \in G$

$$\|gh\|_{\tilde{p}} \leq \|g\|_{\tilde{p}} + \|h\|_{\tilde{p}}.$$

Proof. Since g is an isometry of $\tilde{M}$, we obtain

$$\|gh\|_{\tilde{p}} = d(\tilde{p}, gh\tilde{p}) \leq d(\tilde{p}, g\tilde{p}) + d(g\tilde{p}, gh\tilde{p}) = d(\tilde{p}, g\tilde{p}) + d(\tilde{p}, h\tilde{p}) = \|g\|_{\tilde{p}} + \|h\|_{\tilde{p}}.$$

□

We may think of $\|g\|_{\tilde{p}}$ as the length of the shortest geodesic loop in the homotopy class of loops based at $\tilde{p}$, which are determined by g.

5.9 Theorem (Growth of deck transformations)**.** Remind that M is compact, $\pi : \tilde{M} \to M$ is a universal covering and $\tilde{p} \in \tilde{M}$. For any $r \geq 0$ define

$$S_r := S_r(\tilde{p}) := \{g \in G \,|\, \|g\|_{\tilde{p}} \leq r\}$$

and $d := \operatorname{diam} M$. Then

(i) S_{2d} is a finite generating system of G.

(ii) For $S = S_{3d}$ and $g \in G$ we obtain $d(\|g\|_S - 1) \leq \|g\|_{\tilde{p}} \leq 3d\|g\|_S$.

Proof.

Step 1 (Finiteness): By Lemma 5.6 the set

$$X = \{g(p) \mid g \in G\} = \pi^{-1}(p)$$

is discrete in $\tilde{M}$. Thus for any $r \geq 0$ the set $X \cap \bar{B}_r(\tilde{p})$ is discrete and compact, hence finite. Since $g \in S_r(\tilde{p}) \Leftrightarrow g(\tilde{p}) \subset \bar{B}_r(\tilde{p})$, we obtain that

$$|S_r(\tilde{p})| = |X \cap \bar{B}_r(\tilde{p})|$$

is finite as well. It remains to show, that S_{2d} generates G.

Step 2 (Premilinaries): Let $\tilde{q} \in \tilde{M}$ be arbitary. We claim, there exists $g \in G$ such that

$$d(\tilde{q}, g(\tilde{p})) \leq d.$$

This can be seen as follows: Define $q := \pi(\tilde{q})$. Since M is complete, there exists a minimizing geodesic $c : [0,1] \to M$ satisfying $c(0) = q$, $c(1) = p$. Thus

$$L(c) \leq d(q,p) \leq \operatorname{diam}(M) = d.$$

Let $\tilde{c}$ be the lift of c at $\tilde{q}$. Then $L(\tilde{c}) = L(c)$ and $\pi(\tilde{c}(1)) = c(1) = p$, thus $\tilde{c}(1) \in \pi^{-1}(p)$. Consequently there exists $g \in G$ such that $\tilde{c}(1) = g(\tilde{p})$. Alltogether we obtain

$$d(\tilde{q}, g(\tilde{p})) \leq L(\tilde{c}) = L(c) \leq d.$$

Step 3 (Generating system): Let $g \in G$ be arbitrary, $r := d(\tilde{p}, g(\tilde{p}))$ and $\tilde{c} : [0,r] \to \tilde{M}$ be a unit speed minimizing geodesic from $\tilde{p}$ to $g(\tilde{p})$. Let $\varepsilon > 0$ be arbitrary and let $k \in \mathbb{N}$, such that $k\varepsilon \leq r < (k+1)\varepsilon$. For any $1 \leq i \leq k$ there exists $g_i \in G$ such that

$$d(\tilde{c}(i\varepsilon), g_i(\tilde{p})) \leq d$$

by step 1. Thus

$$\|g_1\|_{\tilde{p}} = d(\tilde{p}, g_1(\tilde{p})) \leq d(\tilde{p}, \tilde{c}(\varepsilon)) + d(c(\varepsilon), g_1(\tilde{p}))) \leq \varepsilon + d.$$

For any $2 \leq i \leq k$ we obtain

$$\begin{aligned}\|g_{i-1}^{-1} g_i\|_{\tilde{p}} &= d(\tilde{p}, g_{i-1}^{-1} g_i(\tilde{p})) = d(g_{i-1}(\tilde{p}), g_i(\tilde{p})) \\ &\leq d(g_{i-1}(\tilde{p}), \tilde{c}((i-1)\varepsilon)) + d(\tilde{c}(i-1)\varepsilon, \tilde{c}(i\varepsilon)) + d(\tilde{c}(i\varepsilon), g_i(\tilde{p})) \leq d + \varepsilon + d = 2d + \varepsilon,\end{aligned}$$

since g_{i-1} is an isometry. Furthermore

$$\|g_k^{-1} g\|_{\tilde{p}} = d(\tilde{p}, g_k^{-1} g(\tilde{p})) = d(g_k(\tilde{p}), g(\tilde{p})) \leq d(g_k(\tilde{p}), c(k\varepsilon)) + d(c(k\varepsilon), g(\tilde{p})) \leq d + \varepsilon,$$

by choice of r. Alltogether we obtain

$$g = g_1(g_1^{-1} g_2 \cdot g_2^{-1} g_3 \ldots g_{k-1}^{-1} g_k) g_k^{-1} g$$

and

$$g_1, g_1^{-1} g_2, g_2^{-1} g_3, \ldots, g_{k-1}^{-1} g_k, g_k^{-1} g \in S_{2d+\varepsilon}.$$

Thus $S_{2d+\varepsilon}$ is a generating system of G, where $\varepsilon > 0$ was arbitrary. By step 1 $|S_r(\tilde{p})| = |X \cap \bar{B}_r(\tilde{p})|$. Thus if ε is sufficiently small $S_{2d} = S_{2d+\varepsilon}$. This proves (i).

Step 4 (Estimate): Again let $\tilde{c} : [0, r] \to \tilde{M}$ be as in step 2 and let $k \in \mathbb{N}$, such that

$$kd \leq r < (k+1)d.$$

Again by step 1 choose $g_i \in G$, $1 \leq i \leq k$, such that

$$d(\tilde{c}(id), g_i(\tilde{p})) \leq d$$

and as in step 3 (replacing ε with d) obtain analogously

$$\|g_1\|, \|g_1^{-1}g_2\|, \ldots, \|g_{k-1}^{-1}g_k\|, \|g_k^{-1}g\| \leq 3d.$$

Thus if $S = S_{3d}$, we obtain

$$\|g\|_S \leq k + 1 \leq \frac{1}{d}\|g\|_{\tilde{p}} + 1$$

by choice of k and $r = \|g\|_{\tilde{p}}$. This establishes the left inequality.
To prove the right one we remark, that by (i) applied to $S = S_{3d}$ there exsits a representation $g = g_1^{\pm 1} \ldots g_l^{\pm 1}$, with $g_i \in S$ and l minimal. This implies

$$\|g\|_{\tilde{p}} = \|g_1^{\pm 1} \ldots g_l^{\pm 1}\|_{\tilde{p}} \leq \|g_1\|_{\tilde{p}} + \ldots + \|g_1\|_{\tilde{p}} \leq 3dl = 3d\|g\|_S.$$

□

We would like formulate step 2 of the preceeding proof as a corollary.

5.10 Corollary. For any $\tilde{p}, \tilde{q} \in \tilde{M}$ ther exists $g \in G$, such that

$$d(\tilde{q}, g(\tilde{p})) \leq d.$$

5.11 Corollary. We assume the same hypothesis as in the preceeding Theorem 5.9 and define

$$N_{\tilde{p}}(R) := \#\{g \in G \mid \|g\|_{\tilde{p}} \leq R\} = |S_R|,$$

where $R \geq 0$ and $S := S_{3d}$. Then

$$N_S\left(\frac{R}{3d}\right) \leq N_{\tilde{p}}(R) \leq N_S\left(\frac{R}{d} + 1\right).$$

In particular the growth of N_S equals the growth of $N_{\tilde{p}}$.

Proof. By Theorem 5.9 we have the implications

$$\|g\|_S \leq \frac{R}{3d} \Longrightarrow \|g\|_{\tilde{p}} \leq 3d\|g\|_S \leq R \qquad \|g\|_{\tilde{p}} \leq R \Longrightarrow \|g\|_S \leq \frac{1}{d}\|g\|_{\tilde{p}} + 1 \leq \frac{R}{d} + 1.$$

□

We are now able to compare the growth of $N_{\tilde{p}}$ with the volume in $\tilde{M}$.

5.12 Theorem (Volume comparison)**.** Remind that M is compact, $\pi : \tilde{M} \to M$ is a universal covering, and let $d := \mathrm{diam}(M)$, $p \in M$, $r = i(p)$, $v := \mathrm{vol}(B_r(\tilde{p})) = \mathrm{vol}(B_r(p))$, $V := \mathrm{vol}(B_d(\tilde{p}))$. Then

$$\frac{1}{v}\,\mathrm{vol}(B_{R+r}(\tilde{p})) \geq N_{\tilde{p}}(R) \geq \frac{1}{V}\,\mathrm{vol}(B_{R-d}(\tilde{p})),$$

where we require $R \geq 0$ for the left inequality and $R \geq d > 0$ for the right one.

Proof.

STEP 1: For any $g \neq h \in G$ we have $g(p) \neq g(h)$ and by Lemma 5.6 $B_r(g(\tilde{p})) \cap B_r(h(\tilde{p})) = \emptyset$. Since $\pi|_{B_r(\tilde{p})}$ is an isometry, we obtain $\mathrm{vol}(B_r(g(\tilde{p}))) = \mathrm{vol}(B_r(p))$. Provided $\|g\|_{\tilde{p}} \leq R$, we obtain

$$\tilde{q} \in B_r(g(\tilde{p})) \Longrightarrow d(\tilde{q}, g(\tilde{p})) < r \Longrightarrow d(\tilde{q}, \tilde{p}) \leq d(\tilde{q}, g(\tilde{p})) + d(g(\tilde{p}), \tilde{p}) < r + R \Longrightarrow \tilde{q} \in B_{R+r}(\tilde{p})$$

thus $B_r(g(\tilde{p})) \subset B_{R+r}(\tilde{p})$. So we obtain

$$\mathrm{vol}(B_{R+r}(\tilde{p})) \geq \mathrm{vol}\left(\bigcup_{g \in S_R(\tilde{p})} B_r(g(\tilde{p}))\right) = N_{\tilde{p}}(R)\,\mathrm{vol}(B_r(p)),$$

since the balls are pairwise disjoint. This proves the left inequality.

Conversely by Corollary 5.10 for any $\tilde{q} \in \tilde{M}$ there exists $g \in G$, such that $d(\tilde{q}, g(\tilde{p})) \leq d$. If $\tilde{q} \in B_{R-d}(\tilde{p})$, we even have

$$\|g\|_{\tilde{p}} = d(\tilde{p}, g(\tilde{p})) \leq d(\tilde{p}, \tilde{q}) + d(\tilde{q}, g(\tilde{p})) \leq R - d + d = R \Rightarrow g \in S_R.$$

Thus

$$B_{R-d}(\tilde{p}) \subset \bigcup_{g \in S_R(\tilde{p})} \bar{B}_d(g(\tilde{p})),$$

which imlies

$$\mathrm{vol}(B_{R-d}(\tilde{p})) \leq N_{\tilde{p}}(R)\,\mathrm{vol}(\bar{B}_d(\tilde{p})).$$

□

5.13 Theorem. Let M be compact and denote by K the sectional curvatures. Then

(i) If $K \leq -a^2$, $a > 0$, then $\pi_1(M)$ grows exponentially.

(ii) If $K \leq 0$, then $\pi_1(M)$ has at least polynomial growth.

Proof.

(i) We may assume $a = 1$, since otherwise we may rescale the metric (c.f. Theorem A.15).[8] By Theorem 5.7 the fundmental group $\pi_1(M)$ is isomorphic to G, where G is the group of covering transformations on the universal Riemannian covering $\pi : \tilde{M} \to M$. By Theorem 5.9 the set $S := S_{3d}$, where $d := \operatorname{diam} M$, is a finite generating system for G. By Corllary 5.10 the growth of N_S equals the growth of $N_{\tilde{p}}$. By Theorem 5.12 the growth of $N_{\tilde{p}}$ equals the growth of $B_R(\tilde{p})$. By Theorem 4.10, we obtain

$$\mathrm{vol}(B_R(\tilde{p})) \geq \mathrm{vol}(B_R(\hat{p})),$$

where $\hat{p} \in M^n_{-1}$ is a point in hyperbolic n space with constant curvature -1. Finally these volumes will be analysed in more detail in the following chapter and we will show in Corollary 6.8 that $B_R(\hat{p})$ growths exponentially.

(ii) This follows by the same token and the well known growth of volume in Euclidean space.

□

[8] This causes no problem since the conclusion of the theorem does not involе the metric explicitely. The topology of M is merely required to admit a metric of such curvature.

6 Volume and Ricci Curvature

In this section let M be a connected complete Riemannian manifold of dimension m.

6.1 Definition (measure zero)**.** A set $A \subset M$ is a *set of measure zero*, if there exists and index set $I \cong \mathbb{N}$ and charts $\varphi_i : U_i \to U_i'$, such that $A \subset \bigcup_{i \in I} U_i$, and for any $i \in I$: $\varphi_i(A \cap U_i) \subset \mathbb{R}^n$ is a set of measure zero w.r.t. the Lebesgue measure.

We remark, that the following results from classical calculus also hold on manifolds:

6.2 Lemma. Let N be a manifold and let $F : M \to N$ be smooth. If $A \subset M$ is a set of measure zero, then $F(A) \subset N$ is a set of measure zero as well. If $f : M \to \mathbb{R}$ is integrable and $A \subset M$ is a set of measure zero

$$\int_M f = \int_{M \setminus A} f.$$

6.3 Corollary. Let $p \in M$. Then $C_T(p) \subset T_pM$ and $C(p) = \exp_p(C_T(p)) \subset M$ are sets of measure zero. Define

$$U := U_p := \{(v,t) \in \mathbb{S}_pM \times \mathbb{R} \mid 0 < t < t_0(v)\} \qquad V := M \setminus (\{p\} \cup C(p)).$$

Then $F : U \to V$, $F(v,t) := \exp_p(tv)$, is a diffeomorphism and

$$\mathrm{vol}(B_R(p)) = \mathrm{vol}(B_R(p) \cap V).$$

We want to calculate The later using F.

Proof. The first statement is a direct consequence of Lemma 6.2 above, since exp is smooth and $C_T(p)$ is a set of measure zero. By Theorem 3.11 F is a diffeomorphism. The equality of the volumes is also a consequence of the Lemma above. $\square$

We enlist some general remarks concerning the tranformation of integrals.

6.4 Definition. Let $X_1, \ldots, X_m \in T_pM$ be a basis. We denote by

$$X_1 \wedge \ldots \wedge X_m := \Big\{ \sum_{i=1}^m t_i X_i \mid t_1, \ldots, t_m \in [0,1] \Big\}$$

the *parallelepiped spanned by* $X_1, \ldots, X_m$.
Let M and N be Riemannian manifolds, $F : M \to N$ be a diffeomorphism and $p \in M$. Then

$$\mathrm{Jac}\, F(p) := \frac{\mathrm{vol}(F_*|_p(X_1|_p) \wedge \ldots \wedge F_*|_p(X_m|_p))}{\mathrm{vol}(X_1|_p \wedge \ldots \wedge X_m|_p)}$$

is the *Jacobian of* F.

6.5 Lemma. Under this hypothesis

$$\mathrm{vol}(X_1 \wedge \ldots \wedge X_m) = \sqrt{\det(\langle X_i, X_j \rangle)}$$

and $\mathrm{Jac}(p)$ does not depend on the choice of basis $X = (X_1, \ldots, X_m)$.

Proof. Since for any permutation $\sigma \in S_m$

$$X_{\sigma(1)} \wedge \ldots \wedge X_{\sigma(m)} = X_1 \wedge \ldots \wedge X_m$$

as equality of sets, we may assume that the map defined by $X_j \mapsto e_j$, $1 \le j \le m$, is an orientation-preserving isomorphism $\varphi : T_pM \to \mathbb{R}^m$ and a global chart of the Riemannian manifold (T_pM, g). Let $P := X_1 \wedge \ldots \wedge X_m$ and $Q := e_1 \wedge \ldots \wedge e_m \subset \mathbb{R}^m$. By definition of the Riemannian metric on T_pM

$$\begin{aligned}
\operatorname{vol}(X_1 \ldots X_m) &= \operatorname{vol}(P) = \int_P 1 dV = \int_P \sqrt{\det{(g_{ij})}} d\varphi^i \wedge \ldots \wedge d\varphi^m \\
&= \int_Q (\varphi^{-1})^* (\sqrt{\det{(g_{ij})}} d\varphi^i \wedge \ldots \wedge d\varphi^m) = \int_Q \sqrt{\det{(\langle X_i, X_j \rangle)}} dx = \sqrt{\det{(\langle X_i, X_j \rangle)}}.
\end{aligned}$$

This implies the second statement: If $Y = (Y_1, \ldots, Y_m)$ is another basis of T_pM, there exists an automorphism A of T_pM, $X_i \mapsto AX_i := Y_i$. Denote by $c_X(A)$ its coordinate matrix as an endomorphism w.r.t. the basis X. The matrices $c_Y(g) := (\langle Y_i, Y_j \rangle)$ and $c_X(g) := (\langle X_i, X_j \rangle)$ are the coordinate matrices of g, which transform by elementary linear algebra as

$$\det(c_Y(g)) = \det(c_X(A)^t c_X(g) c_X(A)) = \det(c_X(A))^2 \det(c_X(g)).$$

A similar result holds for the bases F_*X and F_*Y: The automorphism $F_*AF_*^{-1}$ of $T_{F(p)}N$ transforms $F_*(X_j)$ into $F_*(Y_j)$ and therefore

$$F_*AF_*^{-1}(F_*X_j) = F_*(Y_j) = F_*\Big(\sum_{i=1}^n A_{ij} X_i\Big) = \sum_{i=1}^n A_{ij} F_*(X_i),$$

i.e. $c_{F_*X}(F_*AF_*^{-1}) = c_X(A)$. The factor $\det(c_X(A))^2$ cancels in the fraction of $\operatorname{Jac} F$. $\square$

6.6 Theorem (Transformation theorem for Riemannian manifolds)**.** Let M, N be Riemannian manifolds, $F : M \to N$ be a diffeomorphism and let $f : N \to \mathbb{R}$ be integrabel. Then $(f \circ F) \cdot \operatorname{Jac} F : M \to \mathbb{R}$ is integrabel and

$$\int_M (f \circ F) \operatorname{Jac} F = \int_N f.$$

Proof. Let $\psi : V \to V' \subset \mathbb{R}^n$ be a chart for N. Since F is a diffeomorhpism, $U := F^{-1}(V) \subset M$ is open and $\varphi := \psi \circ F : U \to U' = V'$ is a chart for M. In these charts F is represented by $\operatorname{id} : V' \to V'$, i.e.

$$\psi \circ F \circ \varphi^{-1} = \operatorname{id}.$$

Let $\partial\varphi_i$, $\partial\psi_i$, $1 \le i \le n$, be the corresponding coordinate frame on U rsp. V. Then $F_*(\partial\varphi_i) = \partial\psi_i$ since by construction

$$\begin{aligned}
&\forall q \in N : \forall \alpha \in \mathcal{C}^\infty(N) : (F_*(\partial\varphi_i))|_q(\alpha) = F_*|_{F^{-1}(q)}(\partial\varphi_i))(\alpha) = \partial\varphi|_{F^{-1}(q)}(\alpha \circ F) \\
&= \partial_i(\alpha \circ F \circ \varphi^{-1})|_{(\varphi \circ F^{-1})(q)} = \partial_i(\alpha \circ F \circ F^{-1} \circ \psi^{-1})|_{\psi(q)} = \partial\psi_i|_q(\alpha).
\end{aligned}$$

Let $B \subset U$ be measurable (i.e. $\varphi(B) \subset \mathbb{R}^n$ is measurable). Then

$$\begin{aligned}
&\int_B (f \circ F) \operatorname{Jac} F = \int_{\varphi(B)} (f \circ F \circ \varphi^{-1}) \operatorname{Jac} F \circ \varphi^{-1} \sqrt{\det g_{ij}} \circ \varphi^{-1} d\mathscr{L}^n \\
&= \int_{\varphi(B)} (f \circ \psi^{-1}) \frac{\sqrt{\det{(\langle F_*(\partial\varphi_i), F_*(\partial\varphi_j) \rangle)}} \circ \varphi^{-1}}{\sqrt{\det{(\langle \partial\varphi_i, \partial\varphi_j \rangle)}} \circ \varphi^{-1}} \sqrt{\det{(\langle \partial\varphi_i, \partial\varphi_j \rangle)}} \circ \varphi^{-1} d\mathscr{L}^n \\
&= \int_{\varphi(B)} (f \circ \psi^{-1}) \sqrt{\det{(\langle \partial\psi_i, \partial\psi_j \rangle)}} \circ F \circ \varphi^{-1} d\mathscr{L}^n \\
&= \int_{\psi(F((B))} (f \circ \psi^{-1}) \sqrt{\det{(\langle \partial\psi_i, \partial\psi_j \rangle)}} \circ \psi^{-1} d\mathscr{L}^n = \int_{F(B)} f.
\end{aligned}$$

Thus the transformation rule holds for any measurable subset, which is contained in a single coordinate domain. Now the general case follows directly from the definition of the integral. ☐

6.7 Theorem. Let $R \geq 0$, $p \in M$, $v \in \mathbb{S}_pM$ and let c_v be the geodesic through p with initial velocity v. Let $J_2, \ldots, J_n \in \mathcal{T}(c|_{[0,t_0(v)]})$ be the Jacobi fields along c_v satisfying $J_i(0) = 0$ and $J_i'(0) = E_i$, $2 \leq i \leq n$, where $E_1 := v$, $E_2, \ldots, E_n$ is a parallel ON frame along c_v. If we define

$$j_v : [0, t_0(v)] \to \mathbb{R}, \qquad\qquad j_v(t) := \mathrm{vol}(J_2(t) \wedge \ldots \wedge J_n(t)),$$

then

$$\mathrm{vol}(B_R(p)) = \int_{\mathbb{S}_pM} \int_0^{\min(R,t_0(v))} j_v(t) dt dv. \tag{6.1}$$

Proof. The set U_p from 6.3 is open in the Riemannian product $\mathbb{S}_pM \times \mathbb{R}$ and $F : U \to V$ (as in Corollary 6.3) is a diffeomorphism. Thus by Theorem 6.6

$$\mathrm{vol}(B_R(p)) = \int_{\mathbb{S}_pM} \int_0^{t_0(v)} \chi_{]0,R[}(t) \,\mathrm{Jac}\, F(v,t) dt dv = \int_{\mathbb{S}_pM} \int_0^{\min(R,t_0(v))} \mathrm{Jac}\, F(v,t) dt dv.$$

So we have to calculate $\mathrm{Jac}\, F(v,t)$. Now $\mathbb{S}_pM$ is the round sphere with radius 1 in T_pM, i.e.

$$\forall v \in \mathbb{S}_pM : T_v\mathbb{S}_p \cong \{X \in T_pM \mid X \perp v\}.$$

We have

$$T_{(v,t)}U = T_v\mathbb{S}_pM \oplus T_t\mathbb{R}$$

and this sum is orthogonal (w.r.t. the product metric). [9] We analyse F_* on both summands: For any $\alpha \in \mathcal{C}^\infty(V)$:

$$F_*|_{(v,t)}(\partial_s)(\alpha) = \partial_s(\alpha \circ F)|_{(v,t)} = \partial_s(\alpha \circ \exp_p(sv))|_{s=t} = \dot{c}_v(t)(\alpha)$$

thus $F_*|_{(v,t)}(\partial_s) = \dot{c}_v(t)$. For any $X \in T_v\mathbb{S}_pM \subset T_{(v,t)}U$ we have analogously

$$F_*|_{(v,t)}(X) = \partial_s(\exp_p(t(v + sX)))|_{s=0} =: J(t),$$

where J is the Jacobi field along c_v satisfying $J(0) = 0$ and $J'(0) = X$.
Now let $E_1 := v, E_2, \ldots, E_n$ be an ONB of T_pM. Then $\partial_t, E_2, \ldots, E_n$ is an ONB of $T_{(v,t)}U)$ and consequently

$$\mathrm{Jac}\, F(v,t) = \frac{\mathrm{vol}(F_*\partial_s(t) \wedge F_*E_2(t) \wedge \ldots \wedge F_*E_n(t))}{\mathrm{vol}(\partial_t \wedge E_2(t) \wedge \ldots \wedge E_n(t))} = \mathrm{vol}(\dot{c}_v(t) \wedge J_2(t) \wedge \ldots \wedge J_n(t)),$$

where $J_i(t)$ if the Jacobi field along c_v satisfying $J_i(0) = 0$, $J_i'(0) = E_i$. By Lemma A.9 the fields $J_2, \ldots, J_n$ are perpendicular to $\dot{c}_v(t)$ and in addition $\dot{c}_v(t)$ has unit length. Thus

$$\mathrm{Jac}\, F(v,t) = \mathrm{vol}(J_2(t) \wedge \ldots \wedge J_n(t)) = j_v(t),$$

where this volume is the one of an $(n-1)$-dimenisonal parallelepiped. Thus alltogether

$$\mathrm{vol}(B_R(p)) = \int_{\mathbb{S}_pM} \int_0^{\min(R,t_0(v))} j_v(t) dt dv.$$

☐

[9] We always employ the following convention: Let (M_1, g_1), (M_2, g_2) be two Riemannian manifolds. Then $M_1 \times M_2$ is canonically a manifold and we identify $T_{(p_1,p_2)}(M_1 \times M_2) \cong T_{p_1}M_1 \oplus T_{p_2}M_2$. We obtain a product metric $g := g_1 \oplus g_2$ by

$$\forall X = X_1 + X_2, Y = Y_1 + Y_2 \in T_{p_1}M_1 \oplus T_{p_2}M_2 : g(X,Y) := g_1(X_1,Y_1) + g_2(X_2,Y_2).$$

In this metric $T_{p_1}M_1 \perp T_{p_2}M_2$.

6.8 Corollary. Let M have constant curvature κ and let $E_1 = \dot{c}_v, E_2, \dots, E_n$ be a parallel ONB along c_v. Then $J_i(t) = \mathrm{sn}_\kappa(t)E_i(t)$, so $j_v(t) = \mathrm{sn}_\kappa^{n-1}(t)$. In particular, we obtain for our model spaces M^n_κ and $R < R_\kappa$, that

$$\mathrm{vol}(B_R(p)) = \mathrm{vol}(\mathbb{S}^{n-1}) \int_0^R \mathrm{sn}_\kappa(t)^{n-1} dt.$$

In particular if $\kappa = -1$, then

$$\lim_{R\to\infty} \frac{1}{R} \ln(\mathrm{vol}(B_R(p))) = n - 1.$$

Proof. To prove the first statement we use Theorem 6.7 above and the characterization of Jacobi fields on manifolds of constant curvature (c.f. [2, 10.8]). In addition using the representation of the metric on constant curvature manifolds (c.f. [2, 19.9]), we see, that the g-volume of the g-unit-sphere $\mathbb{S}_p \subset T_pM$ is indeed the same as the Euclidean volume of the Euclidean unit sphere.
If $\kappa = -1$, then $\mathrm{sn}_\kappa = \sinh$ by Lemma 2.7. So by the first part:

$$\begin{aligned} &\lim_{R\to\infty} \frac{1}{R} \ln(\mathrm{vol}(B_R(p))) = \lim_{R\to\infty} \frac{1}{R} \ln\Big(\mathrm{vol}(\mathbb{S}^{n-1}) \int_0^R \sinh(t)^{n-1} dt\Big) \\ &= \lim_{R\to\infty} \frac{\ln(\mathrm{vol}(\mathbb{S}^{n-1}))}{R} + \lim_{R\to\infty} \frac{1}{R} \ln\Big(\int_0^R \sinh(t)^{n-1} dt\Big) \\ &= \lim_{R\to\infty} \frac{1}{R} \ln\Big(\int_0^R \sinh(t)^{n-1} dt\Big). \end{aligned} \tag{6.2}$$

Now notice, that

$$\forall t \in \mathbb{R}_{\geq 0} : \sinh(t) = \frac{1}{2}\Big(e^t - e^{-t}\Big) \leq \frac{1}{2}e^t, \qquad \exists t_0 \in \mathbb{R}_{\geq 0} : \forall t \geq t_0 : \sinh(t) \geq \frac{1}{4}e^t$$

and for any $\lambda, t_1 \in \mathbb{R}_{>0}$

$$\begin{aligned} &\lim_{R\to\infty} \frac{1}{R} \ln\Big(\int_0^R (\lambda e^t)^{n-1} dt\Big) = \lim_{R\to\infty} \frac{1}{R} \ln\Big(\lambda^{n-1} \frac{1}{n-1} e^{t(n-1)}\Big|_{t_1}^R = \lim_{R\to\infty} \frac{1}{R} \ln\Big(e^{R(n-1)} - \underbrace{e^{t_1(n-1)}}_{=:\mu}\Big) \\ &= \lim_{R\to\infty} \frac{1}{R} \ln\Big(e^{R(n-1)} - \mu\Big). \end{aligned} \tag{6.3}$$

Clearly there exists $R_0 \in \mathbb{R}_{>0}$ such that for all $R \geq R_0 : e^{R(n-1)} - \mu \geq \frac{1}{2}e^{R(n-1)}$. Thus

$$n - 1 = \lim_{R\to\infty} \frac{1}{R} \ln(\frac{1}{2}e^{R(n-1)}) \leq \lim_{R\to\infty} \frac{1}{R} \ln\Big(e^{R(n-1)} - \mu\Big) \leq \lim_{R\to\infty} \frac{1}{R} \ln(e^{R(n-1)}) = n - 1.$$

Therefore we may continue (6.3) by $\lim_{R\to\infty} \frac{1}{R} \ln\Big(e^{R(n-1)} - e^{t_1(n-1)}\Big) = n - 1$. By the same token we may now estimate (6.2) to prove the claim. ☐

6.1 Bishop-Gromov Inequality

In this section we will discuss the celebrated Bishop-Gromov Inequality. Before we start let's discuss some preleminaries.

6.9 Convention. In the following we will assume a bounded Ricci curvature and employ the notation

$$\mathrm{Ric}_M \geq (n-1)\kappa :\Leftrightarrow \forall V \in \mathcal{T}(M) : \mathrm{Ric}_M(V, V) \geq (n-1)\kappa |V|^2$$

Under this assumption one may we calculate for $u_v = \frac{1}{n-1}\ln(j_v)'$, that

$$u_v' \leq -u^2 - \kappa \tag{6.4}$$

see 2.3. Equality in this equation implies, that the second fundamental form $U_v(t)$ of the geodesic sphere of radius t around p with respect to the inward pointing unit normal $-\dot{c}_v$, satisfies

$$\operatorname{tr}(U_v(t)^2) = \frac{1}{n-1}(\operatorname{tr}(U_v(t)))^2.$$

Since U_v is symmetric, this implies

$$U_v(t) = u(t)I_v(t), \tag{6.5}$$

where $I_v(t)$ is the identity on $\dot{c}(t)^\perp$. The comparison theory (c.f. Lemma 2.9) states that

$$u \leq \operatorname{ct}_\kappa, \tag{6.6}$$

since if $E_1 = \dot{c}_v$, $E_2, \ldots, E_n$ is a parallel ON frame along c_v and J_i, $2 \leq i \leq n$, are the Jacobi fields along c_v satisfying $J_i(0) = 0$,$J_i'(0) = E_i$, we obtain

$$J_i(t) = tE_i(t) + t^2 X_i(t), \tag{6.7}$$

where the X_i are smooth along c_v (one can reduce this to the corresponding statement of basic calculus by expressing the J_i as a linear combination of the Ej). This implies

$$j_v(t) = t^{n-1} + \varphi(t)t^n, \tag{6.8}$$

where φ is smooth. Therefore

$$\ln(j_v)'(t) = \frac{j_v'}{j_v} = (n-1)u_v \sim \frac{n-1}{t}, t \to 0. \tag{6.9}$$

Furthermore we obtain

$$\lim_{t \searrow 0} \frac{j_v(t)}{\operatorname{sn}_\kappa^{n-1}(t)} = 1 \tag{6.10}$$

uniformly in v. Now let's get back to (6.1). Define

$$f(v,t) := \begin{cases} \frac{j_v(t)}{\operatorname{sn}_\kappa^{n-1}(t)} & ,0 < t < t_0(v) \\ 0 & ,t > t_0(v) \end{cases}$$

and obtain for any $p \in M$, that

$$V_p(R) := \operatorname{vol}(B_R(p)) = \int_{\mathbb{S}_p} \int_0^R f(v,t) \operatorname{sn}_\kappa^{n-1}(t) dt dv. \tag{6.11}$$

6.10 Theorem (Bishop-Gromov Inequality)**.** Let M be complete, connected, $p \in M$, $\operatorname{Ric}_M \geq (n-1)\kappa$, $\kappa \in \mathbb{R}$, $0 \leq r \leq R \leq R_\kappa$, $V_p(R) := \operatorname{vol}(B_R(p))$, and let $V_\kappa(R)$ be the volume of a ball with radius R in M_κ^n. Then

$$\frac{V_p(R)}{V_\kappa(R)} \leq \frac{V_p(r)}{V_\kappa(r)}$$

and equality for one pair $0 < r < R < R_\kappa$ implies, that $B_p(R)$ is isometric to a ball of radius R in M_κ^n. Furthermore

$$\lim_{R \searrow 0} \frac{V_p(R)}{V_\kappa(R)} = 1.$$

6.11 Remark. Equality implies in particular that the sectional curvature on $B_R(p)$ is constant and equal to κ.

Proof. Let j_v as in Theorem 6.7. We obtain:

$$\frac{V_p(R)}{V_\kappa(R)} = \frac{\int_{\mathbb{S}_pM}\int_0^R f(v,t)\operatorname{sn}_\kappa(t)^{n-1}dtdv}{\operatorname{vol}(\mathbb{S}^{n-1})\int_0^R \operatorname{sn}_\kappa(t)^{n-1}dt}. \tag{6.12}$$

STEP 1 (Inequality): We start with

$$\ln\Big(\frac{j_v}{\operatorname{sn}_\kappa^{n-1}}\Big)' = \frac{\operatorname{sn}_\kappa^{n-1}}{j_v}\frac{j_v'\operatorname{sn}_\kappa^{n-1} - (n-1)\operatorname{sn}_\kappa^{n-2}\operatorname{cs}_\kappa j_v}{\operatorname{sn}_\kappa^{2(n-1)}} = \frac{1}{j_v}\frac{(u_v(n-1)j_v)\operatorname{sn}_\kappa^{n-1} - (n-1)\operatorname{sn}_\kappa^{n-2} j_v\operatorname{cs}_\kappa}{\operatorname{sn}_\kappa^{n-1}}$$
$$= \frac{u_v(n-1)\operatorname{sn}_\kappa^{n-1} - (n-1)\operatorname{sn}_\kappa^{n-2}\operatorname{cs}_\kappa}{\operatorname{sn}_\kappa^{n-1}} = (n-1)(u_v - \operatorname{ct}_\kappa) \overset{(6.6)}{\leq} 0.$$

Thus $\ln(\frac{j_v}{\operatorname{sn}_\kappa^{n-1}})$ is monotonously decreasing. Since ln is strictly monotonously increasing , $\frac{j_v}{\operatorname{sn}_\kappa^{n-1}}$ is monotonously decreasing as well. Thus the function $t \mapsto f(v,t)$ is monotonously decreasing on $]0, t_0(v)[$. Since the integral is monotonous as well, we obtain

$$\int_0^R f(v,R)\operatorname{sn}_\kappa(t)^{n-1}dt \leq \int_0^R f(v,t)\operatorname{sn}_\kappa(t)^{n-1}dt. \tag{6.13}$$

Thus

$$\partial_R\Big(\frac{\int_0^R f(v,t)\operatorname{sn}_\kappa(t)^{n-1}dt}{\int_0^R\operatorname{sn}_\kappa(t)^{n-1}dt}\Big) = \frac{f(v,R)\operatorname{sn}_\kappa(R)^{n-1}dt\int_0^R\operatorname{sn}_\kappa(t)^{n-1}dt - \operatorname{sn}_\kappa(R)^{n-1}\int_0^R f(v,t)\operatorname{sn}_\kappa(t)^{n-1}dt}{\Big(\int_0^R\operatorname{sn}_\kappa(t)^{n-1}dt\Big)^2}$$
$$= \frac{\operatorname{sn}_\kappa(R)^{n-1}}{\Big(\int_0^R\operatorname{sn}_\kappa(t)^{n-1}dt\Big)^2}\Big(\int_0^R f(v,R)\operatorname{sn}_\kappa(t)^{n-1}dt - \int_0^R f(v,t)\operatorname{sn}_\kappa(t)^{n-1}dt\Big) \overset{(6.13)}{\leq} 0.$$

Substituting this back into (6.12), we obtain

$$\partial_R\left(\frac{V_p(R)}{V_\kappa(R)}\right) = \int_{\mathbb{S}_pM}\frac{1}{\operatorname{vol}(\mathbb{S}^{n-1})}\partial_R\frac{\int_0^R f(v,t)\operatorname{sn}_\kappa(t)^{n-1}dt}{\int_0^R\operatorname{sn}_\kappa(t)^{n-1}dt}dv \leq 0$$

and thus the function $R \mapsto \frac{V_p(R)}{V_\kappa(R)}$ is monotonously decreasing.

STEP 2 (Limit): Choose an antiderivative F of $f(v,\_)\operatorname{sn}_\kappa(\_)^{n-1}$ and an antiderivative G of $\operatorname{sn}_\kappa(\_)^{n-1}$. By de l'Hospital:

$$\lim_{R\searrow 0}\frac{\int_0^R f(v,t)\operatorname{sn}_\kappa(t)^{n-1}dt}{\int_0^R\operatorname{sn}_\kappa(t)^{n-1}(t)dt} = \lim_{R\searrow 0}\frac{F(R)-F(0)}{G(R)-G(0)} = \lim_{R\searrow 0}\frac{F'(R)}{G'(R)}$$
$$= \lim_{R\searrow 0}\frac{f(v,t)\operatorname{sn}_\kappa(t)^{n-1}}{\operatorname{sn}_\kappa(t)^{n-1}} = \lim_{R\searrow 0} f(v,t) \overset{(6.10)}{=} 1.$$

STEP 3 (Equality): The hypothesis implies $f(v,t) = 1$ for any $v \in \mathbb{S}_pM$ and $0 < t < R$, thus $u = \operatorname{ct}_\kappa$ for any $v \in \mathbb{S}_p$ (on $]0,R[$) and $t_0(v) \geq R$. This implies $U_v = u_v I_v$ as pointed out in (6.5). Now $I_v' = 0$ and therefore

$$R(\_,\dot{c}_v)\dot{c}_v = -U_v' - U_v^2 = -(u_v' + u_v^2)I_v = \kappa I_v.$$

Thus the curvature tensor along c_v is the same as the curvature tensor in the case of constant curvature κ. If E is a parallel unit normal field along c_v. then $\operatorname{sn}_\kappa E$ is a Jacobi field along c_v. This holds for any $v \in \mathbb{S}_p$. Thus for any $q \in M_\kappa^n$, the map

$$\exp_p \circ A \circ \exp_q^{-1} : B_R(q) \to B_R(p),$$

where $A : T_qM_\kappa^n \to T_qM$ is an isometry, is a Riemannian isometry.

$\square$

6.12 Corollary. Let M be complete, connected and let $\mathrm{Ric}_M \geq (n-1)\kappa$, $\kappa > 0$. Then $\mathrm{vol}(M) \leq \mathrm{vol}(M^n_\kappa)$. If $\mathrm{vol}(M) = \mathrm{vol}(M^n_\kappa)$, then M is isometric to M^n_κ.

Proof. Remember that $R_\kappa := \pi/\sqrt{\kappa}$.
STEP 1 (Inequality): By the (strong) Theorem of Bonnet-Myers $\mathrm{diam}(M) \leq R_\kappa$. Therefore

$$M = \bar{B}_p(R_\kappa)$$

and we obtain

$$\mathrm{vol}(M) = \lim_{R \nearrow R_\kappa} \mathrm{vol}(B_p(R)).$$

The same holds for M^n_κ. Now $R \mapsto \frac{V_p(R)}{V_\kappa(R)}$ is monotonously decreasing by 6.10 and ≤ 1. Thus

$$\frac{\mathrm{vol}(M)}{\mathrm{vol}(M^n_\kappa)} = \lim_{R \to R_\kappa} \frac{V_p(R)}{V_\kappa(R)} \leq 1.$$

This shows the first statement.
STEP 2 (Preliminaries): Now let $\mathrm{vol}(M) = \mathrm{vol}(M^n_\kappa)$. Then

$$V_\kappa(R_\kappa) = \mathrm{vol}(M^n_\kappa) = \mathrm{vol}(M) = \mathrm{vol}(\bar{B}_p(R_\kappa) = V_p(R_\kappa)$$

and thus for any $0 < r < R_\kappa$

$$1 = \frac{V_p(R_\kappa)}{V_\kappa(R_\kappa)} \leq \frac{V_p(r)}{V_\kappa(r)} \leq 1.$$

Consequently for any $0 < R < R_\kappa$ the ball $B_R(p) \subset M$ is isometric to $B_R(q) \subset M^n_\kappa$ by Theorem 6.10. An isometry is given by $\exp_p \circ A \circ \exp_q^{-1} : M^n_\kappa \supset B_R(q) \to B_R(p) \subset M$ (as in the proof of 6.10), where $A : T_qM \to T_pM$ is an isometry.
STEP 3 (Constructing the isometry): We would like to show, that for reasons of continuity this map extends to a map $F : \bar{B}_{R_\kappa}(q) = M^n_\kappa \to M = \bar{B}_{R_\kappa(q)}$, which preserves distances and is an isometry on the open ball of radius R. First of all

$$B_{R_\kappa}(q) = \bigcup_{R < R_\kappa} B_R(q),$$

thus F is well-defined and isometric on $B_{R_\kappa}(q)$. In addition $M^n_\kappa \setminus B_{R_\kappa}(q) = \{\hat{q}\}$, where $\hat{q}$ is antipodal to q ist (since we are assuming $\kappa > 0$ M^n_κ is a sphere of radius $1/\sqrt{\kappa}$). For any $q_1, q_2 \in B_{R_\kappa}(q)$ the number $d(q_1, q_2)$ equals the infimum of all lengths of paths from q_1 to q_2, that are contained in $B_{R_\kappa}(q)$. Since F is isometric there, is preserves these lengths and therefore

$$d(F(q_1), F(q_2)) \leq d(q_1, q_2),$$

since the infimum of all lenghts of paths in M connecting $F(q_1)$ and $F(q_2)$ is less or equal to the infimum of all lengths of paths, which connect $F(q_1)$ and $F(q_2)$ in the image of F. Thus F is continuous on $B_{R_\kappa}(q)$ with Lipschitz constant $= 1$. Of course $B_{R_\kappa}(q) \subset \bar{B}_{R_\kappa}(q)$ is dense, thus we may extend F to a continuous map $F : M^n_\kappa \to M$.
STEP 4 (Surjectivity): We will now show that $M \setminus B_{R_\kappa}(p) = \{\hat{p}\}$ for some point $\hat{p} \in M$ and that $F(\hat{q}) = \hat{p}$. For any $v \in \mathbb{S}_pM$ we have $t_0(v) \geq R_\kappa$ by what we have proven so far (due to the isometry of the open balls). Since $\kappa > 0$ we have $t_0(v) \leq R_\kappa$ anyway (since the first conjugate point certainly occurs there) and thus $t_0(v) = R_\kappa$ for any $v \in \mathbb{S}_pM$. Thus $d(p, C(p)) = R_\kappa$ and

$$C(p) = \{\hat{p} \in M \mid d(p, \hat{p}) = R_\kappa\}.$$

Now let $\hat{p} \in C(p)$. Then there is a unit speed minimizing geodesic c from p to $\hat{p}$. Let $v := \dot{c}(0)$, thus $c = c_v$. Then $c|_{[0,R_\kappa[}$ is the image under F of a unit speed geodesic $\tilde{c}$, starting at q with velocity $\dot{\tilde{c}}(0) = A^{-1}v$. Thus

$$F(\hat{q}) = \lim_{R \nearrow R_\kappa} F(\tilde{c})(R) = \lim_{R \nearrow R_\kappa} c(R) = \hat{p},$$

therefore F surjective and $M \setminus B_{R_\kappa}(q) = \{\hat{p}\}$ as claimed.
Step 5 (Global isometry): Thus F not only has Lipschitz constant $= 1$, but also preserves distances: For $p_1, p_2 \in M \setminus \{\hat{p}\}$ one may calculate $d(p_1, p_2)$ as the infimum of all lengths of paths from p_1 to p_2, which do not touch $\hat{p}$.
Step 6 (Smoothness): All unit speed geodesics c_1, c_2 starting from q satisfy

$$\forall i = 1, 2 : F(c_i(R_\kappa)) = F(\hat{q}) = \hat{p}$$

and

$$\langle (F \circ c_1)'(R_\kappa), (F \circ c_2)'(R_\kappa) \rangle = \langle \dot{c}_1(R_\kappa), \dot{c}_2(R_\kappa) \rangle$$

since $F \circ c_i$, $i = 1, 2$, is a geodesic on $[0, R_\kappa]$ (since it is minimizing hence in particular smooth), so:

$$\begin{aligned} \|(F \circ c_1)'(R_\kappa) - (F \circ c_2)'(R_\kappa)\| &= \lim_{R \nearrow R_\kappa} \frac{d(F(c_1(R)), F(c_2(R)))}{R_\kappa - R} = \lim_{R \nearrow R_\kappa} \frac{d(c_1(R), c_2(R))}{R_\kappa - R} \\ &= \|\dot{c}_1(R_\kappa) - \dot{c}_2(R_\kappa)\| \end{aligned}$$

Thus F is smooth in $\hat{q}$ with push-forward $F_*(\dot{c}_i(R_\kappa)) = (F \circ c_i)'(R_\kappa)$.

□

6.13 Theorem (Cheng)**.** Let M be an n-dimensional, complete, connected manifold with curvature $\mathrm{Ric}_M \geq (n-1)\kappa$, $\kappa > 0$. If $\operatorname{diam} M = R_\kappa$, then M is isometric to M^n_κ.

Proof. By the Theorem of Myers [2, 11.8] M is compact. The continuous function $M \times M \to \mathbb{R}$, $(p, q) \mapsto d(p, q)$ attains its supremum. So let $p, q \in M$, such that $d(p, q) = \frac{\pi}{\sqrt{\kappa}} = R_\kappa$. By Theorem 6.10

$$\frac{V_p(R_\kappa)}{V_\kappa(R_\kappa)} \leq \frac{V_p(R_\kappa/2)}{V_\kappa(R_\kappa/2))} \Rightarrow \frac{V_p(R_\kappa)}{V_p(R_\kappa/2)} \leq \frac{V_\kappa(R_\kappa)}{V_\kappa(R_\kappa/2)} = 2, \tag{6.14}$$

where the last equality holds since M^n_κ is a round sphere of radius $1/\sqrt{\kappa}$ and the volume of a hemisphere is exactly half the volume of a sphere. Of course the same holds for q:

$$\frac{V_q(R_\kappa)}{V_q(R_\kappa/2)} \leq 2 \tag{6.15}$$

Now $V_p(R_\kappa) = V_q(R_\kappa) = \mathrm{vol}(M)$, so together with the first two inequalities above:

$$V_p(R_\kappa/2) + V_q(R_\kappa/2)) \geq \frac{1}{2}(V_p(R_\kappa)) + V_q(R_\kappa)) = \mathrm{vol}(M)$$

On the other hand we have by construction $B_p(R_\kappa/2)) \cap B_q(R_\kappa/2)) = \emptyset$, thus

$$V_p(R_\kappa/2) + V_q(R_\kappa/2)) = \mathrm{vol}(M).$$

We have archieved:

$$V_p(R_\kappa/2) \geq \frac{1}{2}\mathrm{vol}(M) \qquad V_q(R_\kappa/2) \geq \frac{1}{2}\mathrm{vol}(M) \qquad V_p(R_\kappa/2) + V_q(R_\kappa/2) = \mathrm{vol}(M)$$

This implies

$$V_p(R_\kappa/2) = \frac{1}{2}\operatorname{vol}(M) = V_q(R_\kappa/2). \tag{6.16}$$

Substituting this into the very first inequality, we obtain

$$1 = \frac{\operatorname{vol}(M)}{2V_p(R_\kappa/2)} \leq \frac{\operatorname{vol}(M^n_\kappa)}{2V_\kappa(R_\kappa/2)} \leq 1,$$

thus equality. Therefore the equality statement of the Bishop-Gromov inequality 6.10 implies that the open balls are isometric. Since their boundary is a set of measure zero, this implies $\operatorname{vol}(M) = \operatorname{vol}(M^n_\kappa)$. Now the statement follows from Corollary 6.12. □

7 Toponogov's Theorem

The Theorem of Alexandrov-Toponogov compares triangles in a manifold, whose sectional curvature is bounded from below by a constant κ, with triangles in M_κ^2. It is a counterpart of Theorem 4.18, which requires the sectional curvature to be bounded from above. There is another important difference: Theorem 4.18 is a statement concerning "small" triangles, where Toponogov's Theorem concerns the global geometry of M.

7.1 Preparation

7.1 Remark. In this section M is a Riemannian manifold, which is complete, connected and has sectional curvatures $K \geq \kappa$, $\kappa \in \mathbb{R}$.

We will need some preparation, before we can start. Let $p \in M$ and for any $v \in \mathbb{S}_p M$ let c_v be the geodesic satisfying $\dot{c}_v(0) = v$ and let $U_v(t)$, $0 < t < t_0(v)$, be the second fundamental form of the geodesic sphere of radius t around p at $c_v(t)$ with respect to the inward pointing unit normal $-\dot{c}_v(t)$. Let J be a Jacobi field along c_v satisfying $J(0) = 0$, $J \perp \dot{c}_v$. Then $J' = U_v J$ on $]0, t_0(v)[$ (see section 1). Thus

$$\langle J', J \rangle = \langle U_v J, J \rangle \leq \|U_v\| \|J\|^2 \tag{7.1}$$

and therefore estimates on U_v yield estimates on J. Now let E be a parallel unit normal field long c_v and let J be the Jacobi field along c_v satisfying $J(0) = 0$, $J'(0) = E(0)$. This implies $J(t) = tE(t) + o(t)$, $t \to 0$. Furthermore

$$\lim_{t \searrow 0} \frac{\langle J', J \rangle}{\|J\|} = \|J\|'(0) = \|J'(0)\| = 1,$$

(this was discussed in another context already, c.f. (4.1)). By Lemma 2.2 and Lemma 2.9, we obtain

$$\langle U_v E, E \rangle \leq \mathrm{ct}_\kappa \ \text{ on }]0, t_0(v)[. \tag{7.2}$$

This implies

$$\lim_{t \searrow 0} t \frac{\langle J', J \rangle}{\langle J, J \rangle}(t) = \lim_{t \searrow 0} t \langle U_v E, E \rangle = 1 \tag{7.3}$$

and

$$\frac{\langle J', J \rangle}{\langle J, J \rangle} \leq \mathrm{ct}_\kappa \ \text{ on }]0, t_0(v)[. \tag{7.4}$$

7.2 Lemma. Let c be a unit speed geodesic in $U := M \setminus (\{p\} \cup C(p))$. Define $r := d(p, c(\_))$ and $e := m_\kappa \circ r$. Then

$$e'' + \kappa e \leq 1.$$

Proof. Using the definition of m_κ and 4.13, we calculate

$$e'' = (m_\kappa' \circ r) \cdot r')' = (m_\kappa'' \circ r)(r')^2 + (m_\kappa' \circ r) \cdot r'' = (\mathrm{cs}_\kappa \circ r)(r')^2 + (\mathrm{sn}_\kappa \circ r) r''.$$

Now

$$r'(t) = \langle \dot{c}(t), \operatorname{grad} d_p|_{c(t)} \rangle,$$

where $d_p := d(p, \_)$. Denote by c_t the unit speed minimizing geodesic from p to $c(t)$ and by $\dot{c}^\perp$ the component of $\dot{c}(t)$ perpendicular to $\dot{c}_t = \operatorname{grad} d_p|_{c_t(r(\_))}$. We calculate

$$\begin{aligned} r''(t) &= \langle \ddot{c}(t), \operatorname{grad} d_p|_{c(t)} \rangle + \langle \dot{c}(t), \nabla_{\dot{c}(t)} \operatorname{grad} d_p|_{c(t)} \rangle \overset{1.4}{=} \operatorname{Hess} d_p(\dot{c}(t), \dot{c}(t)) \\ &\overset{1.5}{=} \operatorname{Hess} d_p(\dot{c}^\perp(t), \dot{c}^\perp(t)) \overset{[2,\text{ p.140}]}{=} \langle U_{\dot{c}_t(0)}(\dot{c}^\perp(t)), \dot{c}^\perp(t) \rangle = \langle J'(r(t)), J(r(t)) \rangle, \end{aligned}$$

where J is the normal Jacobi field along c_t from p to $c(t)$ satisfying $J(0) = 0$, $J(r(t)) = \dot{c}^\perp(t)$ (existence is guaranteed by [2, Exc. 10.2]). This implies

$$\langle J', J \rangle|_{c(t)} \leq \operatorname{ct}_\kappa(r(t)) \|\dot{c}(t)^\perp\|^2.$$

Alltogether we obtain

$$e''(t) \leq (\operatorname{cs}_\kappa \circ r) \langle \dot{c}(t), \operatorname{grad} d_p|_{c(t)} \rangle^2 + (\operatorname{sn}_\kappa \circ r)(\operatorname{ct}_\kappa \circ r)(t) \|\dot{c}^\perp(t)\|^2 = (\operatorname{cs}_\kappa \circ r)(t) \|\dot{c}(t)\| = (\operatorname{cs}_\kappa \circ r)(t).$$

This implies

$$e'' + \kappa e \leq \operatorname{cs}_\kappa \circ r + \kappa \cdot m_\kappa \circ r \overset{4.13}{=} 1.$$

$\square$

7.3 Lemma. Let $c_1 : [0, l_1] \to M^2_\kappa$ and $\bar{c}[0, l] \to M^2_\kappa$ be unit speed geodesics satisfying $\bar{c}_1(l_1) = \bar{c}(0) =: \bar{p}_1$ and let $\bar{\alpha}$ be their angle in $\bar{p}_1$. Let $l_1, l < R_\kappa$ and define $f(\bar{\alpha}_1) := d(\bar{c}_1(0)\bar{c}(l))$. Then f, seen as a function of $\bar{\alpha}_1$ on $[0, \pi]$ satisfies

$$m_\kappa(f(\bar{\alpha}_1)) = m_\kappa(|l_1 - l|) + \operatorname{sn}_\kappa(l_1) \operatorname{sn}_\kappa(l)(1 - \cos(\alpha_1)),$$

is strictly monotonously increasing and

$$f(0) = |l_1 - l| \qquad\qquad f(\pi) = \min\{l_1 + l, 2R_\kappa - (l_1 + l_2).\}$$

Proof. Applying the law of cosines (c.f. Theorem A.13) in this situation, we obtain

$$\operatorname{cs}_\kappa(f(\alpha_1)) = \operatorname{cs}_\kappa(l_1) \operatorname{cs}_\kappa(l) + \kappa \operatorname{sn}_\kappa(l_1) \operatorname{sn}_\kappa(l) \cos(\bar{\alpha}_1).$$

By Lemma 4.13 $\operatorname{cs}_\kappa + \kappa m_\kappa = 1$ and therefore we may restate this formula as

$$\begin{aligned} 1 - \kappa m_\kappa(f(\bar{\alpha}_1)) &= \operatorname{cs}_\kappa(l_1) \operatorname{cs}_\kappa(l) + \kappa \operatorname{sn}_\kappa(l_1) \operatorname{sn}_\kappa(l) \cos(\bar{\alpha}_1) \\ \Leftrightarrow m_\kappa(f(\bar{\alpha}_1)) &= \frac{1}{\kappa} - \frac{1}{\kappa} \operatorname{cs}_\kappa(l_1) \operatorname{cs}_\kappa(l) - \operatorname{sn}_\kappa(l_1) \operatorname{sn}_\kappa(l) \cos(\bar{\alpha}_1) \end{aligned}$$

Using die angle sum identity (c.f. Theorem A.14) we obtain

$$\begin{aligned} m_\kappa(l_1 - l) &= \frac{1}{\kappa}(1 - \operatorname{cs}_\kappa(l_1 - l)) \\ &= \frac{1}{\kappa}(1 - (\operatorname{cs}_\kappa(l_1) \operatorname{cs}_\kappa(l) + \kappa \operatorname{sn}_\kappa(l_1) \operatorname{sn}_\kappa(l))) \\ &= \frac{1}{\kappa} - \frac{1}{\kappa} \operatorname{cs}_\kappa(l_1) \operatorname{cs}_\kappa(l) - \operatorname{sn}_\kappa(l_1) \operatorname{sn}_\kappa(l) \end{aligned}$$

and since this is symmetric in l_1,l the same holds for $m_\kappa(l - l_1)$ and thus for $m_\kappa(|l_1 - l|)$. Alltogether we obtain

$$\begin{aligned} &m_\kappa(|l_1 - l|) + \operatorname{sn}_\kappa(l_1) \operatorname{sn}_\kappa(l)(1 - \cos(\alpha_1)) \\ &= \frac{1}{\kappa} - \frac{1}{\kappa} \operatorname{cs}_\kappa(l_1) \operatorname{cs}_\kappa(l) - \operatorname{sn}_\kappa(l_1) \operatorname{sn}_\kappa(l) + \operatorname{sn}_\kappa(l_1) \operatorname{sn}_\kappa(l)(1 - \cos(\alpha_1)) \\ &= \frac{1}{\kappa} - \frac{1}{\kappa} \operatorname{cs}_\kappa(l_1) \operatorname{cs}_\kappa(l) - \operatorname{sn}_\kappa(l_1) \operatorname{sn}_\kappa(l) \cos(\alpha_1) = m_\kappa(f(\bar{\alpha}_1)). \end{aligned}$$

The other statements follow from the monotonicity of m_κ and this representation. $\square$

7.2 Statement and Proof

7.4 Theorem (Alexandrov-Toponogov)**.** Let M be a complete connected n-manifold with sectional curvatures $K \geq \kappa$, $\kappa \in \mathbb{R}$. In case $\kappa > 0$ let M not be isometric to $\mathbb{S}^n(1/\sqrt{\kappa})$. Then the following equivalent statements hold:

(i) Let $\Delta = (c_1, c_2, c)$ be a geodesic triangle consisting of unit speed minimizing geodesics $c_i : [0, l_i] \to M$, $i = 1, 2$, $c_1(0) = c_2(0) =: p$ and a unit speed geodesic $c : [0, l] \to M$, $c(0) = c_1(l_1) =: p_1$, $c(l) = c_2(l_2) =: p_2$. Let $l \leq l_1 + l_2$ and in case $\kappa > 0$ let $l < R_\kappa$. Then $l_1 + l_2 + l < 2R_\kappa$ and the corresponding comparison triangle $\bar{\Delta}$ in M_κ^2 satisfies

$$d(\bar{p}, \bar{c}(t)) \leq d(p, c(t)).$$

(ii) Let Δ be as in 2 and denote by α_i the angle of Δ in p_i, $i = 1, 2$. Then $l_1 + l_2 + l < 2R_\kappa$ and the comparison triangle $\bar{\Delta}$ in M_κ^2 satisfies

$$\bar{\alpha}_i \leq \alpha_i \ , i = 1, 2.$$

(iii) Let $c_1 : [0, l_1] \to M$ and $c : [0, l] \to M$ be unit speed geodesics where c_1 is minimizing, $l \leq R_\kappa$, $c(0) = c_1(l_1) =: p_1$ and let α_1 be the angle between c_1 and c in p_1. Let $\bar{c}_1$, $\bar{c}$ be corresponding geodesics in M_κ^2 with angle $\bar{\alpha}_1 := \alpha_1$ in $\bar{p}_1 = \bar{c}_1(l_1) = \bar{c}(0)$. Then

$$d(c_1(0), c(l)) \leq d(\bar{c}_1(0), \bar{c}(l)).$$

7.5 Remark.

(i) We will prove the first statement and then show, that it implies the second, which implies the third. We leave it as an exercise to close the circle.

(ii) In the first and in the second statement existence and uniqueness (up to congruence) of $\bar{\Delta}$ is at first only clear in case $\kappa \leq 0$: Since c_1 and c_2 are minimizing $l_1 \leq l + l_2$ and $l_2 \leq l_1 + l$. But then in case $\kappa > 0$ the inequality $l_1 + l_2 + l < 2R_\kappa$ implies existence of $\bar{\Delta}$.

(iii) In case $\kappa > 0$, we obtain $l_1, l_2 \leq \operatorname{diam} M < R_\kappa$ (by Cheng's Theorem 6.13) since M is required not to be isometric to $\mathbb{S}^n(1/\sqrt{\kappa})$. This implies uniqueness (up to congruence) of $\bar{\Delta}$ by Lemma 7.3. The somewhat strange hypothesis that M shall not be isometric to $\mathbb{S}^n(1/\sqrt{\kappa})$ avoids complications in the formulation of the theorem.

(iv) Let $c_1 : [0, l_1] \to M$, $c_2 : [0, l_2] \to M$ be unit speed minimizing geodesics satisfying $c_1(0) = c_2(0) =: p$ and let $c : [0, l] \to M$ be a unit speed geodesic satisfying $c(0) = c_1(l_1)$, $c(l) = c_2(l_2)$ and $l = R_\kappa = \pi/\sqrt{\kappa}$, $\kappa > 0$. Then

$$l_1 + l_2 \leq R_\kappa.$$

This can be seen as follows: Suppose to the contrary that $l_1 + l_2 = l_1 + d(p, c(l)) > R_\kappa = l$. By Bonnet/Myer's Theorem [2, 11.7] $l_1, l_2 \leq R_\kappa$ and by Cheng's Theorem 6.13 and by hypothesis $l_1, l_2 < R_\kappa$. Consider the continuous function $[0, l] \to \mathbb{R}$, $t \mapsto l_1 + d(p, c(t)) + t$. Evaluated at $t = 0$ we obtain

$$l_1 + d(p, c(0)) + 0 = 2l_1 < 2R_\kappa$$

and evaluated at $t = l$ we obtain by hypothesis

$$l_1 + d(p, c(l)) + l > R_\kappa + l = 2R_\kappa.$$

Thus by the intermediate value theorem there exists a smallest $t > 0$ such that $t < l = R_\kappa$ and $l_1 + d(p, c(t)) + t = 2R_\kappa$. Denote by c_t a unit speed minimizing geodesic from p to $c(t)$. Since $t < l = R_\kappa$, we obtain

$$l_1 + d(p, c(t)) = 2R_\kappa - t > R_\kappa = l > t$$

and therefore the triangle $\Delta = (c_1, c_t, c|[0, t])$ satisfies the hypothesis of the first statement. This is a contradiction since the perimeter of Δ is $2R_\kappa$.

Proof.
STEP 1: By Remark 7.5,(iii) we obtain $l_1, l_2 < R_\kappa$. We first assume $l_1 + l_2 + l < 2R_\kappa$. By Lemma 7.3 there exists a comparison triangle $\bar{\Delta}$ in M^2_κ, which is unique up to congruence.
The case $l_1 + l_2 = l$ is trivial: In that case $\bar{\Delta}$ is degenerate and

$$d(\bar{p}, \bar{c}(t)) = \begin{cases} l_1 - t & , 0 \le t \le l_1 \\ t - l_1 & , l_1 \le t \le l \end{cases} \le d(p, c(t)),$$

where in the last step we used the triangle inequality. Therefore we will assume $l_1 + l_2 > l$ in the following. This implies $p \notin \operatorname{im} c$.
STEP 2: In case c does not intersect the cut locus of p, we may argue completely analogously as in the proof of 4.18: Define $r := d(p, c(\_))$, $e := m_\kappa \circ r$ and analogously $\bar{r}$, $\bar{e}$. We obtain $\bar{e} + \kappa \bar{r} = 1$ and by Lemma 7.2

$$e'' + \kappa e \le 1.$$

Define $f := e - \bar{e}$, we obtain

$$f'' + \kappa f \le 0 \qquad\qquad f(0) = f(l) = 0,$$

thus $f \ge 0$ on $[0, l]$ by Lemma 4.21 and thus the statement.
STEP 3: If c intersects the cut locus of p we cannot derive $e'' + \kappa e \le 1$ like this, because $\operatorname{grad} d_p$ is not well-defined in the cut locus. Nevertheless define r and e as above and suppose to the contrary, that f has a negative minimum in $]0, l[$. Since $l < 0$ there exists $\alpha > 0$ such that $l < R_{\kappa+\alpha}$. There exists a fuction $j : [0, l] \to \mathbb{R}$ such that

$$j > 0 \qquad\qquad j'' + (\kappa + \alpha) j = 0, \tag{7.5}$$

for example take $j = \operatorname{sn}_{\kappa+\alpha}(\delta + \_)$ for a sufficiently small $\delta > 0$. Define

$$g := \frac{f}{j} : [0, l] \to \mathbb{R},$$

i.e. $f = gj$. Then g has a negative minimum in some $t_0 \in]0, l[$.
Let σ be a unit speed minimizing geodesic from $p = \sigma(0)$ to $c(t_0)$. For sufficiently small $\eta > 0$ the point $c(t_0)$ is not a cut point of $\sigma(\eta)$ along σ since otherwise $\sigma(\eta)$ were a cut point of $c(t_0)$ along σ^{-1} and σ^{-1} could not be a minimizing geodesic from $c(t_0)$ to p. That obviously contradicts the fact that σ is a minimizing geodesic from p to $c(t_0)$. Define

$$r_\eta := \eta + d(\sigma(\eta), c(\_)) : [0, l] \to \mathbb{R} \tag{7.6}$$

and observe, that

$$r(t) = d(p, c(t)) \le d(p, \sigma(\eta)) + d(\sigma(\eta), c(t)) = \eta + d(\sigma(\eta), c(t)) = r_\eta(t) \tag{7.7}$$
$$r_\eta(t_0) = \eta + d(\sigma(\eta), c(t_0)) = d(p, c(t_0)) = r(t_0), \tag{7.8}$$

since σ is minimizing. Now define

$$e_\eta := m_\kappa \circ r_\eta : [0, l] \to \mathbb{R} \tag{7.9}$$
$$f_\eta := e_\eta - \bar{e} : [0, l] \to \mathbb{R} \tag{7.10}$$

and observe, that due to (7.7) and the monotonicity of m_κ, we obtain

$$f_\eta = e_\eta - \bar{e} \ge e - \bar{e} = f, \tag{7.11}$$
$$f_\eta(t_0) = e_\eta(t_0) - \bar{e}(t_0) = m_\kappa(r_\eta(t_0)) - \bar{e}(t_0) \overset{(7.8)}{=} m_\kappa(r(t_0)) - \bar{e}(t_0) = f(t_0). \tag{7.12}$$

Therefore the function

$$g_\eta := \frac{f_\eta}{j} : [0,l] \to \mathbb{R} \tag{7.13}$$

has a negative minimum in t_0 and we obtain

$$g_\eta(t_0) = \frac{f_\eta(t_0)}{j(t_0)} = \frac{f(t_0)}{j(t_0)} = g(t_0) < 0.$$

We may now modify 4.13 used in the proof of 4.18 in order to obtain

$$\begin{aligned} r'_\eta &= \langle \dot{c}, \operatorname{grad}_{d_{\sigma(\eta)}} \rangle && (7.14)\\ \ddot{r}_\eta &= \operatorname{Hess} d_{\sigma(\eta)}(\dot{c},\dot{c}) \le \operatorname{ct}_\kappa(r_\eta - \eta)\|\dot{c}^\perp\|^2 && (7.15)\end{aligned}$$

and again

$$\dot{r}_\eta^2 + \|\dot{c}^\perp\|^2 = \|\dot{c}\|^2 = 1 \Rightarrow \|\dot{c}^\perp\|^2 = 1 - \dot{r}_\eta^2. \tag{7.16}$$

In particular we obtain

$$(r'_\eta)^2 = 1 - \|\dot{c}^\perp\|^2 \le 1 \qquad\qquad r'_\eta \le 1. \tag{7.17}$$

Therfore the derivatives satisfy:

$$\begin{aligned} e''_\eta &\overset{(7.9)}{=} (m_\kappa \circ r_\eta)'' = (m''_\kappa \circ r_\eta)(r'_\eta)^2 + m'_\kappa \circ r_\eta \cdot r''_\eta && (7.18)\\ &\overset{4.13}{=} (\operatorname{cs}_\kappa \circ r_\eta)(r'_\eta)^2 + \operatorname{sn}_\kappa \circ r_\eta \cdot r''_\eta \\ &\overset{(7.15)}{\le} (\operatorname{cs}_\kappa \circ r_\eta)(r'_\eta)^2 + \operatorname{sn}_\kappa \circ r_\eta \cdot \operatorname{ct}_\kappa(r_\eta - \eta)\|\dot{c}^\perp\|^2 \\ &\overset{(7.16)}{=} (\operatorname{cs}_\kappa \circ r_\eta)(r'_\eta)^2 + \operatorname{sn}_\kappa \circ r_\eta \cdot \operatorname{ct}_\kappa(r_\eta - \eta)(1 - (r'_\eta)^2) \\ &= \operatorname{cs}_\kappa \circ r_\eta + (\operatorname{cs}_\kappa \circ r_\eta)((r'_\eta)^2 - 1) + \operatorname{sn}_\kappa \circ r_\eta \cdot \operatorname{ct}_\kappa(r_\eta - \eta)(1 - (r'_\eta)^2) \\ &= \operatorname{cs}_\kappa \circ r_\eta + (\operatorname{sn}_\kappa \circ r_\eta)(1 - (r'_\eta)^2)(\operatorname{ct}_\kappa(r_\eta - \eta) - (\operatorname{ct}_\kappa \circ r_\eta)) \\ &\overset{4.13}{=} 1 - \kappa\, m_\kappa \circ r_\eta + (\operatorname{sn}_\kappa \circ r_\eta)(1 - (r'_\eta)^2)(\operatorname{ct}_\kappa(r_\eta - \eta) - (\operatorname{ct}_\kappa \circ r_\eta))\end{aligned}$$

We would like to bound this expression at t_0 and therefore choose a small $\varepsilon > 0$, such that

$$\begin{aligned} &\forall t \in [0,l] : d(p, c(t)) \ge 2\varepsilon && (7.19)\\ &\forall t \in [0,l] : d(p, c(t)) + \varepsilon < R_\kappa && (7.20)\\ &\eta < \varepsilon && (7.21)\end{aligned}$$

where the last condition is assured by shrinking η if necessary. This implies

$$\begin{aligned} \forall t \in [0,l] : 2\varepsilon &\overset{(7.19)}{\le} d(p, c(t)) \le d(p, \sigma(\eta)) + d(\sigma(\eta), c(t)) = \eta + d(\sigma(\eta), c(t)) \overset{(7.6)}{=} r_\eta && (7.22)\\ &\le \eta + d(\sigma(\eta), p) + d(p, c(t)) \le 2\eta + \max_{0\le s\le l} d(p, c(s)) \overset{(7.21)}{\le} \max_{0\le s\le l} d(p, c(s)) + 2\varepsilon. \end{aligned}$$

Define:

$$m_1 := \max\{\operatorname{sn}_\kappa(t) \mid 0 \le t \le \max_{0\le s\le l} d(p, c(s)) + 2\varepsilon\} \tag{7.23}$$

$$m_2 := \max\{\operatorname{ct}'_\kappa(t) \mid \varepsilon \le t \le \max_{0\le s\le l} d(p, c(s)) + 2\varepsilon\} \tag{7.24}$$

Equation (7.22) together with definition (7.23) implies that

$$\forall t \in [0,l] : (\mathrm{sn}_\kappa \circ r_\eta)(t) \leq m_1. \tag{7.25}$$

By the mean value theorem and definition (7.24)

$$\mathrm{ct}_\kappa(r_\eta - \eta) - \mathrm{ct}_\kappa(r_\eta) \leq m_2\eta. \tag{7.26}$$

Therefore we can bound (7.18) by

$$e_\eta'' \leq 1 - \kappa\, m_\kappa \circ r_\eta + \underbrace{(\mathrm{sn}_\kappa \circ r_\eta)}_{\leq m_1 \text{ ,by (7.25)}} \underbrace{(1-(r_\eta')^2)}_{\leq 1 \text{ by, (7.17)}} \underbrace{(\mathrm{ct}_\kappa(r_\eta - \eta) - (\mathrm{ct}_\kappa \circ r_\eta))}_{\leq m_2\eta \text{ ,by (7.26)}} \leq 1 - \kappa\, m_\kappa \circ r_\eta + m_1 m_2 \eta. \tag{7.27}$$

We obtain

$$\begin{aligned} &g_\eta''(t_0)j(t_0) + 2\underbrace{g_\eta'(t_0)}_{=0}j'(t_0) + g_\eta(t_0)j''(t_0) + \kappa g_\eta(t_0)j(t_0) = (g_\eta' j + g_\eta j')'(t_0) + \kappa g_\eta(t_0)j(t_0) \\ &= (g_\eta j)''(t_0) + \kappa g_\eta(t_0)j(t_0) \overset{(7.13)}{=} f_\eta''(t_0) + \kappa f_\eta(t_0) \overset{(7.10)}{=} (e_\eta'' - e'')(t_0) + \kappa e_\eta(t_0) - \kappa e_\eta(t_0) \\ &= e_\eta''(t_0) + \kappa e_\eta(t_0) - \underbrace{(e'' - \kappa e_\eta)(t_0)}_{=1} \overset{(7.27)}{\leq} 1 - \kappa m_\kappa \circ r_\eta(t_0) + m_1 m_2 \eta + \kappa e_\eta(t_0) - 1 \\ &\overset{(7.9)}{=} m_1 m_2 \eta \end{aligned}$$

and therefore

$$\begin{aligned} g_\eta''(t_0)j_0(t_0) &\leq m_1 m_2 \eta - g_\eta(t_0)(j''(t_0) + \kappa j(t_0)) \overset{(7.5)}{=} m_1 m_2 \eta + \alpha g_\eta(t_0) j(t_0) \\ &\overset{(7.13)}{=} m_1 m_2 \eta + \alpha f_\eta(t_0) \overset{(7.12)}{=} m_1 m_2 \eta + \alpha f(t_0). \end{aligned}$$

This implies $g_\eta''(t_0) < 0$ for small $\eta > 0$, which contradicts the hypothesis that g has a minimum in t_0. This proves the first statement under the hypothesis $l_1 + l_2 + l < 2R_\kappa$ and thus in particular for $\kappa \leq 0$.
Step 4: In case $\kappa > 0$ remember the hypothesis $l_1 + l_2 \geq l$. Define

$$t_0 := \sup\{t \in [0,l] \mid l_1 + d(p, c(t)) + t < 2R_\kappa\} > 0$$

and for any $t \in]0, t_0[$ let $c_t : [0, d(p,c(t))] \to M$ be a minimizing geodesic from p to $c(t)$. The triangle $\Delta_t = (c_1, c_t, c|[0,t])$ satisfies the hypothesis of the first statement and we obtain

$$l_1 + l_2 + t < 2R_\kappa$$

by definition of t_0. Now $t \leq l < R_\kappa$ and thus

$$\lim_{t \to t_0} \max_{0 \leq s \leq t} d(p, c(s)) = R_\kappa.$$

Thus $\mathrm{diam}(M) = R_\kappa$ and so M is isometric to $\mathbb{S}^n(1/\sqrt{\kappa})$ by Cheng's Theorem 6.13. Contradiction!
Step 5 ("(i)⇒(ii)"): Choose a variation H of $\sigma(t) = c_1(l_1 t)$, $0 \leq t \leq 1$, such that $H(s,0) = p$ and $H(s,1) = c(s)$, $s \geq 0$. We obtain

$$-\cos(\alpha_1) = \langle \dot{c}_1(l), \dot{c}(0) \rangle = \partial_s E(H(s, \_))|_{s=0}.$$

Choose a corresponding variation $\bar{H}$ ind M_κ^2, where $\bar{H}(s,\_)$ is the unique minimizing geodesic from $\bar{p}$ to $\bar{c}(s)$ (notic that $\bar{c}(s)$ is not antipodal to p since $l_1 + l_2 + l < 2R_\kappa$). We obtain

$$-\cos(\bar{\alpha}_1) = \langle \dot{\bar{c}}(l_1), \dot{\bar{c}}(0) \rangle = \partial_s E(\bar{H}(s, \_))|_{s=0}.$$

By (i) and since $\bar{H}(s, \_)$ as constant speed, we obtain

$$E(\bar{H}(s,\_)) = \frac{1}{2}L(\bar{H}(s,\_))^2 = \frac{1}{2}d(\bar{p},\bar{c}(s)) \leq \frac{1}{2}d(p,c(s))^2 \leq \frac{1}{2}L(H(s,\_))^2 \leq E(H(s,\_)),$$

thus $\cos(\alpha_1) \leq \cos(\bar{\alpha}_1)$ and therefore (ii).

Step 6 ("(ii)⇒(iii)"): First assume $l < R_\kappa$. Since $l_1 < R_\kappa$ the curves $\bar{c}_1$ and $\bar{c}$ are minimal in M^2_κ. Die inequality is a direct consequence of the triangle inequality provided $d(c_1(0), c(l)) \leq l - l_1$. In case $l(\alpha_1) = d(c_1(0), c(l)) > l - l_1$ apply (ii) to the triangle (c_1, c_2, c), where $c_2 : [0, l(\alpha_1)] \to M$ is a minimizing geodesic from $c_1(0) = p$ to $c(l)$. By (ii) we obtain $\bar{\alpha}_1 \leq \alpha_1$ for the corresponding comparison triangle. By definition $l(\alpha_1) = f(\alpha_1)$ and thus the statement follows from Lemma 7.3. We have proven (iii) in case $l < R_\kappa$. The case $l = R_\kappa$ follows by taking limits.

□

7.3 Application: Gromov's Theorem

We would like to discuss an application of Toponogov's Theorem to the Fundamental Group. Therefore it is necessary to introduce the concept of a *short basis*. As usual let M be a complete connected Riemannian n-manifold and denote by $\pi : \tilde{M} \to M$ a universal covering. Choose any $\tilde{p} \in \tilde{M}$ and define $p := \pi(\tilde{p}) \in M$. As usual we identify $\pi_1(M, p)$ with the group G of deck transformations of π. Remember (c.f. 5.6), that

$$X = \{g\tilde{p} \mid g \in G\}$$

is a discrete set. In case $0 < \varepsilon < i(p)$ we obtain

$$B_\varepsilon(d\tilde{p}) \cap B_\varepsilon(h\tilde{p}) \neq \emptyset \Longrightarrow g = h.$$

Also remember (c.f. 5.8), that $\|g\|_{\tilde{p}} = d(\tilde{p}, g\tilde{p})$ and

$$\|g\|_{\tilde{p}} = \|g^{-1}\|_{\tilde{p}} \qquad\qquad \|gh\|_{\tilde{p}} \leq \|g\|_{\tilde{p}} + \|h\|_{\tilde{p}}.$$

7.6 Definition (short basis)**.** Define subsets $B_i \subset G$, $i \in \mathbb{N}$, inductively defined as follows:

(i) Define $g_0 := e$, $B_0 := \{g_0\}$.

(ii) Assume $B_i = \{g_0, \ldots, g_i\}$ has been defined. Denote by $G_i \subset G$ the subgroup generated by B_i. We distinguish two cases:
Case 1 ($G_i \neq G$): Define $X_i := \{g\tilde{p} \mid g \in G_i\}$. The set $X \setminus X_i$ is discrete and not empty. Thus there exists at least one element $\tilde{q} \in X \setminus X_i$ having minimal distance to $\tilde{p}$ (notice that $\tilde{q}$ does not have to be unique). There exists a unique $g_{i+1} \in G$ such that $g_{i+1}\tilde{p} = \tilde{q}$. Notice that

$$\|g_{i+1}\|_{\tilde{p}} = \min_{g \in G \setminus G_i} \|g\|_{\tilde{p}}.$$

Define $B_{i+1} := B_i \cup \{g_{i+1}\}$.
Case 2 ($G_i = G$): Define $B_{i+1} := B_i$.

Any

$$B := \bigcup_{i \in \mathbb{N}} B_i \subset G$$

obtained in that way is a *short basis* of G. We call the procedure above the *short basis algorithm*.

7.7 Lemma (Properties of a short basis)**.** Let $B = \{g_0, g_1, g_2, \ldots\}$ be a short basis of G.

(i) For any $i < j$

$$\|g_i\|_{\tilde{p}} \leq \|g_j\|_{\tilde{p}} \leq \|g_i^{-1} g_j\|_{\tilde{p}}.$$

(ii) B is a finite or at least countable generating system for G.

(iii) Let M be compact and $d := \operatorname{diam} M$. Then any $g_i \in B$ satisfies $\|g_i\| \leq 2d$ and furthermore $|B| < \infty$.

Proof.

(i) The first inequality holds by construction. To see the second, notice that $i < j$ implies

$$g_i \in G_{j-1} \qquad\qquad g_j \notin G_{j-1}.$$

Together this implies $g_i^{-1}g_j \notin G_{j-1}$, since otherwise there exists $h \in G_{j-1}$ such that $g_i^{-1}g_j = h$, which implies $g_j = g_i h \in G_{j-1}$. So by construction

$$\|g_j\|_{\tilde{p}} \leq \|g_i^{-1}g_j\|_{\tilde{p}}.$$

(ii) If in the definition of a short basis the case 2 occurs one time, then B is a finite generating system of G by definition. If this does not happen, B is at least a countable set. Let $g \in G$ be arbitrary. Clearly there exists $R > 0$ such that $\|g\|_{\tilde{p}} \leq R$. The set

$$X \cap \bar{B}_R(\tilde{p})$$

is compact and discrete, hence finite. Thus the short basis algorithm treats g after finitely many steps: Either there exists some i such that $g = g_i$ or $g \in G_i$.

(iii) We have shown in Theorem 5.9,(i), that the elements $g \in G$ satisfying $\|g\|_{\tilde{p}} \leq 2d$ already generate G. Since $X \cap \bar{B}_{2d}(\tilde{p})$ is again finite, the statement follows.

$\square$

7.8 Theorem (Gromov)**.** Let M be a complete connected Riemannian n-manifold.

(i) If all the sectional curvatures K of M satisfy $K \geq 0$, then $\pi_1(M)$ has a finite generating system B, such that

$$|B| \leq 5^{\frac{n}{2}}.$$

(ii) If M is compact and if all the sectional curvatures K of M satisfy $K \geq -\lambda^2$ for some $\lambda \in \mathbb{R}$, then $\pi_1(M)$ has a finite generating system B, such that

$$|B| \leq (3 + 2\cosh(2\lambda d))^{\frac{n}{2}},$$

where $d := \operatorname{diam} M$.

Proof.

(i) Choose a universal Riemannian covering $\pi : \tilde{M} \to M$, $\tilde{p} \in \tilde{M}$, and let B be a short basis of M (we employ all the notation from Definition 7.6). Denote by $v_i \in \mathbb{S}_{\tilde{p}}\tilde{M}$ the unit vector pointing in the direction of $g_i\tilde{p}$, i.e. $\exp_{\tilde{p}}(\|g_i\|_{\tilde{p}}v_i) = g_i\tilde{p}$.
STEP 1: We claim, that for any $i < j$, we obtain

$$\alpha_{ij} := \measuredangle(v_i, v_j) \geq \frac{\pi}{3}.$$

This can be seen as follows: Consider the triangle in $\tilde{M}$ obtained by joining p, $g_i\tilde{p}$, $g_j\tilde{p}$ by minimizing geodesics. We would like to apply Toponogov's Theorem 7.4,(ii) to this triangle and compare it with a triangle in Euclidean space. Therefore we remark that its geodesics have lengths $d(\tilde{p}, g_i\tilde{p}) = \|g_i\|_{\tilde{p}}$, $d(\tilde{p}, g_j\tilde{p}) = \|g_j\|_{\tilde{p}}$ and since g_i is an isometry of $\tilde{M}$

$$d(g_i\tilde{p}, g_j\tilde{p}) = d(\tilde{p}, g_i^{-1}g_j\tilde{p}) = \|g_i^{-1}g_j\|_{\tilde{p}} \overset{5.8}{\leq} \|g_i^{-1}\|_{\tilde{p}} + \|g_j\|_{\tilde{p}} = \|g_i\|_{\tilde{p}} + \|g_j\|_{\tilde{p}}.$$

So we may apply 7.4,(ii) and obtain

$$\alpha_{ij} \geq \bar{\alpha}_{ij},$$

where $\bar{\alpha}_{ij}$ is the corresponding angle in the comparison triangle in Euclidean space. By Lemma 7.7 the side with length $\|g_i^{-1}g_j\|$ is the longest. By basic Euclidean geometry, the angle in the vertex opposite to the longest side is $\bar{a}_{ij} \geq \frac{\pi}{3}$.

Step 2: This shows, that the various $B_{1/2}(v_i) \subset B_{3/2}(0_{\tilde{p}}) \subset T_{\tilde{p}}\tilde{M}$ are all disjoint. Denoting by ω_n the volume of the n-dimensional unit sphere, we calculate:

$$\mathrm{vol}\Big(\dot{\bigcup_{0 \leq i \leq |B|-1}} B_{1/2}(v_i) \Big) \leq \mathrm{vol}(B_{3/2}(0_{\tilde{p}})) \Rightarrow |B| \frac{1}{2^n}\omega_n \leq \frac{3^n}{2^n}\omega_n \Rightarrow |B| \leq 3^n.$$

Step 3: This is almost the statement we want to prove. The bound can be sharpened by using both of the following optimizations: First denote by $w_i \in \mathbb{S}_{\tilde{p}}\tilde{M}$ the unit vector poining in direction $g_i^{-1}\tilde{p}$. By the same reasoning as in the previous step $\measuredangle(w_i, w_j) \geq \frac{\pi}{3}$, $i < j$, so the various $B_{\frac{1}{2}}(w_i)$ are disjoint from one another as well. By replacing g_i with g_i^{-1} we obtain, that $\measuredangle(w_i, v_j) \geq \frac{\pi}{3}$, $i < j$, as well.

Second we may replace the fulls balls $B_{1/2}(v_i)$ by their inner halfs $\hat{B}_{1/2}(v_i)$. These satisfy $\hat{B}_{1/2}(v_i) \subset B_{\sqrt{5}/2}(0_{\tilde{p}})$. Now we obtain:

$$\mathrm{vol}\Big(\dot{\bigcup_{0 \leq i \leq |B|-1}} \hat{B}_{1/2}(v_i) \cup \dot{\bigcup_{0 \leq i \leq |B|-1}} \hat{B}_{1/2(w_i)} \Big) \leq \mathrm{vol}(B_{\sqrt{5}/2}(0_{\tilde{p}})) \Rightarrow 2 \cdot \frac{1}{2}|B| \frac{1}{2^n}\omega_n \leq \frac{5^{\frac{n}{2}}}{2^n}\omega_n \Rightarrow |B| \leq 5^{\frac{n}{2}}.$$

(ii) We may assume $\lambda \geq 0$.

Step 1: Again consider the triangle of the geodesics joining the points $\tilde{p}$, $g_i\tilde{p}$, $g_j\tilde{p}$. These geodesics have lengths $l_i := \|g_i\|_{\tilde{p}}$, $l_j := \|g_j\|_{\tilde{p}}$, $l_{ij} := \|g_i^{-1}g_j\|_{\tilde{p}}$. Again we may compare the angle α_{ij} to an angle $\bar{\alpha}_{ij}$ in a corresponding comparison triangle but now in $M^2_{-\lambda^2}$ and obtain $\cos(\alpha_{ij}) \leq \cos(\bar{\alpha}_{ij})$. In $M^2_{-\lambda^2}$ the law of cosines (c.f. A.13 and 2.7) takes the form

$$\cosh(\lambda l_{ij}) = \cosh(\lambda l_i)\cosh(\lambda l_j) + \sinh(\lambda l_i)\sinh(\lambda l_j)\cos(\bar{\alpha}_{ij}).$$

Alltogether we obtain

$$\cos(\alpha_{ij}) \leq \cos(\bar{\alpha}_{ij}) = \frac{\cosh(\lambda l_i)\cosh(\lambda l_j) - \cosh(\lambda l_{ij})}{\sinh(\lambda l_i)\sinh(\lambda l_j)}$$

Step 2: We claim, that the expression on the right hand side is monotonously increasing in l_i on $]0, 2d]$ and therefore define $a := \cosh(\lambda l_j)$, $b := \cosh(\lambda l_{ij})$, $c := \sinh(\lambda l_j)$ and calculate

$$\begin{aligned}
\Big(\frac{\cosh(\lambda t)a - b}{\sinh(\lambda t)c}\Big)' &= \frac{ac\lambda\sinh(\lambda t)\sinh(\lambda t) - c\lambda\cosh(\lambda t)(\cosh(\lambda t)a - b)}{\sinh(\lambda t)^2c^2} \\
&= \frac{ac\lambda\sinh(\lambda t)^2 - ac\lambda\cosh(\lambda t)^2 + bc\lambda\cosh(\lambda t)}{\sinh(\lambda t)^2c^2} \\
&= \frac{ac\lambda\sinh(\lambda t)^2 - ac\lambda(1 + \sinh(\lambda t)^2) + bc\lambda\cosh(\lambda t)}{\sinh(\lambda t)^2c^2} = \lambda\frac{-ac + bc\cosh(\lambda t)}{\sinh(\lambda t)^2c^2}.
\end{aligned}$$

We have to show that the last expression is ≥ 0. Therefore it suffices to show, that $-ac + bc\cosh(\lambda t) \geq 0$. We assume $t > 0$ ($\Rightarrow c \neq 0$) and transform:

$$-ac + bc\cosh(\lambda t) \geq 0 \Longleftrightarrow bc\cosh(\lambda t) \geq ac \Longleftrightarrow b\cosh(\lambda t) \geq a$$

Finally, to show that the last inequality holds, consider:

$$b\cosh(\lambda t) = \cosh(\lambda l_{ij})\cosh(\lambda t) \geq \cosh(\lambda l_j)$$

STEP 3: Since $l_i \leq l_j \leq l_{ij} \leq 2d$ by Lemma 7.7, the monotonicity and $\cosh^2 - \sinh^2 = 1$ implies:

$$\cos(\alpha_{ij}) \leq \frac{\cosh(\lambda l_i)^2 - \cosh(\lambda l_{ij})}{\sinh(\lambda l_j)^2} \leq \frac{\cosh(\lambda l_j)^2 - \cosh(\lambda l_j)}{\sinh(\lambda l_j)^2} = \frac{\cosh(\lambda l_j)^2 - \cosh(\lambda l_j)}{\cosh(\lambda l_j)^2 - 1^2}$$
$$= \frac{\cosh(\lambda l_j)(\cosh(\lambda l_j) - 1)}{(\cosh(\lambda l_j) - 1)(\cosh(\lambda l_j) + 1)} = \frac{\cosh(\lambda l_j)}{\cosh(\lambda l_j) + 1} \overset{(1)}{\leq} \frac{\cosh(2\lambda d)}{\cosh(2\lambda d) + 1},$$

where the last inequality (1) is due to the fact, that the expression in monotonously increasing in l_j as can be seen via

$$\left(\frac{\cosh(\lambda t)}{\cosh(\lambda t) + 1}\right)' = \frac{\lambda \sinh(\lambda t)(\cosh(\lambda t) + 1) - \lambda \sinh(\lambda t) \cosh(\lambda t)}{(\cosh(\lambda t) + 1)^2} = \frac{1}{(\cosh(\lambda t) + 1)^2} > 0.$$

STEP 4: Define

$$\alpha := \arccos\left(\frac{\cosh(2\lambda d)}{\cosh(2\lambda d) + 1}\right) \leq \alpha_{ij}$$

and proceed as in the proof of the first statement: Now the $B_{\sin(\alpha/2)}(v_i) \subset T_{\tilde{p}}\tilde{M}$ are pairwise disjont. Notice that $\cos(2\alpha) = \cos(\alpha)^2 - \sin(\alpha)^2 = 1 - 2\sin(\alpha)^2$ and therefore by definition of α

$$\sin\left(\frac{\alpha}{2}\right)^2 = \frac{1 - \cos(\alpha)}{2} = \frac{1 - \frac{\cosh(2\lambda d)}{\cosh(2\lambda d)+1}}{2} = \frac{1}{2\cosh(2\lambda d) + 2}.$$

Now we obtain $C_i := \hat{B}_{\sin(\alpha/2)}(v_i) \subset B_{\sqrt{1+\sin(\alpha/2)^2}}(0_{\tilde{p}})$ and therefore

$$|B| \leq \frac{(1 + \sin(\alpha/2)^2)^{\frac{n}{2}}}{\sin(\alpha/2)^n} = \left(1 + \frac{1}{\sin(\alpha/2)^2}\right)^{\frac{n}{2}} = (1 + 2\cos(2\lambda d) + 2))^{\frac{n}{2}} = (3 + 2\cos(2\lambda d)))^{\frac{n}{2}}.$$

□

8 Synge's Theorem

Before we discuss further applications of Toponogovs theorem, let's discuss Synge's theorem first. Its proof motivates an estimate on the injectivity radius by Klingenberg.
First of all we discuss some basic facts regarding parallel transport and orientations and remind some elementary concepts from homotopy theory.

8.1 Theorem. Let M be an oriented Riemannian manifold and let $c : [a, b] \to M$ be a piecewise smooth curve. Let $P_c : T_{c(a)}M \to T_{c(b)}M$ be the parallel transport along c. Then P_c is orientation-preserving.

Proof. Since composing paths corresponds to composing parallel transports, we may assume, that c is smooth and that $\operatorname{im} c \subset U$, where $\varphi : U \to U' \subset \mathbb{R}^n$ is a chart for M. Let P_t, $a \leq t \leq b$, be the parallel transport along $c|[a, t]$. Denote by X_i the parallel translates of $\partial\varphi_i$ along c, i.e.

$$X_i(a) = \partial\varphi_i|_{c(a)} \qquad\qquad P_t(\partial\varphi_i|_{c(a)}) = X_i(t).$$

We may expand $X_i = A_i^j \partial\varphi_j$. Then the A_i^j are solutions of an ODE satisfying $A_i^j(a) = \delta_i^j$ and hence smooth in t. The matrix $(A_i^j(t))$ is a coordinate representation of P_t with respect to the $\partial\varphi_i$, $1 \leq i \leq n$, and P_t is invertible. This implies $\det(A_i^j(t)) \neq 0$ for all $t \in [a, b]$. Since $\det(A_i^j(a)) = 1$ this implies $\det(A_i^j(t)) > 0$ for all t. □

8.2 Definition (free homotopy classes)**.** Let $c : [0, 1] \to M$ be a continuous curve. Then c is a *loop*, if $c(0) = c(1)$. A *free homotopy* between two loops c and c' is a continuos map $H : [0, 1] \times [0, 1] \to M$ such that

$$\forall t \in [0, 1] : H(0, t) = c(t) \qquad \forall t \in [0, 1] : H(1, t) = c'(t) \qquad \forall s \in [0, 1] : H(s, 0) = H(s, 1).$$

In that case c and c' are *freely homotopic*. The free homotopy class of c is denoted by $[c]_Z$. The set of all free homotopy classes of curves in M is denoted by $Z(M)$.

8.3 Remark. Notice that this concept is weaker than path homopy. If c and c' are two loops based at a certain point $p \in M$, i.e. $c(0) = c(1) = p = c'(0) = c'(1)$, then c and c' are *path homotopic*, if there exists *path homotopy*, i.e. a free homotopy $H : [0, 1] \times [0, 1] \to M$ which additionally satisfies $\forall s \in [0, 1] : H(s, 0) = p$. The path homotopy class of c is denoted by $[c]_\pi$.

8.4 Lemma. Any two path homotopic loops in M based at any point $p \in M$ are freely homotopic. The canonical inclusion $f : \pi_1(M, p) \to Z(M)$ is surjective and in addition

$$\forall [c]_\pi, [c']_\pi \in \pi_1(M, p) : f([c]_\pi) = f([c']_\pi) \iff [c]_\pi \text{ is conjugate to } [c']_\pi \text{ in } \pi_1(M, p)$$

Proof.
Step 1 (Surjectivity): Let $[c]_Z \in Z(M)$ be arbitrary. Since M is path connected there exists a path $\sigma : [0, 1] \to M$ such that $\sigma(0) = p$ and $\sigma(1) = c(0) = c(1)$. Clearly the composite $\sigma * c * \sigma^{-1}$ is a loop based at p, so $[\sigma * c * \sigma^{-1}]_\pi \in \pi_1(M, p)$. Furthermore $f([\sigma * c * \sigma^{-1}]_\pi) = [c]_Z$. A free homotopy can be constructed as follows: For $s \in [0, 1]$ let $H(s, \_)$ be the path from $\sigma(s)$ to $\sigma(1)$ (traversing σ at threefold speed), then from $\sigma(1) = c(0)$ traversing c at threefold speed to $c(1) = \sigma(1)$ and then from $\sigma(1)$ again at threefold speed back to $\sigma(s)$.
Step 2 (Conjugacy):

□

8.5 Corollary (null-homotopy)**.** A loop c is null-homotopic in $\pi_1(M, p)$ if and only if it is freely null-homotopic.

Proof.
"⇒": Follows directly from the fact that every path homotopy is a free homotopy.
"⇐": By the Lemma 8.4 above, $[c]_Z = [c_p]_Z$ (where c_p is the constant curve at p) implies, that there exists $[\sigma] \in \pi_1(M, p)$ such that

$$[c]_\pi = [\sigma]_\pi [c_p]_\pi [\sigma]_\pi^{-1} = [c_p]_\pi.$$

□

8.6 Lemma. Let M be compact and let $c_0 : [0,1] \to M$ be a loop in M, which is not freely null-homotopic. Then

$$L_0 := \inf\{L(c') \mid c' : [0,1] \to M \text{ is freely homotopic to } c_0\} > 0$$

and there exists a closed geodesic $c : [0,1] \to M$, which is freely homotopic to c_0 and has length L_0.

Proof.
STEP 1 (Existence of c): Let (c_n) be a sequence of loops freely homotopic to c_0 such that $L(c_n) \to L_0$ as $n \to \infty$. Since reparametrization changes neither the homotopy class nor the length, we may assume, that all the c_n have constant speed parametrizations. [10] Since every convergent sequence is bounded, we may assume $L(c_n) \leq L_0 + 1$. We calculate for any $n \in \mathbb{N}$ and any $t, t' \in [0,1]$

$$d(c_n(t), c_n(t')) \leq L(c_n|[t,t']) = \int_t^{t'} \|\dot{c}_n(t)\| dt = L(c_n)|t - t'| \leq (L_0 + 1)|t - t'|$$

and thus c_n is Lipschitz continuous with Lipschitz constant $\mathrm{Lip}(c_n) \leq L_0 + 1$ independent of n. Consequently the theorem of Arzelá-Ascoli implies, that there exists a subsequence converging uniformly to a loop $c : [0,1] \to M$. We assume that this sequence is c_n itself and claim that this loop has the desired properties.
STEP 2 (free homotopy): Let $i(M) > \varepsilon > 0$ (this is possible due to the compactness of M). Uniform convergence implies in particular

$$\exists n \in \mathbb{N} : \forall t \in [0,1] : d(c_n(t), c(t)) \leq \varepsilon.$$

Denote by $c_t : [0,1] \to M$ the unique geodesic from $c_n(t)$ to $c(t)$, $0 \leq t \leq 1$. Then $H : [0,1] \times [0,1] \to M$, $(s,t) \mapsto c_t(s)$ is a free homotopy from c_n to c (smoothness follows from the smoothness of exp and the hypothesis $\varepsilon < i(M)$). Consequently c is freely homotopic to c_n and thus to c_0.
STEP 3 (Length): By definition and step 2 we have $L(c) \geq L_0$. For any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ the curve c_n also has Lipschitz constant $\mathrm{Lip}(c_n) \leq L_0 + \frac{\varepsilon}{3}$. Since $c_n \to c$ uniformly, there exists $N' \geq N$ such that $d(c_n, c) \leq \frac{\varepsilon}{3}$. Thus

$$\forall [t,t'] \in [0,1] : d(c(t), c(t')) \leq d(c(t), c_n(t)) + d(c_n(t), c_n(t')) + d(c_n(t'), c(t')) \leq L_0 + \varepsilon$$

which implies $\mathrm{Lip}(c) \leq L_0$. In particular since c has constant speed as well

$$L(c) = \int_0^1 \|\dot{c}(t)\| dt = \|\dot{c}(0)\| = \lim_{t \searrow 0} \frac{d(c(0), c(t))}{t} \leq \lim_{t \searrow 0} \frac{\mathrm{Lip}(c) t}{t} \leq L_0.$$

Alltogether, this implies $L(c) = L_0$ and therefore c has constant speed L_0.
Since c is freely homotopic to c_0 and c_0 is not null-homotopic, c is not null-homotopic as well and in particular not itself a constant map. Thus $L_0 = L(c) > 0$.

[10] These can be obtained by first reparametrzing the curves by arclength and then scaling them back to $[0,1]$ by a constant factor. This factor equals the length of the curve.

STEP 4 (Geodesic): Suppose to the contrary, that c is not a closed geodesic. Then there exists a $t_0 \in [0,1]$, such that for arbitrariliy small $t_0 > \varepsilon > 0$ the curve $c|[t_0-\varepsilon, t_0+\varepsilon]$ is not minimizing from $c(t_0-\varepsilon)$ to $c(t_0+\varepsilon)$ [11]. If we replace this segment by a minimizing segment, we obtain a closed curve c' which is freely homotopic to c (and thus to c_0) having length $< L_0$. Contradiction!

□

8.7 Theorem (Synge)**.** Let M be a compact connected orientable Riemannian manifold of even dimension. If M has strictly positive sectional curvature everywhere, then M is simply connected.

Proof. Suppose $\pi_1(M) \neq 0$. Then there exists a non-trivial path homotopy class $[c] \in \pi_1(M)$. By Corollary 8.5 this class is also not freely null-homotopic. By Lemma 8.6 it has a representative $c : [0,1] \to M$, which is a non-trivial closed geodesic minimizing the length over its own free homotopy class.
By Lemma 8.9 proven below there exists a periodic parallel unit length vector field X along c. Define $H :]-\varepsilon, \varepsilon[\times [0,1] \to M$ by $(s,t) \mapsto \exp_{c(t)}(sX(t))$. Then $c_s := H(s, \_)$ is freely homotopic to c (since H itself is a free homotopy). Now consider the index form (c.f. [2, 10.14] along c and calculate

$$\partial_s^2(L(c_s))|_{s=0} = I(X,X) = -\int_0^1 D_t^2 X + Rm(X, \dot{c}, \dot{c}, X) dt = -\int_0^1 K(X, \dot{c}) dt < 0$$

by hypothesis. This contradicts the fact, that c is length minimizing (c.f. [2, 10.13]). □

8.8 Remark. With a similar argumentation one can show a converse: If M is compact, connected of odd dimension and has strictly positive sectional curvature, then M is orientable. Argueing as in 8.1 one shows, that it does not depend on the free homotopy class of the closed curve c whether or not P_c preserves the orientation. If M is not orientable, there exists a geodesic c as in the proof of 8.7, such that P_c does not preserve the orientation.

8.9 Lemma. Let M be oriented and of even dimension. Let c be a nontrivial closed geodesic in M. Then there exists a periodic unit length parallel normal vector field along c.

Proof. We may assume $c : [0,1] \to M$ has constant speed. Denote by $P_c : T_pM \to T_pM$, $p := c(0)$, the parallel transport along c. By Theorem 8.1 P_c is orientation-preserving, i.e. $\det(P_c) > 0$. Since c is a closed geodesic $P_c(\dot{c}(0)) = \dot{c}(1) = \dot{c}(0)$ by uniqueness of geodesics (the parallel translate of $\dot{c}(0)$ is $\dot{c}(t)$). This implies alltogether, that P_c restricts to an orientation-perserving isometry on $V := \dot{c}(0)^\perp$. By hypothesis $\dim V$ is odd. Basic linear algebra [12] implies the existence of a unit vector $X_0 \in V$ satisfying $P_c(X_0) = X_0$. Let X be the parallel translate of X_0 along c satisfying $X(0) = X_0$. Then

$$X(1) = P_c(X_0) = X_0 = X(0).$$

□

[11] In case $t_0 \in \{0,1\}$ one may periodically extend c to all of $\mathbb{R}$.

[12] More precisely: By choosing any positive ONB, we may consider a coordinate representation A of P_c acting as an isometry on $\mathbb{R}^n$. Since n is odd, there exists at least one real eigenvalue λ. Since A is an isometry, for any unit eigenvector v

$$|\lambda| = |\lambda||v| = |\lambda v| = |Av| = |v| = 1,$$

thus $\lambda = \pm 1$. It may be possible, that A is not diagonalizable over $\mathbb{R}$ and may have complex eigenvalues. But for a complex eigenvalue μ, its conjugate $\bar{\mu}$ is also a zero of the characteristic polynomial. So the number of complex eigenvalues is even and its product is positive since $\mu\bar{\mu} = |\mu|^2$. Therefore the product of real eigenvalues must also equal 1, since otherwise the determinant of A was negative, which contradicts our assumption, that A is orientation preserving. Therefore at least one real eigenvalue λ must equal $+1$.

9 The Sphere Theorem

The Sphere Theorem by Rauch, Klingenberg and Berger is one of the most celebrated theorems in global Riemannian geometry (aside from the theorem of Gauss-Bonnet for surfaces).

9.1 Theorem (Sphere Theorem)**.** Let M be an n-dimensional compact simply connected manifold. If there exists a $\delta \in \mathbb{R}$ such that all sectional curvatures K of M satisfy

$$\frac{1}{4} < \delta \leq K \leq 1,$$

then M is homeomorphic to $\mathbb{S}^n$.

9.2 Remark.

(i) The bounds for the curvature are optimal, i.e. the theorem is wrong for $\delta = \frac{1}{4}$. Up to normalization the standard metric on $\mathbb{C}\,\mathbb{P}^n$ is a counterexample (c.f. chapter 10). This is the reason why $\mathbb{C}\,\mathbb{P}^n$ is sometimes called "the roundest space which is not a sphere".

(ii) The question wether or not M is also diffeomorphic to $\mathbb{S}^n$ had been subject to much research for a long time. The problem was the existence of *exotic spheres* proven by Milnor. These are spaces which equal $\mathbb{S}^n$ topologically, but have a different smooth structure. In 2007 Brendle and Schoen were able to prove, that M is in fact always diffeomorphic to $\mathbb{S}^n$ with its standard smooth structure. Their result is known as the *"Differentiable Sphere Theorem"* and uses the Ricci flow.

Before we are able to prove this theorem we require some preparation.

9.3 Lemma. Let M be compact with sectional curvature $K \leq \kappa$, $\kappa \in \mathbb{R}$. Then

$$i(M) \geq \min\{R_\kappa, \frac{l}{2}\},$$

where l is the length of a shortest non-trivial geodesic in M.

Proof. Since M is compact, there exists $p \in M$ and $v \in \mathbb{S}_p M$ such that

$$i(M) = i(p) = t_0(v).$$

Define $q := \exp_p(t_0(v)v)$. By Corollary 4.4 there are two possibilities:
CASE 1: If p is conjugate to q along a minimizing geodesic, there exists a Jacobi field J along c such that $J(0) = 0 = J(t_0(v))$. By Theorem 4.7, this implies $t_0(v) \geq R_\kappa$.
CASE 2: If there exists a closed geodesic c through p and q such that $c(0) = c(2t_0(v)) = p$, $c(t_0(v)) = q$, then $L(c) = 2t_0(v)$.

□

9.4 Lemma (Klingenberg, even dimensional)**.** Let M be compact, oriented, of even dimension and let all the sectional curvatures K satisfy $0 < K \leq \kappa$. Then

$$i(M) \geq R_\kappa.$$

Proof. Suppose to the contrary that $i(M) < R_\kappa$. By Lemma 9.3 there exists a shortest closed geodesic c on M with length $l = 2i(M) < 2R_\kappa$. By Lemma 8.9 there exist a periodic parallel normal unit length vector field X along c. The map $]-\varepsilon, \varepsilon[\times [0,1] \to M$

$$(s,t) \mapsto c_s(t) := \exp_{c(t)}(sX(t))$$

is a free homotopy and a smooth variation of the geodesic c. As in the proof of Synge's Theorem 8.7, the hypothesis $K > 0$ implies, that $I(X, X) < 0$ and therefore $s \mapsto L(c_s)$ as a local maximum at $s = 0$. So for small $s > 0$

$$L(c_s) < L(c) = l = 2i(M) < 2R_\kappa$$

and

$$d(c_s(t), c_s(0)) \leq \frac{1}{2} L(c_s) < i(M),$$

since c_s is a closed curve. Thus there exists a unique minimizing geodesic $\sigma_{s,t} : [0, 1] \to M$ joining $c_s(0)$ and $c_s(t)$, $0 \leq t \leq 1$. The map $[0, 1]^3 \to M$, $(s, t, r) \mapsto \sigma_{s,t}(r)$ is smooth. For any fixed s there exists $t(s)$ such that $\forall t \in [0, 1] : L(\sigma_{s,t(s)}) \geq L(\sigma_{s,t})$ due to the compactness of $[0, 1]$. By construction c is not a trivial one-point curve. Hence c_s is non trivial, so by construction $\sigma_{s,t(s)}$ is nontrivial, thus $0 < t(s) < 1$. Now fix s and define the variation $H : [0, 1]^2 \to M$, $(t, r) \mapsto \sigma_{s,t}(r)$. The length of this variation has a maximum at $t = t(s)$ by definition and thus the first variation formula implies

$$0 = \partial_t L(H_t)|_{t=t(s)} = \langle \partial_t H(t, 1)|_{t=t(s)}, \dot{\sigma}_{s,t(s)}(1)\rangle = \langle \partial_t c_s(t)|_{t=t(s)}, \dot{\sigma}_{s,t(s)}(1)\rangle$$

and therefore

$$\dot{\sigma}_{s,t(s)}(1) \perp \dot{c}_s(t(s)).$$

Since c is a closed geodesic $\forall t \in [0, 1] : d(c(0), c(t)) = \min\{t, 1 - t\}$ and therefore

$$\lim_{s \to 0} t(s) = \frac{1}{2}.$$

Any accumulation point of $(\sigma_{s,t(s)})_s$ as $s \to 0$ is a minimizing geodesic $\sigma : [0, 1] \to M$ joining $c(0)$ and $c(\frac{1}{2})$ which satisfies

$$\dot{\sigma}(1) \perp c(\frac{1}{2}).$$

So we have found three different minimizing geodesics joining $p := c(0)$ and $q := c(\frac{1}{2})$. By Theorem 4.3 p and q are conjugate, which contradicts $d(p, q) < R_\kappa$. □

9.5 Lemma (Klingenberg)**.** Let M be a compact simply connected manifold with sectional curvatures K satisfying

$$\frac{1}{4} < K \leq 1.$$

Then

$$i(M) \geq \pi.$$

9.6 Remark. The proof of this lemma is rather extensive. In odd dimensions it requires Morse theory, which is beyond the scope of this script to cover and we will therefore go without it.
In even dimensions we argue, that Lemma 9.4 for $\kappa = 1$ implies

$$i(M) \geq R_\kappa = \frac{\pi}{\sqrt{\kappa}} = \pi.$$

9.7 Lemma (Berger)**.** Let M be complete and connected, $p \in M$. Let the function $d_p := d(p, \_)$ have a local maximum in q. Then for every $v \in T_q M$, $v \neq 0$, there exists a minimizing geodesic c from q to p such that

$$\measuredangle(\dot{c}(0), v) \leq \frac{\pi}{2}.$$

Proof. Let σ be the geodesic through q with initial velocity v and let $\|v\| = 1$. We distinguish two cases.

CASE 1: There exists a sequence of reals $t_i \searrow 0$ corresponding to a sequence c_i of minimizing geodesics from $\sigma(t_i)$ to p such that

$$\measuredangle(\dot{c}_i(0), \dot{\sigma}(t_i)) \leq \frac{\pi}{2}.$$

We may identify the sequence c_i with a sequence in TM by $c_i \mapsto \dot{c}_i(0)$. This sequence is contained in a compact subset and therefore has an accumulation point, which corresponds to a minimizing geodesic c from q to p satisfying

$$\measuredangle(\dot{c}(0), v) \leq \frac{\pi}{2}.$$

CASE 2: Since M is complete the only way how this can fail is, that there exists $\varepsilon > 0$, such that for all $t \in]0, \varepsilon[$ all minimizing geodesics (there exists at least one such geodesic c_t) from $\sigma(t)$ to p satisfy

$$\measuredangle(\dot{c}_t(0), \dot{\sigma}(t)) > \frac{\pi}{2}.$$

We may assume that c_t has a constant speed parametrization $c_t : [0,1] \to M$ and remark that $(t,r) \mapsto c_t(r)$ is eventually discontinuous in t. Nevertheless for any fixed t we may choose a smooth variation $H :] - \delta, \delta[\times[0,1] \to M$, $(s,r) \mapsto c_{t,s}(r)$, satisfying $c_{t,s}(0) = \sigma(t-s)$ and $c_{t,s}(1) = p$. By the first variation formula

$$\begin{aligned} \partial_s(Lc_{t,s})|_{s=0} &= -\langle \partial_s H(s,r)|_{(s,r)=(0,0)}, \dot{c}_t(0)\rangle = -\langle \partial_s \sigma(t-s)|_{s=0}, \dot{c}_t(0)\rangle \\ &= \langle \dot{\sigma}(t), \dot{c}_t(0)\rangle = \|\dot{\sigma}(t)\| \|\dot{c}_t(0)\| \cos(\measuredangle(\dot{\sigma}(t), \dot{c}_t(0))) < 0 \end{aligned}$$

by hypothesis. Thus for small $s > 0$, we obtain $L(c_{t,s}) < L(c_t)$. But this implies for such s

$$d(p, \sigma(t-s)) \leq L(c_{t,s}) < L(c_t) = d(p, \sigma(s))$$

so $d(p, \sigma(t))$ is monotonously decreasing as $t \searrow 0$. This is absurd since d_p has a local maximum in q.

□

In the following we will explain Bergers construction of a homeomorphism $\mathbb{S}^n \to M$ if M satisfies the hypothesis of the sphere theorem.

9.8 Lemma (Existence of hemispheres)**.** Let M be compact, connected, let all the sectional curvatures K satisfy $K \geq \delta > 0$ and let $\mathrm{diam}(M) \geq \frac{\pi}{2\sqrt{\delta}}$. Let $p, q \in M$ such that $d(p,q) = \mathrm{diam}(M)$. Then

$$\bar{B}_{\frac{\pi}{2\sqrt{\delta}}}(p) \cup \bar{B}_{\frac{\pi}{2\sqrt{\delta}}}(q) = M.$$

Proof. We may assume $\delta = 1$ (since otherwise we may scale the metric, c.f. A.15). Let $x \in M$ and assume $x \notin \bar{B}_{\frac{\pi}{2}}(q)$, i.e. $d(x,q) \geq \frac{\pi}{2}$. Let c_1 be a minimizing geodesic from q to x. By Berger's Lemma 9.7 there exists a minimizing geodesic c from q to p such that $\alpha := \measuredangle(\dot{c}(0), \dot{c}_1(0)) \leq \frac{\pi}{2}$. Now the statement follows from Toponogov's Theorem 7.4,(iii): If M is isometric to a sphere the statement is trivial anyway. By the Theorem of Bonnet/Myers we obtain $l := L(c), l_1 := L(c_1) \leq \pi$. Therefore Toponogov's Theorem is applicable and we obtain

$$d(p,x) \leq d(\bar{c}(l), \bar{c}_1(l_1)).$$

Now $\bar{c}, \bar{c}_1$ are geodesics in the sphere $\mathbb{S}^n$ having lengths l,l_1. Connecting the points $\bar{c}(l)$, $\bar{c}_1(l_1)$ with a minimizing geodesic $\bar{c}_2$ of length l_2 we obtain a spherical triangle and therefore by the law of cosines (c.f. A.13)

$$\cos(l_2) = \cos(l)\cos(l_1) + \sin(l)\sin(l_1)\cos(\alpha) \geq 0,$$

since $l, l_1 \geq \frac{\pi}{2}$ by construction. Alltogether

$$d(p,x) \leq d(\bar{c}(l), \bar{c}_1(l_1)) = l_2 \leq \frac{\pi}{2}.$$

□

9.9 Lemma (Existence of an equator)**.** Let M be compact, connected and simply connected and let all the sectional curvatures K of M satisfy $\frac{1}{4} < \delta \leq K \leq 1$. Then $\operatorname{diam}(M) \geq \pi$. If $p, q \in M$ such that $d(p,q) = \operatorname{diam}(M)$, for any unit speed minimizing geodesic c starting at p there exists a unique $t \in]0, \frac{\pi}{2\sqrt{\delta}}]$ such that $d(p, c(t)) = d(q, c(t))$.

Proof. By Klingenberg's Lemma we obtain

$$\operatorname{diam}(M) \geq i(M) \overset{9.5}{\geq} \pi.$$

Now let $p, q \in M$ such that $d(p,q) = \operatorname{diam}(M)$ and define $t_\delta := \frac{\pi}{2\sqrt{\delta}}$.
Step 1 (Existence of t): Define $f : [0, t_\delta] \to \mathbb{R}$ by

$$t \mapsto d(p, c(t)) - d(q, c(t)).$$

It is clear, that this function is continuous. We obtain

$$f(0) = d(p, c(0)) - d(q, c(0)) = -d(q,p) < 0.$$

Since $t_\delta < \pi$, we obtain that $c|[0, t_\delta]$ is minimizing and therefore $x := c(t_\delta) \in \partial B_{t_\delta}(p)$.
We claim that $x \in \bar{B}_{t_\delta}(q)$: Otherwise $x \notin \bar{B}_{t_\delta}(q)$, so there exists an open neighbourhood U near x such that $U \cap \bar{B}_{t_\delta}(q) = \emptyset$. But by construction $x \in \partial \bar{B}_{t_\delta}(p)$ and so by Lemma 9.8 and the definition of the topological boundary $U \cap \bar{B}_{t_\delta}(q) \neq \emptyset$, which is a contradiction.
Thus we obtain

$$f(t_\delta) = d(p, c(t_\delta)) - d(q, c(t_\delta)) \geq t_\delta - t_\delta = 0.$$

By the intermediate value theorem there exists $t \in]0, t_\delta]$ such that $f(t) = 0$.
Step 2 (Uniqueness): Assume there are $0 < t_0 < t_1 \leq t_\delta < \pi$ with the desired property, i.e.

$$t_0 = d(p, c(t_0)) = d(q, c(t_0)), \qquad t_1 = d(p, c(t_1)) = d(p, c(t_1)).$$

Let c_0 be a unit speed minimizing geodesic from q to $c(t_0)$. The composite curve $\sigma := c_0 * c|[t_0, t_1]$ has length

$$L(\sigma) = L(c_0) + L(c|[t_0, t_1]) = t_0 + (t_1 - t_0) = t_1$$

and joins q and $c(t_1)$. By hypothesis σ is minimizing and therefore a geodesic. Thus σ is smooth and therefore $\dot{c}_0(t_0) = \dot{c}(t_0)$. Since $c_0(t_0) = c(t_0)$ by construction the geodesics c and c_0 are equal on their common domain of definition. In particular

$$p = c(0) = c_0(0) = q,$$

which implies $\operatorname{diam}(M) = d(p,q) = 0 < \pi$. Contradiction!

$\square$

9.10 Remark. The uniqueness of t in the previous Theorem 9.9 implies, that we can interpret this as a continuous function $t : \mathbb{S}_p M \to]0, \frac{\pi}{2\sqrt{\delta}}]$. Since the roles of p and q may be interchanged, we also obtain a continuous function $t : \mathbb{S}_q M \to]0, \frac{\pi}{2\sqrt{\delta}}]$.

Proof. (of the Sphere Theorem) Following Berger we construct a homeomorphism $h : \mathbb{S}^n \to M$.
Step 1 (Construction of h): Since M is compact there exist $p, q \in M$ such that

$$d(p,q) = \operatorname{diam}(M) \geq i(M) \overset{9.5}{\geq} \pi.$$

With t defined as in the Remark 9.10 above, define $f : T_pM \setminus \{0\} \to \bar{B}_{\frac{\pi}{2\sqrt{\delta}}}(0_p) \subset T_pM$ by

$$v \mapsto t\Big(\frac{v}{\|v\|}\Big)\frac{v}{\|v\|}.$$

For any $\alpha > 0$, this function satisfies $f(\alpha v) = f(v)$ and

$$d(\exp_p(f(v)), p) = d(\exp_p(f(v)), q)$$

by construction. Choose an isometry $I : T_{\bar{p}}\mathbb{S}^n \to T_pM$, for some $\bar{p} \in \mathbb{S}^n$. Denote by $\bar{q} := -\bar{p}$ the antipodal point of $\bar{p}$. Define $h : \mathbb{S}^n \to M$ by

$$x \mapsto \begin{cases} p & , x = \bar{p} \\ \exp_p\left(\frac{d(x,\bar{p})}{\pi/2} \cdot (f \circ I \circ \exp_{\bar{p}}^{-1})(x)\right) & , \ d(x,\bar{p}) \leq \frac{\pi}{2} \\ \exp_q\left(\frac{d(x,\bar{q})}{\pi/2} \cdot (\exp_q^{-1} \circ \exp_p \circ f \circ I \circ \exp_{\bar{p}}^{-1})(x)\right) & , \ d(x,\bar{q}) \leq \frac{\pi}{2} \\ q & , x = \bar{q} \end{cases}$$

Step 2 (Continuity of h): Define $M^+ := \bar{B}_{\frac{\pi}{2\sqrt{\delta}}}(p)$ and $M^- := \bar{B}_{\frac{\pi}{2\sqrt{\delta}}}(q)$. First of all notice, that if $x = \bar{p}$ obviously $d(x, \bar{p}) = 0$ and therefore the two upper cases in the definition of h agree and define a continuous function $h^+ : H^+ \to M^+ \subset M$ from the upper hemisphere of $\mathbb{S}^n$ to an upper hemisphere of M. Analogously the lower two cases define a continuous function $h^- : H^- \to B_{\frac{\pi}{2\sqrt{\delta}}}(q) \subset M$. On the equator $H^+ \cap H^-$, we have $d(x, \bar{p}) = d(x, \bar{q}) = \pi/2$ and therefore $h^+|_{H^+\cap H^-} = h^-|_{H^+\cap H^-}$ and thus h is continuous.

Step 3: Careful analysis of the definition of h reveals that it is injective and surjective. Since $\mathbb{S}^n$ is compact and M is hausdorff, this automatically implies, that h^{-1} is continuous (c.f. [4, 4.25]) and thus h is a homeomorphism.

$\square$

10 Complex Projective Space

10.1 Preliminaries

10.1 Remark. We denote by $\langle \_, \_ \rangle_h : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$

$$\langle u, v \rangle_h := \sum_{i=1}^{n} u_i \bar{v}_i$$

the canonical hermitian form on $\mathbb{C}^n$. Since $\mathbb{C}^n$ is canonically isomorphic to $\mathbb{R}^{2n}$, we can also use the Euclidean scalar product $\langle \_, \_ \rangle_e \to \mathbb{R} : \mathbb{R}^n \times \mathbb{R}^n$

$$\langle x, y \rangle_e := \sum_{i=1}^{n} x_i y_i$$

for $2n$, i.e.

$$\forall u, v \in \mathbb{C}^n : \langle u, v \rangle_e := \langle \mathrm{Re}(u), \mathrm{Re}(v) \rangle_e + \langle \mathrm{Im}(u), \mathrm{Im}(v) \rangle_e.$$

For any $u, v \in \mathbb{C}^n$ remember the following relations from Linear Algebra:

(i) $\langle u, v \rangle_h = \langle \mathrm{Re}(u), \mathrm{Re}(v) \rangle_e + \langle \mathrm{Im}(u), \mathrm{Im}(v) \rangle_e + i(\langle \mathrm{Im}(u), \mathrm{Re}(v) \rangle_e - \langle \mathrm{Re}(u), \mathrm{Im}(v) \rangle_e)$

(ii) $\mathrm{Re}\langle u, v \rangle_h = \langle u, v \rangle_e$

(iii) $u \perp_h v \Rightarrow u \perp_e v$

(iv) $u \perp_e v$ and $u \perp_e iv \Longrightarrow u \perp_h v$.

(v) $\|u\| := \|u\|_e = \|u\|_h$.

We will constantly index expressions with e resp. h to indicate dependence on the euclidean or hermitian form unless it is clear that the expression is independent on that index. For example if $U \subset \mathbb{R}^{2n}$ is a complex subspace and $x \in \mathbb{R}^{2n}$, we denote $x \perp U$ to express, that x is perpendicular to U, since $x \perp_e U \Leftrightarrow x \perp_h U$.

10.2 Definition (Complex projective space)**.** The n-dimensional *complex projective space* is the set of all lines in $\mathbb{C}^{n+1}$, i.e.

$$\mathbb{C}\,\mathbb{P}^n := \{z \in \mathbb{C}^n \setminus \{0\}\}/ \sim,$$

where $z_1 \sim z_2 :\Leftrightarrow \exists \lambda \in \mathbb{C} : z_2 = \lambda z_1$.
We may also write $\mathbb{C}\,\mathbb{P}^n = \mathbb{S}^{2n+1}/ \sim$, where $z_1 \sim z_2 :\Leftrightarrow \exists r \in \mathbb{S}^1 \subset \mathbb{C} : z_2 = r z_1$. In both cases the equivalence class of a point $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ is denoted by $[z_0 : \ldots : z_n]$.

10.3 Remark (Complex structure)**.** Remember that $\mathbb{C}\,\mathbb{P}^n$ is a manifold in a canonical way: For any $0 \leq i \leq n+1$ define $U_i := \{[z_0, \ldots, z_n] | z_i \neq 0\}$ (this is well-defined) and $\varphi_i : U_i \to \mathbb{C}^n$, $[z_0, \ldots, z_n] \mapsto (\frac{1}{z_i}(z_0, \ldots, \hat{z}_i, \ldots, z_n))$ (this is also well-defined). These maps determine an atlas: Define $\psi_i : \mathbb{C}^n \to U_i$, $(z_1, \ldots, z_n) \mapsto [z_0 : \ldots : 1 : \ldots : z_n]$ (where the 1 is at position i). We obtain:

$$\psi_i(\varphi_i([z_0 : \ldots : z_n])) = \psi_i(\frac{1}{z_i}(z_0, \ldots, \hat{z}_i, \ldots, z_n)) = [\frac{z_0}{z_i} : \ldots : \frac{1}{z_i} : \ldots : \frac{z_n}{z_i}] = [z_0 : \ldots : z_n]$$

$$\varphi_i(\psi_i(z_1, \ldots, z_n)) = \psi_i([z_1 : \ldots : 1 : \ldots : z_n]) = (z_1, \ldots, z_n)$$

thus $\psi_i = \varphi_i^{-1}$. If $i \neq j$, then the transition map on $\varphi_j(U_j)$ is given by

$$\varphi_j(\psi_i(z_1, \ldots, z_n)) = \varphi_j([z_1 : \ldots : 1 : \ldots : z_n]) = \frac{1}{z_j}(z_1, \ldots, \hat{z}_j, \ldots, 1, \ldots, z_n)$$

This is biholomorphic, so we have defined a complex structure on $\mathbb{C}\,\mathbb{P}^n$. In particular $\mathbb{C}\,\mathbb{P}^n$ is a real manifold and $\dim_{\mathbb{R}} \mathbb{C}\,\mathbb{P}^n = 2 \dim_{\mathbb{C}} \mathbb{C}\,\mathbb{P}^n = 2n$. Throughout this chapter we will usually think of $\mathbb{C}\,\mathbb{P}^n$ as $\mathbb{S}^{2n+1}/ \sim$ and as a real manifold.

10.4 Definition (Hopf circles)**.** The projection $\pi : \mathbb{S}^{2n+1} \to \mathbb{C}\,\mathbb{P}^n$ is called *Hopf map*. Its fibres are called *Hopf circles*, i.e. for any $p \in \mathbb{C}\,\mathbb{P}^n$ we call $\pi^{-1}(p)$ a Hopf circle. Since π is surjective, we may also think of $\mathbb{C}\,\mathbb{P}^n$ as the set of all Hopf circles.

10.5 Lemma. For any $x \in \mathbb{S}^{2n+1}$ let $H(x)$ be the Hopf circle through x, i.e. $H(x) := \pi^{-1}(\{\pi(x)\})$. Then

$$H(x) = \{e^{i\varphi}x \mid \varphi \in \mathbb{R}\} \qquad\qquad T_xH(x) \cong \mathbb{R}ix.$$

Proof. The first equation holds by definition. Notice that $\mathbb{R} \to H(x)$, $\varphi \mapsto H(x)(\varphi) := e^{i\varphi}x$ is a curve in $H(x)$ through x. We have

$$\dot{H}(x)(0) = ix \neq 0$$

and since $H(x)$ is a 1-dimensional manifold, we obtain the statement concerning the tangential space. □

10.6 Lemma. The Hopf map $\pi : \mathbb{S}^{2n+1} \to \mathbb{C}\,\mathbb{P}^n$ is a smooth submersion. For any $x \in \mathbb{S}^n$

$$\ker \pi_*|_x \cong \mathbb{C}\,x =: L_x.$$

Consequently the restriction of $\pi_*|_x$ to any complement of L_x is an isomorphism.

Proof. By construction π is smooth. Let $c : I \to \mathbb{S}^{2n+1}$ be a smooth curve such that $c(0) = x$. In case $c(I) \subset H(x)$, the curve $\pi \circ c$ is constant and therefore

$$0 = (\pi \circ c)'(0) = \pi_*|_x(\dot{c}(0)) \Longrightarrow \dot{c}(0) \in \ker \pi_*|_x.$$

So $T_xH(x) \subset \ker \pi_*|_x$ and $\dim \ker \pi_*|_x \geq 1$.
On the other hand, there exists $0 \leq i \leq n$ such that $x_i \neq 0$, where we think here of x as $x \in \mathbb{S}^{2n+1} \subset \mathbb{C}^{\,n+1}$. By composing with the chart we obtain

$$\varphi_i \circ \pi \circ c = \frac{1}{c_i}(c_0, \ldots, \hat{c}_i, \ldots, c_n).$$

If we set $c_i := x_i$ we still have n complex parameters left which generate $2n$ real linear independent vectors. So $\dim \operatorname{im} \pi_*|_x = 2n$ and now the statement follows for dimensional reasons. □

10.7 Definition (distance)**.** We define a distance on $\mathbb{C}\,\mathbb{P}^n$ by

$$d(p,q) := \inf\{d_{\mathbb{S}}(x,y) | x \in p, y \in q\},$$

where $d_{\mathbb{S}}$ is the distance on $\mathbb{S}^{2n+1}$.

10.8 Lemma. This distance has the following properties:

(i) The infimum is allways attained, i.e. there exist $x \in p$, $y \in q$ such that $d(p,q) = d_{\mathbb{S}}(x,y)$.

(ii) The value $d_{\mathbb{S}}(x,y)$ is the angle between x and y.

(iii) The distance satisfies $d(p,q) \leq \frac{\pi}{2}$, so in particular

$$\operatorname{diam} \mathbb{C}\,\mathbb{P}^n \leq \frac{\pi}{2}.$$

Proof.

(i) This is a direct consequence of the compactness of $\pi^{-1}(p) \times \pi^{-1}(q) \subset \mathbb{S}^{2n+1}$.

(ii) Since x, y are contained in a plane, we can apply a suitable rotation and may as well assume that $x, y \in \mathbb{S}^1 \subset \mathbb{C}$ and that $x = 1$. In that case the curve $c : [0, \arg(y)] \to \mathbb{S}^1$, $t \mapsto e^{it}$, is a minimizing geodesic joining x and y. Therefore $d_{\mathbb{S}}(x,y) = L(c) = \arg(y) = \arccos(\langle x, y \rangle) = \measuredangle(x,y)$.

(iii) This is due to the simple fact, that $x \in p$ implies $-x \in p$ by definition.

$\square$

We would like to construct a Riemannian metric on $\mathbb{C}\,\mathbb{P}^n$, such that the induced distance is d.

10.9 Lemma. Let $c : I \to \mathbb{S}^{2n+1}$ be a geodesic, i.e. a great circle traversed with constant speed. If there exists $t_0 \in I$, such that c is perpendicular to $H(c(t_0))$ in $c(t_0)$, then for all $t \in I$, c is perpendicular to $H(c(t))$ in $c(t)$.

Proof. We may write c as $c(t) = \cos(t)x + \sin(t)v$, where $x, v \in \mathbb{S}^{2n+1}$. Thus $\ddot{c}(t) = -c(t)$ and we obtain

$$\partial_t \langle ic(t), \dot{c}(t)\rangle_h = \langle i\dot{c}(t), \dot{c}(t)\rangle_h + \langle i\dot{c}(t), \ddot{c}(t)\rangle_h = i\|c(t)\| - i\|c(t)\| = 0.$$

Thus $t \mapsto \langle ic(t), \dot{c}(t)\rangle_e = \mathrm{Re}\langle ic(t), \dot{c}(t)\rangle_h$ is constant, zero at one point, and hence identically zero. $\square$

10.10 Lemma. Let $p, q \in \mathbb{C}\,\mathbb{P}^n$ and $x \in p$, $y \in q$ such that $d_{\mathbb{S}}(x, y) = d(p, q) =: \delta > 0$. Let $c : [0, \delta] \to \mathbb{S}^{2n+1}$ the unique $(\delta \leq \pi/2)$ unit speed geodesic from x to y . Then c is perpendicular to the Hopf circles, i.e.

$$\forall t \in [0, \delta] : \dot{c}(t) \perp T_{c(t)}H(c(t)).$$

Proof. Let $c_\varphi : [0, \delta] \to \mathbb{S}^{2n+1}$ be the minimizing geodesic from x to $e^{i\varphi}y$, $-\varepsilon < \varphi < \varepsilon$. Since $\delta < \pi = i(\mathbb{S}^{2n+1})$ the map $H : [-\varepsilon, \varepsilon] \times [0, \delta] \to \mathbb{S}^{2n+1}$, $(s, t) \mapsto c_\varphi(t)$ is smooth in φ und t (since we could express this in terms of the exponential map). By the first variation formula and our choice of $x \in p$ and $y \in q$, we obtain (since $e^{i\varphi} \in q$):

$$0 = \partial_\varphi L(c_\varphi)|_{\varphi=0} = \langle \dot{c}(\delta), \partial_\varphi H(0, \delta)|_{\varphi=0}\rangle_e = \langle \dot{c}(\delta), (ie^{i\varphi}y)|_{\varphi=0}\rangle_e = \langle \dot{c}(\delta), iy\rangle_e.$$

Thus $\dot{c}(\delta) \perp_e iy$ and therefore it is perpendicular to $\mathbb{R}iy = T_yH(y)$. Using Lemma 10.9, we obtain the statement. $\square$

10.11 Remark. The curve $\mathbb{R} \to \mathbb{S}^{2n+1}$, $\varphi \mapsto e^{i\varphi}c$, is a geodesic of length δ from $e^{i\varphi}x \in p$ to $e^{i\varphi}y$. Together with Lemma 10.10 we have reason to strongly suspect, that $\pi \circ c$ will be a geodesic in $\mathbb{C}\,\mathbb{P}^n$ w.r.t. the Riemannian metric we have yet to construct.

Before we construct this Riemannian metric on $\mathbb{C}\,\mathbb{P}^n$, we can already guess the cut locus.

10.12 Lemma. Let $p \in \mathbb{C}\,\mathbb{P}^n$, $x \in p$, $y, z \perp \mathbb{C}\,x$, $\|x\| = \|y\| = 1$, and let $\varphi, \psi \in \mathbb{R}$. Define the curves $c_1, c_2 : \mathbb{R} \to \mathbb{S}^{2n+1}$ by

$$c_1(s) := e^{i\psi}(\cos(s)x + \sin(s)y) \qquad\qquad c_2(t) := \cos(t)e^{i\varphi}x + \sin(t)z.$$

Then $\pi \circ c_1$,$\pi \circ c_2$ are both curves in $\mathbb{C}\,\mathbb{P}^n$ starting at p. If there exists $0 < s_0, t_0 \leq \frac{\pi}{2}$ such that $c_1(s_0) = c_2(t_0)$, we obtain $s_0 = t_0$ and $e^{i\psi}y = z$. In case $s_0, t_0 < \frac{\pi}{2}$, we have $e^{i\psi} = e^{i\varphi}$. (If $s_0, t_0 = \pi/2$, there exists no additional condition for ψ.)

Proof. By definition $\pi(c_1(0)) = \pi(e^{i\psi}x) = p = e^{i\varphi}x = c_2(0)$. Analysing

$$\underbrace{\cos(s_0)e^{i\psi}x}_{\in\mathbb{C}\,x} + \underbrace{\sin(s_0)e^{i\psi}y}_{\in(\mathbb{C}\,x)^\perp} = c_1(s_0) = c_2(t_0) = \underbrace{\cos(t_0)e^{i\varphi}x}_{\in\mathbb{C}\,x} + \underbrace{\sin(t_0)z}_{\in(\mathbb{C}\,x)^\perp}$$

we deduce

$$\cos(s_0)e^{i\psi}x = \cos(t_0)e^{i\varphi}x \qquad\qquad \sin(s_0)e^{i\psi}y = \sin(t_0)z.$$

Since $s_0, t_0 > 0$ and $|e^{i\psi}| = \|y\| = \|z\| = 1$ the second equality implies $s_0 = t_0$. In case $s_0, t_s < \frac{\pi}{2}$, we obtain $\cos(s_0), \cos(t_0) \neq 0$ and thus the first equality implies $e^{i\psi} = e^{i\varphi}$. $\square$

10.2 Fubini-Study Metric

Now we will construct the Riemannian metric on $\mathbb{C}\,\mathbb{P}^n$.

10.13 Definition. Let

$$H(n) := \{A \in \mathbb{C}^{n\times n} \mid A^* = A\},$$

where $A^* = \bar{A}^t$. This is an $\mathbb{R}$ vector space (of dimension $2\frac{(n+1)n}{2} - n = n^2$), which we now endow with the scalar product [13]

$$\langle\langle A, B\rangle\rangle := \frac{1}{2}\operatorname{Re}\operatorname{tr}(AB) = \frac{1}{2}\operatorname{Re}\sum_{i,j=1}^{n} a_{ij}\bar{b}_{ij}.$$

If $e_1, \dots, e_n \in \mathbb{C}^n$ is a hermitian basis w.r.t. $\langle\_,\_\rangle_h$ (i.e. $\langle e_i, e_j\rangle_h = \delta_{ij}$), then (since $A^* = A$):

$$\langle\langle A, B\rangle\rangle = \frac{1}{2}\operatorname{Re}\sum_{i=1}^{n}\langle ABe_i, e_i\rangle_h = \frac{1}{2}\operatorname{Re}\sum_{i=1}^{n}\langle Be_i, Ae_i\rangle_h.$$

10.14 Definition. Define the map $U : \mathbb{S}^{2n+1} \to H(n+1)$ by

$$x \mapsto \langle\_, x\rangle_h x,$$

where we think of x as an element of $\mathbb{C}^{n+1}$ and of $\langle\_, x\rangle_h x$ as the matrix of this map w.r.t. the canonical basis of $\mathbb{C}^{n+1}$.

10.15 Lemma. The map U has the following properties.

(i) Written as a system of columns the matrix $U(x)$ is given by $(\bar{x}_0 x, \dots, \bar{x}_n x)$.

(ii) We have indeed $U(x) \in H(n+1)$.

(iii) For any $x \in \mathbb{S}^{2n+1}$ the map $U(x)$ is the $\langle\_,\_\rangle_h$-orthogonal projection $\mathbb{C}^{n+1} \to \mathbb{C}\,x$.

(iv) U is smooth.

(v) We have $\forall x \in \mathbb{S}^{2n+1} : \forall \varphi \in \mathbb{R} : U(e^{i\varphi}x) = U(x)$.

Proof.

(i) By definition

$$U(x)(e_i) = \langle e_i, x\rangle_h x = \bar{x}_i x.$$

(ii) We calculate

$$(U(x)^*)_{ij} = (\bar{U}(x))_{ji} = \overline{\bar{x}_i x_j} = x_i\bar{x}_j = U(x)_{ij}.$$

(iii) By definition.

(iv) Follows from (i).

(v) For any $z \in \mathbb{C}^{n+1}$:

$$U(e^{i\varphi}x)(z) = \langle z, e^{i\varphi}x\rangle_h e^{i\varphi}x = e^{-i\varphi}\langle z, x\rangle_h e^{i\varphi}x = U(x)(z).$$

□

10.16 Definition (Veronese map)**.** The map $V : \mathbb{C}\,\mathbb{P}^n \to H(n+1)$, $V(p) := U(x)$, where $x \in p$ is arbitrary, is called *Veronese map.*

10.17 Lemma. The Veronese map has the following properties.

(i) V is well-defined.

[13] Notice that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ and $\operatorname{tr}(AA) = (AA)^i_i = A^i_k A^k_i = \sum_{i,k} A^i_k \bar{A}^i_k = \sum_{i,k}|A^i_k|^2$.

(ii) V is smooth.

(iii) For any $p \in \mathbb{C}\,\mathbb{P}^n$ we have $\|V(p)\|^2 = \frac{1}{2}$. The image of V (=image of U) is contained in the sphere $\mathbb{S}^{n(n+2)}(1/\sqrt{2}) \subset H(n+1)$.

Proof. The first two properties follow immediately from 10.15.
To see (iii), first notice that $V(p)$ is an orthogonal projection onto a subspce of (complex) dimension one. So $V(p)$ has eigenvalue 1 with onefold multiplicity and the eigenvalue 0 whith n-fold multiplicity. By the spectral theorem there exists a hermitian ONB $v_0, \ldots, v_n$ of eigenvector to these eigenvalues and therefore

$$\|V(p)\|^2 = \langle\langle V(p), V(p)\rangle\rangle = \frac{1}{2}\operatorname{Re}\sum_{i=0}^{n}\langle V(p)(v_i), V(p)(v_i)\rangle = \frac{1}{2}.$$

□

10.18 Lemma. Let $x \in p \in \mathbb{C}\,\mathbb{P}^n$ and $y \perp \mathbb{C}\,x$.

(i) The push-forward is given by

$$U_*|_x(y) = \langle \_, y\rangle_h x + \langle \_, x\rangle_h y$$

and in particular $\|U_*|_x(y)\| = \|y\|$.

(ii) The map $\pi_*|_x : L_x \to T_p\mathbb{C}\,\mathbb{P}^n$, $L_x := (\mathbb{C}\,x)^\perp$, is an isomorphism.

(iii) The map $V_*|_{\pi(x)} = U_*|_x$ is of maximal rank $2n$.

(iv) We have

$$\operatorname{im}(V_*|_{\pi(x)}) = \operatorname{im}(U_*|_x).$$

Proof.

(i) Let $c : \mathbb{R} \to \mathbb{S}^{2n+1}$ be the geodesic satisfying $c(0) = x$, $\dot{c}(0) = y$. Then for all $z \in \mathbb{C}^{n+1}$, we obtain

$$(U_*|_x(y))(z) = \partial_t(U(c(t))(z)|_{t=0} = \partial_t\langle z, c(t)\rangle_h c(t)|_{t=0} = \langle z, y\rangle_h x + \langle z, x\rangle_h y.$$

Let $e_0, \ldots, e_n$ be an hermitian ONB of $\mathbb{C}^{n+1}$. Since $y \perp \mathbb{C}\,x$ and $\langle x, x\rangle_h = 1$, we obtain

$$\begin{aligned}\|U_*|_x(y)\|^2 &= \frac{1}{2}\operatorname{Re}\Big(\sum_{i=0}^{n}\langle\langle e_i, y\rangle_h x + \langle e_i, x\rangle_h y, \langle e_i, y\rangle_h x + \langle e_i, x\rangle_h y\rangle_h\Big)\\ &= \frac{1}{2}\operatorname{Re}\Big(\sum_{i=0}^{n}\langle e_i, y\rangle_h\overline{\langle e_i, y\rangle_h} + \langle e_i, x\rangle_h\overline{\langle e_i x\rangle_h}\langle y, y\rangle_h\Big) = \frac{1}{2}\sum_{i=0}^{n}\bar{y}_i y_i + \frac{\langle y, y\rangle_h}{2}\sum_{i=0}^{n}\bar{x}_i x_i = \|y\|^2.\end{aligned}$$

(ii) By construction of $\mathbb{C}\,\mathbb{P}^n$, c.f. Lemma 10.6.

(iii) Follows from $\dim\mathbb{C}\,\mathbb{P}^n = 2n \leq \dim H(n+1)$.

(iv) By definition of V.

□

10.19 Definition (Fubini-Study metric)**.** We consider $H(n+1)$ as a Riemannian manifold. The metric on $\mathbb{C}\,\mathbb{P}^n$ obtained by pulling back this metric along $V : \mathbb{C}\,\mathbb{P}^n \to H(n+1)$ is the *Fubini-Study metric.* From now on we will assume, that $\mathbb{C}\,\mathbb{P}^n$ is endowed with this Riemannian metric.

10.20 Remark. Notice, that the tensor field obtained by pulling back the metric in $H(n+1)$ along V is indeed a Riemannian metric, because V has full rank by Lemma 10.18. With this additional

structure we may now state, that $\pi : \mathbb{S}^{2n+1} \to \mathbb{C}\,\mathbb{P}^n$ is a Riemannian submersion[14] (c.f. Lemma 10.18) and $V : \mathbb{C}\,\mathbb{P}^n \to H(n+1)$ is Riemannian immersion, i.e. an isometric immersion.
In fact V is a bit more: First of all, it is a an injective immersion since if $x, x' \in \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ there exists $0 \leq i \leq n$ such $x_i \neq 0$. Therefore

$$U(x) = U(x') \Rightarrow \bar{x}_i x = \bar{x}'_i x' \Rightarrow x = \frac{\bar{x}'_i}{\bar{x}_i} x' \Rightarrow \pi(x) = \pi(x').$$

Since $\mathbb{S}^{2n+1}$ is compact and π is smooth, $\mathbb{C}\,\mathbb{P}^n = \pi(\mathbb{S}^{2n+1})$ is compact as well. This implies altogether that V is a smooth embedding with closed image (c.f. [3, 7.4]).

10.21 Remark. Unwinding all the definitions and canonical identifications, the situation is the following:

$$\begin{tikzcd}
\mathbb{S}^{2n+1} \arrow[r, "U"] \arrow[d, "\pi"] & H(n+1) \arrow[d, "\mathrm{id}"] \\
\mathbb{C}\,\mathbb{P}^n \arrow[r, "V"] & H(n+1)
\end{tikzcd}$$

One should bear in mind that V is an isometry onto its image and by definition a vector space isometry $H(n+1) \to H(n+1)$ is an isometry as a map between Riemannian manifolds.
Let $x \in p \in \mathbb{C}\,\mathbb{P}^n$. The situation at the tangent spaces is the following:

$$\begin{tikzcd}
\mathbb{R}^{2n+2} \supset (\mathbb{R}x)^{\perp_e} \arrow[r, "\sim"] \arrow[d] & T_x\mathbb{S}^{2n+1} \arrow[r, "U_*|_x"] \arrow[d, "\pi_*|_x"] & T_{U(x)}H(n+1) \arrow[r] \arrow[d] & H(n+1) \subset \mathbb{C}^{(n+1)\times(n+1)} \arrow[d] \\
\mathbb{R}^{2n+1} \supset (\mathbb{C}\,x)^{\perp_h} \arrow[r, "\sim"] & T_p\mathbb{C}\,\mathbb{P}^n \arrow[r, "V_*|_p"] & T_{V(p)}H(n+1) \arrow[r] & H(n+1) \subset \mathbb{C}^{(n+1)\times(n+1)}
\end{tikzcd}$$

10.22 Theorem. Let $c : \mathbb{R} \to \mathbb{S}^{2n+1}$ be a unit speed geodesic, which is perpendicular to the Hopf circles. Then $\bar{c} := \pi \circ c$ is a geodesic in $\mathbb{C}\,\mathbb{P}^n$.

Proof. First consider the curve $V \circ \bar{c} = V \circ \pi \circ c = U \circ c : \mathbb{R} \to H(n+1) \subset \mathbb{R}^{(n+1)^2}$. Its ordinary acceleration is given by:

$$\begin{aligned}
&(U \circ c)''(t) = \partial_t(U_*|_{c(t)}(\dot{c}(t))) \overset{10.18}{=} \partial_t(\langle _, \dot{c}(t)\rangle_h c(t) + \langle _, c(t)\rangle_h \dot{c}(t)) \\
&= \langle _, \ddot{c}(t)\rangle_h c(t) + \langle _, \dot{c}(t)\rangle_h \dot{c}(t) + \langle _, \dot{c}(t)\rangle_h \dot{c}(t)) + \langle _, c(t)\rangle_h \ddot{c}(t) = -2\langle _, c(t)\rangle_h c(t) + 2\langle _, \dot{c}(t)\rangle_h \dot{c}(t),
\end{aligned}$$

where in the last step, we have used $\ddot{c}(t) = -c(t)$ (which follows from the fact that c is a great circle). Now we compute the part tangential to $U(\mathbb{S}^{2n+1}) = V(\mathbb{C}\,\mathbb{P}^n) \subset H(n+1)$. To that end let $e_0, \dots, e_n$ be a hermitian ONB of $\mathbb{C}^{n+1}$, $x := c(t)$, $y := \dot{c}(t)$ and $z \in (\mathbb{C}\,x)^\perp = T_{\pi(x)}\mathbb{C}\,\mathbb{P}^n$. So $\langle x, z\rangle_h = 0$ by construction and $\langle x, y\rangle_h = 0$, since on the one hand $\langle x, y\rangle_e = \langle c(t), \dot{c}(t)\rangle_e = 0$ by construction of the tangent bundle and $\langle y, ix\rangle_e = 0$, since c is perpendicular to the Hopf circles. Thus

$$\begin{aligned}
\langle\langle (U \circ c)''(t), U_*|x(z)\rangle\rangle &= \frac{1}{2}\operatorname{Re}\sum_{i=0}^{n} \langle -2\langle e_i, x\rangle_h x + 2\langle e_i, y\rangle_h y, \langle e_i, z\rangle_h x + \langle e_i, x\rangle_h z\rangle_h \\
= \operatorname{Re}\sum_{i=0}^{n} -\langle e_i, x\rangle_h \overline{\langle e_i, z\rangle_h}\langle x, x\rangle_h &+ \langle e_i, y\rangle_h \overline{\langle e_i, x\rangle_h}\langle y, z\rangle_h = \operatorname{Re}(\langle -x, z\rangle_h + \langle y, x\rangle_h \langle y, z\rangle_h) = 0.
\end{aligned}$$

Since V is an isometry onto its image, this implies that $\bar{c}$ is an isometry in $\mathbb{C}\,\mathbb{P}^n$. $\square$

[14] Reminder: A submersion $f : (M, g) \to (N, h)$ is a *Riemannian submersion*, provided $f_* : (\ker f_*)^\perp \to TN$ is an isometry.

10.23 Theorem. The geodesics of $\mathbb{C}\,\mathbb{P}^n$ satisfy:

(i) All geodesics in $\mathbb{C}\,\mathbb{P}^n$ are of type $\pi \circ c$, where c is a geodesic in $\mathbb{S}^{2n+1}$, which is perpendicular to the Hopf circles.

(ii) The metric d from 10.7 is the distance induced by the Fubini-Study metric.

(iii) All geodesics in $\mathbb{C}\,\mathbb{P}^n$ having length $\leq \pi/2$ are minimizing.

(iv) Diameter and injectivity radius satisfy

$$\operatorname{diam}(\mathbb{C}\,\mathbb{P}^n) = i(\mathbb{C}\,\mathbb{P}^n) = \frac{\pi}{2}.$$

Proof.

(i) Let $x \in p \in \mathbb{C}\,\mathbb{P}^n$ and let $\bar{c} : I \to \mathbb{C}\,\mathbb{P}^n$ be a geodesic. Then $\dot{\bar{c}}(0) \in T_p\mathbb{C}\,\mathbb{P}^n$. Since $\pi_*|_x : L_x \to T_{\pi(x)}\mathbb{C}\,\mathbb{P}^n$ is an isomorphism, there exists a unique geodesic $c : I \to \mathbb{S}^{2n+1}$ satisfying $c(0) = x$, $\dot{c}(0) \in L_x$, $\pi_*|_x(\dot{c}(0)) = \dot{\bar{c}}(0)$. This geodesic is perpendicular to the Hopf circle through x at x and therefore by Lemma 10.9 it is perpendicular to the Hopf circles everywhere. By the previous Theorem 10.22 $\pi \circ c$ is a geodesic in $\mathbb{C}\,\mathbb{P}^n$ and therefore $c = \pi \circ \bar{c}$.

(ii) Denote by d_g the metric induced by the Fubini-Study metric on $\mathbb{C}\,\mathbb{P}^n$ for a moment. Since $\mathbb{C}\,\mathbb{P}^n = \pi(\mathbb{S}^{2n+1})$ is compact, it is complete in particular. So for any $p, q \in \mathbb{C}\,\mathbb{P}^n$ there exists a geodesic $\bar{c} : [0,1] \to \mathbb{C}\,\mathbb{P}^n$ joining p and q such that $d_g(p,q) = L(\bar{c})$. Now take $x \in p$, $y \in q$ such that $d(p,q) = d_{\mathbb{S}}(x,y)$ and a geodesic $c : [0,1] \to \mathbb{R}$ joining x and y. By Lemma 10.10 this geodesic is perpendicular to the Hopf circles. Consequently $\pi \circ c$ is a geodesic in $\mathbb{C}\,\mathbb{P}^n$ joining p and q. On the other hand let $\tilde{c} : [0,1] \to \mathbb{S}^{2n+1}$ be the unique geodesic lift of $\bar{c}$. Since π is a Riemannian submersion and the length of a geodesic from $[0,1]$ equals the length of its initial velocity, we obtain

$$L(\bar{c}) \leq L(\pi \circ c) = L(c) \leq L(\tilde{c}) = L(\bar{c})$$

and thus alltogether $L(\bar{c}) = L(\pi \circ c)$. Therefore

$$d_g(p,q) = L(\bar{c}) = L(\pi \circ c) = L(c) = d_{\mathbb{S}}(x,y) = d(p,q).$$

(iii) This follows from (i), (ii) and Lemma 10.12.

(iv) Follows from what we have proven so far.

□

10.3 Isometries

We denote by

$$U(n) := \{C \in \mathbb{C}^{\,n} \mid C^{-1} = \bar{C}^t\}$$

the unitary group.

10.24 Theorem (Isometries)**.**

(i) The conjugation $U(n+1) \times H(n+1) \to H(n+1)$, $(C, A) \mapsto CAC^{-1}$ is a well-defined group action of $U(n+1)$ on $H(n+1)$ by isometries.

(ii) The group $U(n+1)$ acts as $U(n+1) \times \mathbb{C}\,\mathbb{P}^n \to \mathbb{C}\,\mathbb{P}^n$, $(C, p) \mapsto C(p)$ on $\mathbb{C}\,\mathbb{P}^n$ by isometries.

(iii) Let $e_0, \ldots, e_n$ be a hermitian ONB of $\mathbb{C}^{\,n+1}$. Then complex conjugation on $\mathbb{C}^{\,n+1}$ w.r.t. that basis induces an isometry on $\mathbb{C}\,\mathbb{P}^n$.

Proof.

(i) Let $A, B \in H(n+1)$, $C \in U(n+1)$. On the one hand

$$(CAC^{-1})^* = (\bar{C}^{-1})^t \bar{A}^t \bar{C}^t = CAC^{-1}$$

and thus $CAC^{-1} \in H(n+1)$. On the other hand

$$\langle\langle CAC^{-1}, CBC^{-1}\rangle\rangle = \frac{1}{2}\operatorname{Re}\operatorname{tr}(CAC^{-1}CBC^{-1}) = \frac{1}{2}\operatorname{Re}\operatorname{tr}(AB) = \langle\langle A, B\rangle\rangle.$$

(ii) First of all

$$\forall C \in U(n+1) : \forall x \in \mathbb{S}^{2n+1} : \|Cx\|^2 = \langle Cx, Cx\rangle_h = \langle \bar{C}^t Cx, x\rangle_h = \langle x, x\rangle_h = \|x\|^2$$

and thus C restricts to a map $\mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1}$. If $y = e^{i\varphi}x \in \mathbb{S}^{2n+1}$, we have (since C is $\mathbb{C}$-linear), that $C(y) = e^{i\varphi}C(x)$. Thus C induces a well-defined map $\mathbb{C}\mathbb{P}^n \to \mathbb{C}\mathbb{P}^n$. The differential of C (as a map $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$) is C itself and we claim, that

$$\forall p \in \mathbb{C}\mathbb{P}^n : CV(p)C^{-1} = V(C(p)).$$

By (i) and the definition of the Fubini-Study metric, this shows that C is an isometry of $\mathbb{C}\mathbb{P}^n$. To show this claim, notice that by definition of V it suffices to verify the equation $CU(x)C^{-1} = U(C(x))$ for one $x \in p$. Since this implies that the right arrow in the commutative diagram

$$\begin{array}{ccc} \mathbb{C}\mathbb{P}^n & \xrightarrow{V} & H(n+1) \\ \downarrow{\scriptstyle C} & & \downarrow \\ \mathbb{C}\mathbb{P}^n & \xrightarrow{V} & H(n+1) \end{array}$$

is the conjugation with C, which is an isometry by (i). By hypothesis $C \in U(n+1)$ and thus

$$\begin{aligned}(CU(x)C^{-1})_{ij} &= (CU(x)\bar{C}^t)_{ij} = (CU(x))^i_k(\bar{C}^t)^k_j = C^i_l U(x)^l_k (\bar{C}^t)^k_j = C^i_l U(x)^l_k \bar{C}^j_k \\ &= C^i_l \bar{x}_l x_k \bar{C}^j_k = \bar{C}^j_k \bar{x}_k C^i_l x_l = \overline{Cx}_j (Cx)_i = U(C(x))_{ij}\end{aligned}$$

(we always sum over all indices but i and j).

(iii) We proceed in a similar fashion. First we verify that complex conjugation with respect to the canonical basis is an isometry. Therefore we will show, that

$$\begin{array}{ccc} \mathbb{C}\mathbb{P}^n & \xrightarrow{V} & H(n+1) \\ \downarrow{\scriptstyle -} & & \downarrow{\scriptstyle -} \\ \mathbb{C}\mathbb{P}^n & \xrightarrow{V} & H(n+1) \end{array},$$

i.e. $V(\bar{p}) = \overline{V(p)}$. Choose $x \in p$ and calculate

$$V(\bar{p}) = U(\bar{x}) = (x_0\bar{x}, \ldots, x_n\bar{x}) = \overline{(\bar{x}_0 x, \ldots, \bar{x}_n x)} = \overline{U(x)} = \overline{V(p)}.$$

Clearly any two matrices $A, B \in H(n+1)$ satisfy

$$\langle\langle \bar{A}, \bar{B}\rangle\rangle = \frac{1}{2}\operatorname{Re}\operatorname{tr}(\bar{A}\bar{B}) = \frac{1}{2}\operatorname{Re}\operatorname{tr}(\overline{AB}^t) = \langle\langle A, B\rangle\rangle.$$

and therefore complex conjugation is an isometry.
If $B = (b_0, \ldots, b_n)$ is any other hermitian basis of $\mathbb{C}^{n+1}$ and $c_B(x)$ is the coordinate vector of x w.r.t. B, we obtain $x = Bc_B(x)$. Denote by $\Psi_B : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ the complex conjugation wr.t. B. Then

$$\Psi_B(x) = B\overline{c_B(x)} = B\overline{B^{-1}(x)} = B\bar{\bar{B}}^t\bar{x} = BB^t\bar{x}.$$

Since $B \in U(n+1)$, we obtain $C := BB^t \in U(n+1)$ as well and therefore

$$V(\Psi_B(p)) = C\overline{V(p)}C^{-1}$$

and the statement follows from (ii) and what we have proven so far.

□

10.4 Curvature

10.25 Lemma. Let $x \in p \in \mathbb{C}\,\mathbb{P}^n$ and $Y, Z \in T_p\mathbb{C}\,\mathbb{P}^n$, $Y = \pi_*|_x(y)$, $Z = \pi_*|_x(z)$, where $y, z \in L_x = (\mathbb{C}\,x)^\perp$, $\|Y\| = 1$.

(i) If $z \in (\mathbb{C}\,y)^\perp$, then
$$R(Z,Y)Y = Z.$$

(ii) If $z \in \mathbb{R}iy$, then
$$R(Z,Y)Y = 4Z.$$

(If $z \in \mathbb{R}y$, then $R(Z,Y)Y = 0$.)

Proof.

(i) In that case we may choose an hermitian ONB $e_0 = x, e_1 = y, e_2 = z, e_3 \ldots, e_n$ of $\mathbb{C}^{n+1}$. Then $R^{n+1} := \mathrm{Lin}_{\mathbb{R}}(e_0, \ldots, e_n)$ is a real subspace of dimension $n+1$ and we obtain an inclusion of its unit sphere $S^n \hookrightarrow \mathbb{S}^{2n+1}$. Obviously this S^n is totally geodesic in $\mathbb{S}^{2n+1}$: A great circle in $\mathbb{S}^{2n+1}$ tangential to S^n at one point is contained in S^n. We claim $iR^{n+1} \perp_e R^{n+1}$: Since for any $0 \le k, l \le n$ either $k \neq l$, i.e. $ie_k \perp_h e_l \Rightarrow ie_k \perp_e e_l$ or $k = l$, in which case
$$\langle ie_k, e_k\rangle_e = \langle \mathrm{Re}(ie_k), \mathrm{Re}(e_k)\rangle_e + \langle \mathrm{Im}(ie_k), \mathrm{Im}(e_k)\rangle_e = -\langle \mathrm{Im}(e_k), \mathrm{Re}(e_k)\rangle_e + \langle \mathrm{Re}(e_k), \mathrm{Im}(e_k)\rangle_e = 0.$$
Thus the great circles of this S^n are perpendicular to the Hopf circles of $\mathbb{S}^{2n+1}$, since for any $x \in S^n$ we have $T_xS^n \subset R^{n+1} \perp_e iR^{n+1} \supset T_xH(x)$. So the restriction $\pi : S^n \to \mathbb{C}\,\mathbb{P}^n$ is an isometric immersion and together with 10.22 we obtain, that π induces a totally geodesic isometric embedding $RP^n \to \mathbb{C}\,\mathbb{P}^n$, where $RP^n = S^n/\sim$, $x \sim -x$ here. We assume that RP^n is endowed with the canonical metric obtained from S^n. [15] By hypothesis the vectors Y, Z are tangential in $p = \pi(x)$ to that RP^n. Thus we may calculate the curvature inside RP^n and use the formulas for its metric of constant sectional curvature $+1$ to obtain (c.f. [2, 8.10])
$$R(Z,Y)(Y) = \langle Y, Y\rangle Z - \langle Z, Y\rangle Y = Z.$$

(ii) Now we consider the case $Z = \pi_*|_x(iy)$. For $e_0, \ldots, e_n$ as in (i), we now obtain $C^2 := \mathrm{Lin}_{\mathbb{C}}(e_1, e_2) \cong \mathbb{C}^2 \subset \mathbb{C}^{n+1}$. As above the corresponding $CP^1 \subset \mathbb{C}\,\mathbb{P}^n$ is totally geodesic and isometrically embedded. Now for any $p, q \in \mathbb{C}\,\mathbb{P}^1$ there exists an isometry mapping p to q (take one from $U(2)$ for example), thus the surface CP^1 has constant sectional curvature (since the curvature tensor is natural). Since CP^1 is diffeomorphic to $\mathbb{S}^2$, the curvature has to be positive (otherwise we could pull back the metric of CP^1 to $\mathbb{S}^2$ and we would have constructed a metric of nonpositive sectional curvature on $\mathbb{S}^2$, which would imply that $\mathbb{S}^2$ would be diffeomorphic to $\mathbb{R}^2$ by the Theorem of Cartan-Hadamard, which is obviously impossible). Since $\mathrm{diam}(\mathbb{C}\,\mathbb{P}^1) = \frac{\pi}{2}$, the curvature is 4. (This can be seen by comparing the pullback metric from CP^1 on $\mathbb{S}^2$ with the standard metric using Theorem A.15.) Therfore (again c.f. [2, 8.10]), we obtain
$$R(Z,Y)(Y) = 4Z.$$

$\square$

10.26 Theorem. Let $p \in \mathbb{C}\,\mathbb{P}^n$ be arbitrary. The endomorphism $I_p : T_p\mathbb{C}\,\mathbb{P}^n \to T_p\mathbb{C}\,\mathbb{P}^n$, $I_p(Y) := \pi_*|_x(iy)$, where $x \in p$ and $y \in L_x$ such that $\pi|_x(y) = Y$ is well-defined and $I_p^2 = -\mathrm{id}_{T_p\mathbb{C}\,\mathbb{P}^n}$.

[15] Since $\mathbb{R}\,\mathbb{P}^n$ is obtained from $\mathbb{S}^n$ by factoring out the antipodal action, the projection $\mathbb{S}^n \to \mathbb{R}\,\mathbb{P}^n$ is a local diffeomorphism, which we declare to be a local isometry. With respect to this metric $\mathbb{R}\,\mathbb{P}^n$ has constant sectional curvature $+1$ exactly like $\mathbb{S}^n$.

In the canonical coordinates of $\mathbb{C}\,\mathbb{P}^n$ the tensor field I corresponds to the multiplication with i. Alltogether we obtain:

$$R(Z,Y)(Y) = \langle Y,Y\rangle Z - \langle Z,Y\rangle Y + 3\langle Z,IY\rangle IY$$

$$R(X,Y)(Z) = \langle Y,Z\rangle X + \langle IY,Z\rangle IX - \langle X,Z\rangle Y - \langle IX,Z\rangle IY + 2\langle X,IY\rangle IZ$$

This completely determines the curvature tensor. For the sectional curvature K we obtain

$$1 \le K \le 4$$

Both bounds are sharp for $n \ge 2$.

We have identified two types of totally geodesic submanifolds. Somewhat more general one can show:

10.27 Theorem (totally geodesic submanifolds)**.** If $L \subset \mathbb{C}^{\,n+1}$ is a totally real subspace, i.e. $iL \perp_e L$, of dimension $k \le n+1$, then $\pi(L \cap S^{2n+1})$ is a totally geodesic $\mathbb{R}\,\mathbb{P}^{k-1}$ with canonical metric.
If $L \subset \mathbb{C}^{\,n+1}$ is a complex subspace with complex dimension k, then $\pi(L \cap S^{2n+1})$ is a totally geodesic $\mathbb{C}\,\mathbb{P}^{k-1}$ with Fubini-Study metric.

10.28 Theorem (symmetric space)**.** $\mathbb{C}\,\mathbb{P}^n$ endowed with Fubini-Study metric is a *symmetric space*: For every $p \in \mathbb{C}\,\mathbb{P}^n$ there exists an isometry S_p, the reflection at p, such that $S_p(p) = p$ and $S_{p*}|_p = -\,\mathrm{id}$.

Proof. Let $x \in p$ and $L_x = (\mathbb{C}\,x)^\perp$. Let $C \in U(n+1)$ be a matrix satisfying $Cx = x$ and $C|_{L_x} = -\,\mathrm{id}_{L_x}$. Then the isometry induced by C has the desired properties: First of all $C \in U(n+1)$, since any vector in C^{n+1} has a unique representation $\lambda x + u \in (\mathbb{C}\,x) \oplus (\mathbb{C}\,x)^\perp$ and for any two such vectors

$$\begin{aligned}&\langle C(\lambda x + u), C(\lambda' x + u')\rangle_h = \langle \lambda Cx + Cu, \lambda' Cx + Cu'\rangle_h = \langle \lambda x - u, \lambda' x - u'\rangle_h \\ &= \lambda\bar{\lambda}'\langle u,u'\rangle_h = \langle \lambda x + u, \lambda' x + u'\rangle_h\end{aligned}$$

Thus C acts as an isometry by 10.24 and has the desired properties. $\square$

10.29 Theorem. We have

$$\forall Y \in T_p\mathbb{C}\,\mathbb{P}^n : \forall C \in U(n+1) : C_*|_p(I_pY) = I_{C(p)}C_*(Y),$$

where C is identified with its action as an isometry on $\mathbb{C}\,\mathbb{P}^n$.

10.30 Theorem. Furthermore

$$\forall Y,Z \in T_p\mathbb{C}\,\mathbb{P}^n : C_*|_p(DI)_p(Y,Z) = (DI)_{C(p)}(C_*Y, C_*Z)$$

$DI(Y,Z) := D_Y I(Z) - I(D_Y z)$

If we substitute the isometry S_p for C, we obtain

$$-(DI_p)(Y,Z) = (DI)_p(-Y,-Z) = (DI)_p(Y,Z)$$

thus $DI = 0$. The tensor field I is parallel and therefore $\mathbb{C}\,\mathbb{P}^n$ is a *Kähler manifold*. One can also show $DR = 0$, c.f. next section.

11 Locally symmetric Spaces

11.1 Remark. Remember that for any $v \in T_pM$, c_v denotes the geodesic satisfying $c_v(0) = p$ and $\dot{c}_v(0) = v$. Also remember that the curvature tensor $R = R_v$ along c_v is defined by

$$R_v(X) = R(X, \dot{c}_v)\dot{c}_v$$

and is a smooth field of endomorphisms along c_v.

11.2 Lemma. The covariant derivative R_v' of the curvature tensor along c_v vanishes if and only if for all parallel vector fields $X, Y \in \mathcal{T}(c_v)$, the map $(X, Y) \mapsto \langle R(X), Y\rangle$ is constant.

Proof. Since R_v' is a tensor field it vanishes if and only if it vanishes on a parallel ONB. Notice, that for parallel vector fields X, Y

$$\partial_t\langle R(X), Y\rangle = \langle D_t(R(X)), Y\rangle + \langle R(X), D_tY\rangle = \langle (D_tR)(X), Y\rangle + \langle R(D_tX), Y\rangle = \langle (D_tR)(X), Y\rangle.$$

This implies the statement. $\square$

11.3 Definition (locally symmetric space)**.** Let M be a connected Riemannian manifold. Then M is a *locally symmetric space*, provided that for any $p \in M$ there exists a neighbourhood U near p and an isometry $S_p : U \to U$, such that

$$S_p(p) = p \qquad\qquad (S_p)_*|_p = -\operatorname{id}_{T_pM}.$$

We say M is a *symmetric space* , in case we may choose $U = M$ for all $p \in M$. We call S_p the *geodesic reflection* .

11.4 Remark.

(i) Isometries map geodesics to geodesics. Therefore S_p reflects the geodesics through p to geodesics through p in the opposite direction. Therefore S_p is called the *geodesic reflection*.

(ii) By shrinking U if necessary we may always assume $U = B_\varepsilon(p)$ for a sufficiently small $\varepsilon > 0$.

(iii) Geodesic reflections are unique, if they exist (c.f. [2, 5-7]). Therefore the notion of a (globally) symmetric space is well-defined and does not depend on a particular choice of geodesic reflection, since there is only one.

11.5 Theorem (Characterization of locally symmetric spaces)**.** Let M be a connected Riemannian manifold. The following are equivalent.

(i) M is a locally symmetric space.

(ii) The covariant differential ∇R of the Riemannian curvature R satisfies $\nabla R = 0$.

(iii) For all $v \in \mathbb{S}M$: $R_v' = 0$.

Proof.
"(i)⇒(ii)": Since S_p is a local isometry and the connection as well as the curvature respect local isometries, we obtain

$$\begin{aligned}&-\nabla R(W, X, Y, Z)|_p = (S_p)_*|_p(\nabla R(W, X, Y, Z)) = \nabla R((S_p)_*|_pW, (S_p)_*|_pX, (S_p)_*|_pY, (S_p)_*|_p, Z)\\ &= \nabla R(-W, -X, -Y, -Z)|_p = \nabla R(W, X, Y, Z)|_p\end{aligned}$$

and therefore $\nabla R(W, X, Y, Z) = 0$.
"(ii)⇒(iii)": This is clear since $R_v'(X) = \nabla R(X, \dot{c}_v, \dot{c}_v, \dot{c}_v)$ (c.f. Lemma A.5 and A.7).

"(iii)⇒(i)": This direction will require Lemma 11.6 below, the proof of which is done seperately afterwards.
Let $p \in M$ and $\varepsilon \leq i(p)$. Then $\exp_p : B_\varepsilon(0_p) \to B_\varepsilon(p)$ is a diffeomorphism. Define $S_p : B_\varepsilon(0) :\to B_\varepsilon(p)$

$$q \mapsto \exp_p(-\exp_p^{-1}(q)).$$

Obviously S_p is a diffeomorphism satisfying $S_p(p) = p$ and $(S_p)_*|_p = -\operatorname{id}_{T_pM}$.
The decicive point is to show, that S_p is an isometry. To that end we will describe its differential using Jacobi fields (c.f. Lemma A.8). Let $p \neq q \in B_\varepsilon(p)$, $0 < r = d(p,q) < \varepsilon$, $v \in \mathbb{S}_pM$, such that $q = \exp_p(rv)$ (the case $p = q$ is trivial). Let $X \in T_qM$ be arbitrary and let J be the Jacobi field along c_v satisfying

$$J(0) = 0 \qquad\qquad J(r) = X.$$

(c.f. [2, Exc. 10.2]). Since

$$X = \exp_{p_*}|_{rv}\Big(r\frac{1}{r}\exp_p^{-1}{}_*|_q(X)\Big)$$

this implies

$$J'(0) = \frac{1}{r}(\exp_p^{-1})_*|_q(X)$$

by uniqueness of Jacobi fields and again A.8. In a similiar fashion, we obtain

$$(S_p)_*|_q(X) = (\exp_{p_*}|_{-rv} \circ I_*|_{rv} \circ \exp_{p_*}^{-1}|_q)(X) = \exp_{p_*}|_{-rv}(r\frac{-1}{r}\exp_p^{-1}{}_*|_q(X)) = \bar{J}(r),$$

where $\bar{J}$ is the Jacobi field along along c_{-v} satisfying

$$\bar{J}(0) = 0 \qquad\qquad \bar{J}'(0) = -\frac{1}{r}\exp_p^{-1}{}_*|_q(X) = -J'(0)$$

and $I := -\operatorname{id}_{T_pM}$. Since $t \mapsto J(-t)$ is a Jacobi field as well, this implies $\bar{J}(r) = J(-r)$. Thus all that remains to show is $\|J(-r)\| = \|J(r)\|$, since this implies

$$\|(S_p)_*|_q(X)\| = \|\bar{J}(r)\| = \|J(-r)\| = \|J(r)\| = \|X\|.$$

In order to prove $\|J(-r)\| = \|J(r)\|$, we finally use Lemma 11.6 to represent J as stated there. Since $J(0) = 0$, we obtain $a_1 = 0$ and $\forall\, 2 \leq i \leq n : b_i = 0$. Thus

$$\|J(r)\|^2 = b_1^2 r^2 + \sum_{i=2}^{n} \alpha_i^2 \operatorname{sn}_{\alpha_i}(r)^2 = \|J(-r)\|$$

as one can see using the antisymmetry of $\operatorname{sn}_{\alpha_i}$ proven in Lemma 2.7,(iii). □

11.6 Lemma. Let M be a Riemannian manifold, such that for all $p \in M$, $v \in \mathbb{S}_pM$, we have $R_v' = 0$. Then there exists a parallel ONB $E_1 = \dot{c}_v, E_2, \dots, E_n$ along c_v satisfying

$$\langle R_v E_i, E_j \rangle = \delta_{ij}\alpha_i,$$

where $\alpha_i \in \mathbb{R}$ and $\alpha_1 = 0$. With respect to this basis the Jacobi fields along c_v are given as linear combinations

$$J = (a_1 + b_1 t)\dot{c}_v + \sum_{j=2}^{n} (a_i \operatorname{sn}_{\alpha_1} + b_i \operatorname{cs}_{\alpha_i})E_i,$$

where $a_i, b_i \in \mathbb{R}$ and $\operatorname{sn}_{\alpha_i}$, $\operatorname{cs}_{\alpha_i}$ are as in Definition 2.6.

Proof.
STEP 1: Start with an arbitrary parallel ONB Sei $\tilde{E}_1 = \dot{c}_v, \tilde{E}_2, \ldots, \tilde{E}_n$ along c_v. By Lemma 11.2 we obtain

$$\langle R_v(E_i), E_j \rangle = \text{const} =: r_{ij},$$

where $R_v(E_1) = 0$. Now $v^\perp \subset T_pM$ and $f : v^\perp \to v^\perp$, $X \mapsto R(X,v)v$ is a symmetric endomorphism as can be seen immediately using the symmetries of the curvature tensor (c.f. [2, 7.6]), i.e.

$$\langle R(X,v)v, v \rangle = Rm(X,v,v,v) = 0 \Rightarrow f(X) \in v^\perp$$

and

$$\begin{aligned} \langle f(X), Y \rangle &= \langle R(X,v)v, Y \rangle = Rm(X,v,v,Y) = Rm(v,Y,X,v) \\ &= Rm(Y,v,v,X) = \langle R(Y,v)v, X \rangle = \langle X, f(Y) \rangle \end{aligned}$$

By the spectral theorem there exists an ONB $E_1(0) = \tilde{E}_1(0) = v, E_2(0), \ldots, E_n(0)$ consisting of eigenvectors to eigenvalues $\alpha_1 = 0, \alpha_2, \ldots, \alpha_n$ of f, which we may extend by parallel translation along c_v to a parallel orthonormal frame $\{E_i\}$. This frame satisfies

$$\langle R_v(E_i), E_j \rangle = \alpha_i \delta_{ij}$$

and thus we have proven the first statement.
STEP 2: To see the second statement we first remarkt, that the fields in question span a space of dimension $2n$. It therefore suffices to verify that they are all Jacobi fields. We just calculate

$$R(J, \dot{c}_v)\dot{c}_v = \sum_{j=2}^{n} (a_i \operatorname{sn}_{\alpha_i} + b_i \operatorname{cs}_{\alpha_i}) R_v(E_i) = \sum_{j=2}^{n} (a_i \operatorname{sn}_{\alpha_i} + b_i \operatorname{cs}_{\alpha_i}) \alpha_i E_i$$

and on the other hand

$$\begin{aligned} J' &= b_1 \dot{c}_v + \sum_{j=2}^{n} (a_i \dot{\operatorname{sn}}_{\alpha_i} + b_i \dot{\operatorname{cs}}_{\alpha_i}) E_i \\ J'' &= \sum_{j=2}^{n} (a_i \ddot{\operatorname{sn}}_{\alpha_i} + b_i \ddot{\operatorname{cs}}_{\alpha_i}) E_i = -\sum_{j=2}^{n} \alpha_i (a_i \operatorname{sn}_{\alpha_i} + b_i \operatorname{cs}_{\alpha_i}) E_i. \end{aligned}$$

□

11.7 Example. Any connected space of constant sectional curvature κ is a locally symmetric space. Thus all considerations in this chapter apply in particular to the model spaces M^n_κ. To see this, we remind, that in this case the curvature tensor is given by $R(X,Y)Z = \kappa(\langle Y,Z \rangle X - \langle X,Z \rangle Y)$, c.f. [2, 8.10]. Thus the covariant derivative satisfies:

$$\begin{aligned} &(\nabla R)(X,Y,Z,W) = (\nabla_W R)(X,Y,Z) = \nabla_W (R(X,Y,Z)) - R(\nabla_W X, Y, Z) - R(X,Y,\nabla_W Z) \\ &= \kappa(\langle \nabla_W Y, Z \rangle X + \langle Y, \nabla_W Z \rangle X + \langle Y,Z \rangle \nabla_W X - \langle \nabla_W X, Z \rangle Y - \langle X, \nabla_W Z \rangle Y - \langle X,Z \rangle \nabla_W Y \\ &- \langle Y,Z \rangle \nabla_W X + \langle \nabla_W X, Z \rangle Y - \langle \nabla_W Y, Z \rangle X + \langle X,Z \rangle \nabla_W Y - \langle Y, \nabla_W Z \rangle X + \langle X, \nabla_W Z \rangle Y) \\ &= 0. \end{aligned}$$

11.8 Lemma (Parallel transport)**.** Let $c : [a,b] \to M$ be a piecewise smooth curve and denote by P the parallel translation along c. If M is a locally symmetric space, then P commutes with R, i.e.

$$P(R(X,Y)Z) = R(PX, PY)(PZ).$$

Proof. We simply verify, that for parallel vector fields $X, Y, Z \in \mathcal{T}(c)$ the condition $\nabla R = 0$ ensures, that

$$\begin{aligned} 0 &= (\nabla R)(X, Y, Z, \dot{c}) = (\nabla_{\dot{c}} R)(X, Y, Z) \\ &= \nabla_{\dot{c}}(R(X, Y, Z)) - R(\nabla_{\dot{c}} X, Y, Z) - R(X, \nabla_{\dot{c}} Y, Z) - R(X, Y, \nabla_{\dot{c}} Z) = \nabla_{\dot{c}}(R(X, Y, Z)) \end{aligned}$$

and therefore $R(X, Y, Z)$ is parallel as well. □

11.9 Theorem. Let $M, \bar{M}$ be locally symmetric spaces of dimension n. Let $p \in M$, $\bar{p} \in \bar{M}$ and let $I : T_pM \to T_{\bar{p}}\bar{M}$ be an isometry such that

$$\forall X, Y, Z \in T_pM : I(R(X, Y)(Z)) = \bar{R}(IX, IY)(IZ). \tag{11.1}$$

Let $\varepsilon > 0$, such that $\exp_p : B_\varepsilon(0_p) \to B_\varepsilon(p)$ is a diffeomorphism and $B_\varepsilon(0_{\bar{p}}) \subset \mathcal{E}_p$. Then the map $F := \exp_{\bar{p}} \circ I \circ \exp_p^{-1} : B_\varepsilon(p) \to B_\varepsilon(\bar{p})$ is a local isometry satisfying $F(p) = \bar{p}$ und $F_*|_p = I$.

Proof. By hypothesis the map F is well defined and satisfies

$$F(p) = (\exp_{\bar{p}}(I(\exp_p^{-1}(p)) = \exp_{\bar{p}}((I(0))) = \bar{p} \qquad F_*|_p = (\exp_{\bar{p}})_*|_0 \circ I_*|_0 \circ (\exp_p)_*^{-1}|_p = I.$$

Thus all that remains to show is that for any $q \in B_\varepsilon(p)$ the map $F_*|_q$ is an isometry. For $q = p$ this holds by hypothesis since $F_*|_q = I$. So let $q = \exp_p(rv)$, $\|v\| = 1$ and $0 < r = d(p, q) < \varepsilon$. For any $X \in T_qM$ there exists a unique Jacobi field J along c_v satisfying

$$J(0) = 0 \qquad\qquad J(r) = X.$$

As is the proof of Theorem 11.5 we may apply Lemma A.8 to conclude $J'(0) = \frac{1}{r}(\exp_p^{-1}{}_*|_q)(X)$. This implies

$$F_*|_q(X) = (\exp_{\bar{p}})_*|_{I(rv)} \circ I_*|_{rv} \circ (\exp_p)_*^{-1}|_q(X) = \exp_{\bar{p}*}|_{rI(v)}\Big(rI(\frac{1}{r}(\exp_p)_*^{-1}|_q(X))\Big) = \bar{J}(r),$$

where $\bar{J}$ is the Jacobi field along c_{Iv} satisfying

$$\bar{J}(0) = 0 \qquad\qquad \bar{J}'(0) = I(\frac{1}{r}(\exp_p)_*^{-1}|_q(X)) = I(J'(0)).$$

By Lemma 11.6 there exists a parallel ONB along c_v such that

$$\langle R_v(E_i), E_j \rangle = \delta_{ij}\alpha_i,$$

where $\alpha_1 = 0$. The parallel translates $\bar{E}_i$ of $I(E_i(0))$ are a parallel ONB along c_{Iv}. Denoting by $\bar{R}_{Iv}$ the curvature tensor along c_{Iv} in $\bar{M}$, we obtain

$$\begin{aligned} \langle \bar{R}_{Iv}(\bar{E}_i), \bar{E}_j \rangle &\overset{11.2}{=} \langle \bar{R}_{Iv}(\bar{E}_i(0)), \bar{E}_j(0) \rangle = \langle R(I(E_i(0)), Iv)(Iv), I(E_j(0)) \rangle \\ &\overset{(11.1)}{=} \langle I(R(E_i(0), v)v), I(E_j(0)) \rangle = \langle R(E_i(0), v)v, E_j(0) \rangle = \langle R_v(E_i(0)), E_j(0) \rangle = \alpha_i\delta_{ij}. \end{aligned}$$

Thus if we represent J as a linear combination of the $\{E_i\}$ as in Lemma 11.6, the representation of $\bar{J}$ with respect to the $\{\bar{E}_i\}$ is the same. Thus

$$\|F_*|_q(X)\| = \|\bar{J}(r)\| = \|J(r)\| = \|X\|$$

and therefore F is an isometry. □

11.10 Remark. In case M and $\bar{M}$ are complete $\mathcal{E}_{\bar{p}} = T_{\bar{p}}M$ and the condition for ε reads $0 < \varepsilon \le i(p)$. In case $\varepsilon \le i(\bar{p})$ as well, F is a global isometry.

11.11 Theorem (Cartan)**.** Let M, $\bar{M}$ be complete locally symmetric spaces of dimension n and let M be simply connected. Let $p \in M$, $\bar{p} \in \bar{M}$ and $I : T_pM \to T_{\bar{p}}\bar{M}$ be an isometry satisfying

$$\forall X, Y, Z \in T_pM : I(R(X,Y)(Z)) = \bar{R}(IX, IY)(IZ). \tag{11.2}$$

Then there exists a local isometry $F : M \to \bar{M}$ satisfying $F(p) = \bar{p}$ and $F_*|_p = I$.

Proof.
Step 1 (Strategy): In general the cut locus of M is not empty, i.e. $i(p) < \infty$ and we cannot argue that directly as in 11.9. Instead of the exponential map $\exp_p$ we will consider broken geodesics starting at p. On the space Ω of these curves we will construct a map $\Phi : \Omega \to \bar{M}$ and show how this induces the local isometry $F : M \to \bar{M}$.
Step 2 (Concerning broken geodesics): Let $c : [a,b] \to M$ be a broken geodesic. By definition there exists a subdivision

$$a = t_0 < t_1 < \ldots < t_k = b$$

of $[a,b]$, such that $c|[t_{i-1}, t_i]$, $1 \leq i \leq k$, is a geodesic. We call this a *geodesic subdivision*. Define $X_i := P_i^{-1}(\dot{c}^+(t_i))$, where $P_i := P|_{c|[t_0,t_i]} : T_{c(a)}M \to T_{c(t_i)}M$, $0 \leq i \leq k-1$, is the parallel transport along $c|[t_0, t_i]$. We call the X_i the *directions of* c. Obviously c is uniquely determined by its directions $X_0, \ldots, X_{k-1} \in T_{c(a)}M$ and the subdivision $t_0 < \ldots < t_k$, i.e. for a given geodesic subdivision and vectors $X_0, \ldots, X_{k-1} \in T_{c(a)}M$ there exists precisely one broken geodesic $c : [a,b] \to M$ with these data.
Step 3 (Definition of Φ): Now we consider our locally symmetric space M and define

$$\Omega := \{c : [0,1] \to M \mid c(0) = p,\ c \text{ is a broken geodesic}\}.$$

Let $c \in \Omega$ and let $0 = t_0 < t_1 < \ldots < t_k = 1$ be a geodesic subdivision for c. As in step 2, we obtain vectors

$$X_0 = \dot{c}^+(t_0), X_1, \ldots, X_{k-1} \in T_pM.$$

Define $\bar{c} : [0,1] \to \bar{M}$ to be the broken geodesic with respect to the same subdivision $0 = t_0 < t_1 < \ldots < t_k = 1$ satisfying $\bar{c}(0) = \bar{p}$ and having the directions

$$\forall 0 \leq i \leq k-1 : \bar{X}_i := I(X_i).$$

Define $\Phi : \Omega \to \bar{M}$ by

$$\Phi(c) := \bar{c}(1).$$

Obviously Φ is well-defined since $\bar{c}$ does not depend on the geodesic subdivision (any two such subdivisions have a common refinement.)
Step 4: If $\varphi : [0,1] \to [0,1]$ is a piecewise affine linear and globally continuous function satisfying $\varphi(0) = 0$ und $\varphi(1) = 1$, then $c \circ \varphi \in \Omega$ and $\overline{c \circ \varphi}(1) = \bar{c}(1)$.
This is due to the fact, that we may assume the subdivision where φ is piecewise affine linear to be a geodesic subdivision for c. Then φ just reparametrizes the various geodesics $c|[t_{i-1}, t_i]$.
Step 5 (Main step): We will show the following: Let $c_0, c_1 \in \Omega$ and $0 = t_0 < \ldots < t_k = 1$ be a geodesic subdivision for c_0 and c_1. Assume there exists an $i \leq k-2$, such that

$$c_0|_{[0,t_i]} = c_1|_{[0,t_i]} \qquad \text{and} \qquad c_0|_{[t_{i+2},t_k]} = c_1|_{[t_{i+2},t_k]}.$$

Define the points

$$\begin{array}{llll} q := c_0(t_i) = c_1(t_i) & q_0 := c_0(t_{i+1}) & q_1 := c_1(t_{i+1}) & q_2 := c_0(t_{i+2}) = c_1(t_{i+2}) \\ \bar{q} := \bar{c}_0(t_i) = \bar{c}_1(t_i) & \bar{q}_0 := \bar{c}_0(t_{i+1}) & \bar{q}_1 := \bar{c}_1(t_{i+1}) & \bar{q}_2 := \bar{c}_0(t_{i+2}) = \bar{c}_1(t_{i+2}), \end{array}$$

assume that $0 < \varepsilon \leq i(q), i(\bar{q})$ and that $c_0|[t_i, t_{i+2}]$ and $c_1|[t_i, t_{i+2}]$ are both contained in $B_\varepsilon(q)$. Then

$$\bar{c}_0(1) = \bar{c}_1(1).$$

We prove this in two substeps.

Step 5.1: To simplify notation let $c := c_0|[0, t_i]$ and let $\bar{c}$ be the broken geodesic starting from $\bar{p}$ with geodesic subdivision $0 = t_0 < t_1 \ldots < t_i$ and directions $\bar{X}_0, \ldots, \bar{X}_{i-1}$, i.e. $\bar{c} = \bar{c}_0|[0, t_i] = \bar{c}_1|[0, t_i]$. Denote by P_c,$P_{\bar{c}}$ the parallel translation along c, $\bar{c}$ and define

$$I_c := P_{\bar{c}} \circ I \circ P_c^{-1} : T_q M \to T_{\bar{q}} \bar{M},$$

which is an isometry by construction. By Lemma 11.8 and (11.2) it satisfies

$$I_c(R(X, Y)(Z)) = \bar{R}(I_c X, I_c Y)(I_c Z).$$

By Theorem 11.9 there exists an isometry $F_c : B_\varepsilon(q) \to B_\varepsilon(\bar{q})$ satisfying $F_c(q) = \bar{q}$ and $(F_c)_*|_q = I_c$. By definition of $\bar{c}_j$, $j = 0, 1$,

$$\dot{\bar{c}}_j^+(t_i) = I_c(\dot{c}_j^+(t_i)) = (F_c)_*|_q(\dot{c}_j^+(t_i)).$$

Now $c_j|[t_i, t_{i+1}]$ is a geodesic and F_c is an isometry. Thus

$$\bar{c}_j|[t_i, t_{i+1}] = F_c \circ (c_j|[t_i, t_{i+1}]). \tag{11.3}$$

Step 5.2: Now we calculate

$$\begin{aligned} P_{\bar{c}_j|[0,t_{i+1}]} \circ I \circ P^{-1}_{c_j[0,t_{i+1}]} &= P_{\bar{c}_j|[t_i,t_{i+1}]} \circ I_c \circ P^{-1}_{c_j[t_i,t_{i+1}]} = P_{\bar{c}_j|[t_i,t_{i+1}]} \circ (F_c)_*|_q \circ P^{-1}_{c_j[t_i,t_{i+1}]} \\ &\overset{(*)}{=} (F_c)_*|_{q_j} \circ (P_{c_j[t_i,i_{i+1}]}) \circ (P_{c_j|[t_i,t_{i+1}]})^{-1} = (F_c)_*|_{q_j}. \end{aligned}$$

(*): This commutativity holds due to the fact, that F_c is an isometry and the Riemannian connection is natural (c.f. [2, 5.6]). In particular the push-forward of an isometry preservers parallelity.
As above we conclude for $j = 0, 1$

$$\dot{\bar{c}}_j^+(t_{i+1}) = (F_c)_*|_{q_j}(\dot{c}_j^+(t_{i+1})) \qquad \bar{c}_j|[t_{i+1}, t_{i+2}] = F_c \circ c_j|[t_{i+1}, t_{i+2}].$$

Repeating this procedure for the indices $i + 1$,$i + 2$, we obtain

$$P_{\bar{c}_j|[0,t_{i+2}]} \circ I \circ P^{-1}_{c_j|[0,t_{i+2}]} = (F_c)_*|_{q_2}$$

and therefore we obtain alltogether $\bar{c}_0|[t_{i+2}, t_k] = \bar{c}_1|[t_{i+2}, t_k]$.

This implies $\bar{c}_0(1) = \bar{c}_1(1)$ and proves this step.

Step 6: Next we will show, that for any $c_0, c_1 \in \Omega$

$$c_0(1) = c_1(1) \Longrightarrow \bar{c}_0(1) = \bar{c}_1(1).$$

By hypothesis M is simply connected. Thus there exists a path homotopy $H : [0, 1] \times [0, 1] \to M$ from c_0 to c_1. The idea is to successively homotop c_1 to c_2 in a way such that the previous step guarantees, that the curves in between are mapped to the same point under Φ.
Since $H([0, 1] \times [0, 1]) \subset M$ is compact, there exists ε such that

$$\{i(q)|q \in \operatorname{im} H\} \geq \varepsilon > 0.$$

For a sufficiently large k the image of squares in $[0, 1] \times [0, 1]$ with edge length $\frac{1}{k}$ under H is contained in balls with radius $\frac{\varepsilon}{2}$ centered at the vertices of the squares. By increasing k further if necessary, we may assume, that dass $c_j|[\frac{i-1}{k}, \frac{i}{k}]$ is a geodesic for any $1 \leq i \leq k$, $j = 0, 1$. Let $0 \leq m \leq k^2$ and $0 \leq l \leq k$, such that $lk \leq m \leq (l + 1)k$. Define a sequence of points $x_0, \ldots, x_{2k} \in [0, 1] \times [0, 1]$ as follows: Let $x_0 := (0, 0)$, move l steps to the top til $(0, \frac{l}{k})$, then to the right til $(1 - \frac{m-lk}{k}, \frac{l}{k})$, one step to the top again til $(1 - \frac{m-lk}{k}, \frac{l+1}{k})$, then to the right til $(1, \frac{l+1}{k})$ and then to the top until $x_{2k} := (1, 1)$. Now let σ_m the broken geodesic joining successively the points $H(x_{i-1})$ and $H(x_i)$. Then

$$\sigma_0(t) = \begin{cases} c_0(2t) = H(0, 2t) & , 0 \leq t \leq \frac{1}{2} \\ H(2t - 1, 1) & , \frac{1}{2} \leq t \leq 1, \end{cases}$$

which equals the curve c_0 up to reparametrization. Now step 4 implies, that $\bar\sigma_0(1) = \bar c_0(1)$ (since $H(s,1) = \text{const}$). Using step 5 we obtain $\bar\sigma_{m-1}(1) = \bar\sigma_m(1)$, $1 \le m \le k^2$. Finally

$$\sigma_{k^2}(t) = \begin{cases} c_0(2t) = H(2t,0) & ,0 \le t \le \frac{1}{2} \\ c_1(2t-1) = H(1,2t-1) & ,\frac{1}{2} \le t \le 1, \end{cases}$$

which equals c_1 up to reparametrization thus by step 4 $\bar\sigma_{k^2}(1) = \bar c_1(1)$ (since $H(s,0) = \text{const}$). The step is proven.

STEP 7 (Construction of F): We claim that Φ induces a local isometry $F : M \to \bar M$ satisfying $F(p) = \bar p$ and $F_*|_p = I$.

This can be seen as follows: Let $q \in M$ and let $c \in \Omega$ be a broken geodesic from p to q and $c \in \Omega$ be a broken geodesic from p to q. Define

$$F(q) := \bar c(1).$$

By step 6 the map F is well-defined.

We have to show, that it is a local isometry: Let $\bar q := F(q)$, $0 < \varepsilon < i(q), i(\bar q)$. For any $r \in B_\varepsilon(q)$ let $c_{qr} : [0,1] \to B_\varepsilon(q)$ be the unique minimizing geodesic from q to r. Defining

$$c_r(t) := \begin{cases} c(2t) & ,0 \le t \le \frac{1}{2} \\ c_{qr}(2t-1) & ,\frac{1}{2} \le t \le 1 \end{cases}$$

we obtain $F(r) = \bar c_r(1)$. Denote by $F_c : B_\varepsilon(q) \to B_\varepsilon(\bar q)$ the isometry satisfying $F_c(q) = \bar q$ and $(F_c)_*|_q = I_c$. As in step 5 we see, that

$$F(r) \overset{(11.3)}{=} F_c(r),$$

thus $F|B_\varepsilon(q) = F_c$. Therefore F is a local isometry.

□

11.12 Remark. In case $\bar M$ is simply connected as well, we may interchange the roles of M and $\bar M$: We obtain a local isometry $\bar F : \bar M \to M$ satisfying $\bar F(\bar p) = p$ and $\bar F_*|_{\bar p} = I^{-1}$. Thus $\bar F \circ F$ is a local isometry from M to M satisfying $(\bar F \circ F)(p) = p$ and $(\bar F \circ F)_*|_p = \mathrm{id}_{T_pM}$. Thus $\bar F \circ F = \mathrm{id}_M$ (c.f. [2, 5-7] and analogously $F \circ \bar F = \mathrm{id}_{\bar M}$. Thus F is a diffeomorphism, hence a global isometry.

11.13 Remark. F is a Riemannian universal covering (c.f. [2, 11.6]).

11.14 Corollary. Let M be complete, simply connected with constant sectional curvature $\kappa > 0$. Then M is isometric to $\mathbb{S}^n(1/\sqrt{\kappa})$, $n = \dim M$.

11.15 Remark. The corresponding statement in case $\kappa \le 0$ is already a consequence of Theorem 11.9. This subsequently justifies the notation M^n_κ from Definition 4.16: Any other model space would be isometric to the spaces we have defined there and would be universally covered by them.

12 Symmetric Spaces

Remember the Definition 11.3 of a symmetric space: A connected Riemannian manifold M is a *symmetric space* (or just "symmetric"), if for any $p \in M$ there exists an isometry S_p of M, called *geodesic reflection*, such that $S_p(p) = p$ and $S_{p*}|_p = -\mathrm{id}_{T_pM}$. There are some easy consequences, we may directly obtain from this definition. As usual M is a connected Riemannian n-manifold.

12.1 Lemma. If M is symmetric, then M is homogenous. If M is homogenous, then M is complete. Thus any symmetric space is complete.

Proof.

(i) Let M be symmetric and $p, q \in M$ be arbitrary. Since M is connected there exists a broken geodesic $c : [0,1] \to M$ joining p and q. Let $0 = t_0 < \ldots < t_k$ be a geodesic subdivision of c and define $F_i := S_{c(\frac{t_i + t_{i-1}}{2})}$, $1 \leq i \leq k$. Then $F_i(c(t_{i-1})) = c(t_i)$. Consequently

$$(F_k \circ F_{k-1} \circ \ldots \circ F_1)(p) = q.$$

Thus M is homogenous.

(ii) Let M be homogenous and suppose to the contrary that M is not complete. Then there exists a maximal unit speed geodesic $c : I \to M$ such that $I \neq \mathbb{R}$. We may assume that $0 \in I$ and $t_0 := \sup I < \infty$. Let $p := c(0)$ and $0 < \varepsilon < i(p)$. Then $t_1 := t_0 - \frac{\varepsilon}{2} \in I$ and $q := c(t_1)$ is well-defined. Since M is homogenous there exists an isometry $F : M \to M$ such that $F(p) = q$. Thus $i(q) = i(p) > \varepsilon$. Therefore the unit speed geodesic c_0 through q is defined on at least $[0, \varepsilon]$. Thus $\sigma : [0, t_1 + \varepsilon] \to M$

$$t \mapsto \begin{cases} c(t) & , 0 \leq t \leq t_1, \\ c_0(t - t_1) & , t_1 - \varepsilon \leq t \leq t_1 + \varepsilon \end{cases}$$

is well-defined and extends the maximal geodesic c. Contradiction!

□

12.2 Definition (Transvection)**.** Let M be symmetric and $c : \mathbb{R} \to M$ be a geodesic. For any $t \in \mathbb{R}$ the isometry

$$T^t := S_{c(t/2)} \circ S_{c(0)} : M \to M$$

is a *transvection*. The $(T^t)_{t \in \mathbb{R}}$ are a one-parameter subgroup of isometries of M, which translate c.

Notice that for any $s, t \in \mathbb{R}$

$$S_{c(t)}(c(s)) = c(2t - s).$$

12.3 Theorem (Properties of transvections)**.** For any $s, t \in \mathbb{R}$ the transvections defined above satisfy

(i) $T^t(c(s)) = c(s + t)$,

(ii) $T^t_*|_{c(s)} : T_{c(s)}M \to T_{c(s+t)}M$ is the parallel translation along $c|[s, s + t]$,

(iii) $T^t \circ T^s = T^{t+s}$.

Proof.

(i) By definition

$$T^t(c(s)) = S_{c(t/2)}(S_{c(0)}(c(s))) = S_{c(t/2)}(c(-s)) = c(t/2 + (t/2 - (-s))) = c(t + s).$$

(ii) Apply the isometry $S_{c(s/2)}$ and notice that

$$S_{c(s/2)}(c(s)) = c(0) \qquad\qquad S_{c(s/2)}(c(s + t)) = c(-t).$$

Since isometries preserve parallelity, it suffices to show that $T^t_*|_{c(0)}T_{c(0)}M \to T_{c(-t)}M$ is the parallel translation. So let $X \in \mathcal{T}(c)$ be parallel. Define $Y \in \mathcal{T}(c)$ by $r \mapsto T^r_*|_{c(0)}(X(0))$. By definition

$$Y(0) = T^0_*|_{c(0)}(X(0)) = ({S_{c(0)}}_*|_{c(0)} \circ {S_{c(0)}}_*|_{c(0)})(X(0)) = X(0),$$

so both fields argee at 0. Furthermore

$$T^r_*|_{c(0)}(\dot{c}(0)) = \partial_r(T^r \circ c)(0) = \partial_r(c(0+r)) = \dot{c}(r).$$

Thus we obtain alltogether

$$Y'(r) = \nabla_{\dot{c}(r)} T^r_*|_{c(0)}(X(0)) = \nabla_{T^r_*|_{c(0)}(\dot{c}(0))} T^r_*|_{c(0)}(X(0)) = T^r_*|_{c(0)} \nabla_{\dot{c}(0)} X(0) = 0$$

and therefore Y is parallel. This implies the statement.

(iii) By construction $T^t \circ T^s$ and T^{t+s} are both isometries satisfying

$$(T^t \circ T^s)(c(0)) = c(s+t) = T^{s+t}(c(0))$$

Denote by P the parallel translation and notice, that

$$(T^t \circ T^s)_*|_{c(0)} = T^t_*|_{c(s)} \circ T^s_*|_{c(0)} \overset{\text{(ii)}}{=} P_{c|[s,s+t]} \circ P_{c|[0,s]} = P_{c|[0,s+t]} \overset{\text{(ii)}}{=} T^{s+t}_*|_{c(0)}.$$

Thus $T^t \circ T^s = T^{s+t}$ by uniqueness of Riemannian isometries.

□

We now give a recipe how to cook up symmetric spaces. In fact one can show that all symmetric spaces are of this form. We will assume the reader to be familiar with the basic concepts of Lie groups and Lie algebras. Some of these facts are discussed in more detail in the next chapter (also c.f. [3, 9,20]).

12.4 Theorem (Symmetric Space Construction Theorem)**.** Let M be a connected manifold, G be a Lie group and $\rho : G \times M \to M$ be a transitive left action. Let $p \in M$ and assume there exists a smooth involutive group automorphism $\sigma : G \to G$ satisfying

$$F_0 \subset H \subset F,$$

where

$$F := F^\sigma := \{g \in G \mid \sigma(g) = g\} \qquad H := G_p := \{g \in G \mid g(p) = p\}$$

and F_0 is the component of F containing the identity $e \in F_0 \subset F$. Then the following hold:

(i) Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebra of the subgroup $H \subset G$. Then

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \sigma_* X = X\}$$

and if we define $\mathfrak{m} := \{X \in \mathfrak{g} \mid \sigma_* X = -X\}$, we obtain

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

and

$$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h} \qquad [\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m} \qquad [\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}.$$

(ii) Define $\rho : G \to \mathrm{Diff}(M)$, $\rho(g)(\_) := \rho(g,\_)$, and $\pi : G \to M$, $g \mapsto \rho(g)(p)$. Then π is a submersion satisfying

$$\ker \pi_*|_g = {L_g}_*|_e(\mathfrak{h}) \qquad \pi_*|_g : {L_g}_*|_e(\mathfrak{m}) \xrightarrow{\sim} T_{\pi(g)}M$$

and for any $h \in H$, $X \in \mathfrak{g}$

$$\pi \circ L_g = \rho(g) = \pi \qquad \pi_*|_e(\mathrm{Ad}_h(X)) = \rho(h)_*|_p(\pi_*|_e(X)).$$

(iii) The equation

$$S(\rho(g)(p)) = \rho(\sigma(g))(p)$$

defines an involutive diffeomorphism $S : M \to M$ satisfying

$$S \circ \pi = \pi \circ \sigma.$$

(iv) Let $\langle \_, \_ \rangle$ be a scalar product on $\mathfrak{m}$ which is invariant under every Ad_h, $h \in H$. Then

$$\langle (\rho(g)_*|_p \circ \pi_*|_e)(X), ((\rho(g))_*|_p \circ \pi_*|_e)(Y) \rangle_{\pi(g)} := \langle X, Y \rangle$$

defines a G-invariant Riemannian metric on M. With respect to this metric M is a symmetric space and the geodesic reflections S are given by $S_p = S$ and for any $q = \rho(g)(p)$

$$S_q = \rho(g) \circ S \circ \rho(g)^{-1}.$$

(v) For any $X \in \mathfrak{g}$ let $\tilde{X}$ be the Killing field on M defined by

$$\tilde{X}(q) := \partial_t(\rho(e^{tX})(q))|_{t=0}.$$

Then

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \tilde{X}(p) = 0\} \qquad \mathfrak{m} = \{X \in \mathfrak{g} \mid \nabla\tilde{X}(p) = 0\}.$$

For any $X, Y, Z \in \mathfrak{m}$

$$R(\pi_*|_e X, \pi_*|_e X)(\pi_*|_e Z) = -[Z, [Y, X]].$$

(vi) For any $X \in \mathfrak{m}$ the map $t \mapsto \gamma_X(t) := \pi(e^{tX})$ is the geodesic through p with initial veolcity $\pi_*|_e(X)$ and e^{tX} is the transvection along γ_X.

(vii) Let A be a tensor on $\mathfrak{m}$, which is invariant under all Ad_h, $h \in H$. Then A induces a G-invariant parallel tensor field on M via $\pi_*|_e$.

Proof. We only sketch the proof and leave some of the easy steps as an exercise.

(i) Since σ_* is involutive, $\mathfrak{g}$ is the sum of eigenspaces to eigenvalues $+1$ and -1. Since $F_0 \subset H \subset F$, $\mathfrak{h}$ is the eigenspace to $+1$.

(ii) For any $k \in G$

$$(\pi \circ L_g)(k) = \pi(gk) = \rho(gk)(p) = \rho(g)(\rho(k)(p)) = \rho(g)(\pi(k)) = (\rho(g) \circ \pi)(k).$$

Furthermore

$$\begin{aligned}\pi_*|_e(\mathrm{Ad}_h(X)) &= \partial_t(\pi(he^{tX}h^{-1}))|_{t=0} = \partial_t(\rho(h)\rho(e^{tX})\rho(h^{-1})(p))|_{t=0} \\ &= \partial_t(\rho(h)\rho(e^{tX})(p))|_{t=0} = \rho(h)_*\pi_*|_e(X).\end{aligned}$$

This proves the last two statements from (ii). The rest is left as an exercise.

(iii) We show that S is well-defined: If $\rho(g)(p) = \rho(k)(p)$, then $k = gh$, where $h \in H$. Since $H \subset F$

$$\rho(\sigma(k))(p) = \rho(\sigma(gh))(p) = \rho(\sigma(g)h)(p) = \rho(\sigma(g))\sigma(h)(p) = \rho(\sigma(g))(p).$$

Furthermore

$$S(\pi(g)) = S(\rho(g)(p)) = \rho(\sigma(g))(p) = (\pi \circ \sigma)(g),$$

thus $S \circ \pi = \pi \circ \sigma$. This implies that S is smooth. Now S is involutive and therefore a diffeomorphism of M.

(iv) Since $\langle\_,\_\rangle$ is invariant under every Ad_h, $h \in H$, the Riemannian metric on M is well-defined. It is easy to see, that it is smooth and G-invariant as well. We show that S is an isometry: Let $q = \pi(g) = \rho(g)(p) \in M$ and $X_0 = \rho(g)_*|_p \pi_*|_e(X)$, $Y_0 = \rho(g)_*|_p \pi_*|_e(Y) \in T_qM$, $X, Y \in \mathfrak{m}$. Since

$$S \circ \rho(g) \circ \pi = S \circ \pi \circ L_g = \pi \circ \sigma \circ L_g = \pi \circ L_{\sigma(g)} \circ \sigma = \rho(\sigma(g)) \circ \pi \circ \sigma$$

we obtain

$$\begin{aligned}\langle S_*|_q(X_0), S_*|_q(Y_0)\rangle_{S(q)} &= \langle \rho(\sigma(g))_*|_p \pi_*|_e(\sigma_*(X)), \rho(\sigma(g))|_*|_p \pi_*|_e(\sigma_* Y)\rangle_{S(q)} \\ &= \langle \sigma_* X, \sigma_* Y\rangle = \langle X, Y\rangle = \langle X_0, Y_0\rangle_q\end{aligned}$$

by definition of the Riemannian metric.

(v) For any $X, Y \in \mathfrak{g}$, we obtain

$$\begin{aligned}\tilde{Y}(\pi(e^{tX})) &= \tilde{Y}(\rho(e^{tX}(p))) = \partial_s(\rho(e^{sY})\rho(e^{tX})(p)|_{s=0} = \partial_s(\rho(e^{tX})\rho(e^{-tX}e^{sY})\rho(e^{tX})(p))|_{s=0} \\ &= \rho(e^{tX})_*|_p \pi_*|_e(\mathrm{Ad}^{-1}_{e^{tX}}(Y))|_{s=0}.\end{aligned}$$

For any $X, Y, Z \in \mathfrak{m}$, we obtain

$$\begin{aligned}\tilde{X}_p\langle \tilde{Y}, \tilde{Z}\rangle &= \partial_t\langle \tilde{Y}(\pi(e^{tX})), \tilde{Z}(\pi(e^{tX}))\rangle|_{t=0} \\ &= \partial_t\langle \rho(e^{tX})_*\pi_*|_e(\mathrm{Ad}^{-1}_{e^{tX}})(Y), \rho(e^{tX})_*\pi_*|_e(\mathrm{Ad}^{-1}_{e^{tX}}(Z))\rangle|_{t=0} \\ &= \partial_t\langle \pi_*|_e\,\mathrm{Ad}^{-1}_{e^{tX}}(Y), \pi_*|_e(\mathrm{Ad}^{-1}_{e^{tX}}(Z))\rangle|_{t=0},\end{aligned}$$

since $\rho(e^{tX})$ is an isometry (G-invariance). Now

$$\mathrm{Ad}^{-1}_{e^{tX}}(Y) = e^{-t\,\mathrm{ad}\,X}(Y) = Y - t[X,Y] + \frac{t^2}{2}[X,[X,Y]] - \dots$$

and the $2k$-th term, $k \geq 1$, on the right hand side is in $\mathfrak{h}$ and the others are in $\mathfrak{m}$. Therefore

$$\tilde{X}_p\langle \tilde{Y}, \tilde{Z}\rangle = \partial_t\langle Y + \frac{t^2}{2}\cdot\text{remainder}, Z + \frac{t^2}{2}\cdot\text{remainder}\rangle|_{t=0} = 0.$$

This implies $D\tilde{Y}(p) = 0$ since $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$ and $\mathfrak{h} = \{X \in \mathfrak{g} | \tilde{X}(p) = 0\}$ by definition of H.

(vi) The previous statement implies that $c(t) := \pi(e^{tX})$, $X \in \mathfrak{m}$, is geodesic in X. Now

$$S_{\pi(e^{\frac{t}{2}X})}(\pi(e^{-sX})) = \pi(e^{(t+s)X}),$$

thus c is geodesic in t as well (the reflections are isometries). Since $\sigma(e^{tX}) = e^{-tX}$ the rest follows from (iii).

(vii) Exercise, c.f. (iv).

□

12.5 Example.

(i) $M = \mathbb{C}\,\mathbb{P}^n$, $G = U(n+1)$, σ is conjugation with $\begin{pmatrix}1 & 0 \\ 0 & -E_n\end{pmatrix} = A \in U(n+1)$, $p =$ Hopf circle through $(1, 0, \dots, 0)$.

(ii) $M = G_k(n)$, the Grassmannian manifold of k-planes in $\mathbb{R}^n$, $G = O(n)$, σ is conjugation with $\begin{pmatrix}E_k & 0 \\ 0 & -E_{n-k}\end{pmatrix} = A \in O(n)$, $p =$ linear span of $e_1, \dots, e_k$.

Find $\mathfrak{m}$ in these examples, describe S and look for suitable scalar products on $\mathfrak{m}$. Analyse more examples! If $G = SL(n,\mathbb{R})$, then $H = SO(n)$ for suitable M and σ. Find them!

13 Lie Groups

In this chapter G is a Lie group with Lie algebra $\mathfrak{g}$. We will explain some of the basic notions concerning Lie theory below, but assume the reader to have at least heard of it before. More on these basics can be found in [3, 9,20].

13.1 Definition (Lie group)**.** A *Lie group* is a smooth manifold and a group, such that the group multiplication

$$G \times G \to G, (gh) \mapsto gh$$

is smooth and the inversion

$$G \to G, g \mapsto g^{-1}$$

is a smooth diffeomorphism.

13.2 Example.

(i) $(\mathbb{R}^n, +)$

(ii) $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) | \det(A) \neq 0\}$

(iii) $O(n) = \{A \in GL_n(\mathbb{R}) | AA^t = E_n\}$

(iv) $U(n)$, $SU(n)$, $SL_n(\mathbb{C})$ and further classical matrix groups.

(v) The *Heisenberg group*

$$H_{2m+1} := \left\{ \begin{pmatrix} 1 & x^t & z \\ 0 & 1 & y \\ 0 & \ddots & 1 \end{pmatrix} \middle| x, y \in \mathbb{R}^m, z \in \mathbb{R} \right\} \subset GL_{m+2}(\mathbb{R}).$$

Identifying $(x, y, z) \in \mathbb{R}^{2m+1}$ with the matrix from H_{2m+1} above, then H_{2m+1} corresponds to $\mathbb{R}^{2m+1}$ with the non abelian group structure

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + \langle x, y' \rangle).$$

Warning: Sometimes other coordinates are used for H_{2m+1}.

13.3 Definition (Translation and conjugation)**.** Let G be a Lie group and $g \in G$.

(i) The map $L_g : G \to G$, $h \mapsto gh$, is the *left-translation with* g. This is a diffeomorphism with inverse $L_g^{-1} = L_{g^{-1}}$.

(ii) The map $R_g : G \to G$, $h \mapsto hg$, is the *right-translation with* g. This is a diffeomorphism with inverse $R_g^{-1} = R_{g^{-1}}$.

(iii) The map $C_g : G \to G$, $h \mapsto ghg^{-1}$, is the *conjugation with* g. This is a diffeomorphism with inverse $C_g^{-1} = C_{g^{-1}}$. Of course

$$C_g = L_g \circ R_{g^{-1}} = R_{g^{-1}} \circ L_g.$$

13.4 Definition (left-invariance)**.** A vector field $X \in \mathcal{T}(G)$ is *left invariant*, if

$$\forall g, h \in G : X \circ L_g = L_{g_*} \circ X,$$

i.e. if it is L_g-related to itself. The space $\mathfrak{g}$ of all left-invariant vector fields on G is the *Lie algebra of* G.

Similar a vector field is *right-invariant*, if it is R_g-related to itself.

13.5 Remark.

(i) The vector space $\mathfrak{g}$ of left invariant vector fields on G can be canonically identified with T_eG, i.e. the map $\} \to T_eG$, $X \mapsto X_e$, is an isomorphism with inverse $T_eG \to \mathfrak{g}$, $X_e \mapsto (L_g)_* X_e$, (c.f. [3, 4.20]).

(ii) If $\{X_1, \ldots, X_n\}$ is a basis of T_eG, then $\{L_{g*}X_1, \ldots L_{g*}X_n\}$ is a basis of T_gG. Thus there exists a continuous global frame on G. Therefore the tangential bundle TG is trivial, i.e. G is parallelizable and orientable. (c.f. [3, 5.15] und [3, 13.5]).

13.6 Lemma. The Lie algebra is closed unter Lie brackets, i.e.

$$\forall X, Y \in \mathcal{T}(G) : X, Y \in \mathfrak{g} \Longrightarrow [X, Y] \in \mathfrak{g},$$

(c.f. [3, 4.18]).

Proof. By hypothesis Y is left invariant. Thus for all $g, p \in G$, $f \in \mathcal{C}^\infty(G)$

$$(Y(f) \circ L_g)(p) = Y|_{gp}(f) = L_{g*}(Y|_p)(f) = Y|_p(f \circ L_g)$$

and therefore

$$X|_g(Y(f)) = L_{g*}(X_e)(Y(f)) = X|_e(Y(f) \circ L_g) = X|_e(Y(f \circ L_g)).$$

Alltogether

$$\begin{aligned} &[X,Y]|_g(f) = X|_g(Y(f)) - Y|_g(X(f)) = X|_e(Y(f \circ L_g)) - Y|_e(X(f \circ L_g)) \\ &= [X,Y]|_e(f \circ L_g) = L_{g*}([X,Y])(f). \end{aligned}$$

$\square$

13.7 Theorem. Let G be a Lie group. The left invariant vector fields $\mathfrak{g}$ on G together with the Lie bracket $[\_, \_] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ are not only a vector space over $\mathbb{R}$, but also a *Lie algebra*, i.e. the Lie bracket satisfies

(i) $[\_, \_]$ is bilinear.

(ii) $\forall X, Y \in \mathfrak{g} : [X, Y] = -[Y, X]$.

(iii) $\forall X, Y, Z \in \mathfrak{g} : [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ ("Jacobi identity").

13.8 Remark.

(i) $\mathfrak{g}$ is a Lie subalgebra of the Lie algebra of all smooth vector fields on G and thus the theorem above follows from Lemma 13.6.

(ii) Since T_eG may be identified with $\mathfrak{g}$, T_eG is a Lie algebra as well. The Lie bracket can be described as follows: If $X, Y \in T_eG$ define left invariant extensions $\tilde{X}, \tilde{Y} \in \mathcal{T}(G)$, $\tilde{X}_g := L_{g*}X$, calculate $[\tilde{X}, \tilde{Y}] \in \mathfrak{g}$, and obtain $[X, Y] := [\tilde{X}, \tilde{Y}]|_e$.

13.9 Example. Let $G := GL_n(\mathbb{R})$, $B \in M_n(\mathbb{R}) = T_eG$, $X \in G \Rightarrow L_{X*}(B) = \partial_t(X \cdot (E_t + tB)|_{t=0} = X \cdot B$. Thus $X \mapsto XB$ is the left invariant vector field with value B in $e = E_n$; denote this by V_B. If $B, C \in T_eG$, then

$$[B, C] = [V_B, V_C]|_e = \partial_t V_C(e + tB)|_{t=0} - \partial_t(V_B(e + tV))|_{t=0} = BC - CB$$

This formula holds for all Lie subgroups of $GL_n(\mathbb{R})$ as well (c.f. [3, 4.23]).

13.10 Lemma. If X is a left invariant vector field on a Lie group G and F_X^t is the maximal flow of X, then

$$\forall g \in G : F_X^t(g) = gF_X^t(e).$$

Proof. On the one hand
$$\partial_t(gF^t_X(e)) = L_{g*}|_{F^t_X(e)}\partial_t(F^t_X(e)) = L_{g*}|_{F^t_X(e)}X|_{F^t_X(e)} = X|_{gF^t_X(e)}$$
and on the other hand
$$gF^t_X(e)|_{t=0} = g.$$
Thus $gF^t_X(e)$ is the integral curve of X through g. □

13.11 Corollary. Left invariant vector fields are complete (c.f. [3, 20.1]).

Proof. Certainly there exists $\varepsilon > 0$, such that the integral curve $F^t_X(e) : [-\varepsilon, \varepsilon] \to G$ is defined. Let $g \in G$ and $t \in \mathbb{R}$, such that $F^t_X(g)$ is defined. Then
$$F^{t\pm\varepsilon}_X(g) = F^{\pm\varepsilon}_X(F^t_X(g)) \overset{13.10}{=} F^t_X(g)F^{\pm\varepsilon}_X(g)$$
is defined as well. Thus $F^t_X(g)$ is defined for all $t \in \mathbb{R}$. □

13.12 Definition (One parameter subgroup)**.** A one parameter subgroup of G (a "1PSG") is a Lie group homomorphism $\alpha : (\mathbb{R}, +) \to G$, i.e. α is smooth and satisfies $\alpha(0) = e$ and
$$\forall s, t \in \mathbb{R} : \alpha(s + t) = \alpha(s)\alpha(t).$$

13.13 Theorem. The map $\Psi : \{1\mathrm{PSG}\} \to T_eG$, $\alpha \mapsto \dot\alpha(0)$ is a bijection with inverse $\Phi : T_eG \to \{1\mathrm{PSG}\}$, $X \mapsto F^t_X(e)$. (c.f. [3, 20.2]).

Proof. Certainly
$$\forall X \in T_eG : (\Psi \circ \Phi)(X) = \Psi(F^t_X) = \dot F^t_X(e) = X \Longrightarrow \Psi \circ \Phi = \mathrm{id}\,.$$
Conversely if α is a 1PSG, the left invariant extension $\tilde X$ of $\dot\alpha(0)$ satisfies
$$(\Phi \circ \Psi)(\alpha) = \Phi((\dot\alpha(0))) = F^t_{\tilde X}(e).$$
We have to show, that α is the integral curve of $\tilde X$:
$$\dot\alpha(t_0) = \partial_t(\alpha(t_0 + t))|_{t=0} = \partial_t(\alpha(t_0)\alpha(t))|_{t=0} = L_{\alpha(t_0)_*}\dot\alpha(0) = \tilde X|_{\alpha(t_0)}.$$
□

13.14 Definition (Lie exponential map)**.** For any $X \in \mathfrak{g}$ let $e^{tX} := F^t_X(e) = \Phi(X)$ be the 1PSG through $e \in G$ with initial velocity X. This is the *exponential map of a Lie group.*

13.15 Lemma. The exponential map e of a Lie group G satisfies
$$\forall X, Y \in \mathfrak{g} : [X, Y] = \partial_t(\partial_s(e^{tX}e^{sY}e^{-tX})|_{s=0})|_{t=0} = \partial_t C_{e^{tX}*}|_e(Y)|_{t=0}.$$

Proof. First notice that for any $t \in \mathbb{R}$
$$\partial_s(e^{tX}e^{sY}e^{-tX})|_{s=0} = \partial_s(C_{e^{tX}}(e^sY))|_{s=0} = C_{e^{tX}*}|_e(Y) \in T_eG, \tag{13.1}$$
therefore we may take ∂_t inside T_eG and the right side makes sense. Using a general rule for vector fields X, Y (c.f. [3, 18.20]]), we obtain
$$\begin{aligned}[X, Y]|_p &= \mathscr{L}_XY|_p = \partial_t({F^{-t}_X}_*(Y|_{F^t_X(p)}))|_{t=0} = \partial_t({F^{-t}_X}_*(\partial_s(F^s_Y)|_{F^t_X(p)}))|_{s=0}|_{t=0}\\ &= \partial_t(\partial_s(F^{-t}_X(F^s_Y(F^t_X(p)))))|_{s=0}|_{t=0}.\end{aligned}$$
In our case Lemma 13.10 ensures, that $F^t_X(p) = pe^{tX}$. Thus
$$\begin{aligned}[X, Y]|_e &= \partial_t(\partial_s(F^{-t}_X(F^s_Y(F^t_X(e)))))|_{s=0}|_{t=0} = \partial_t(\partial_s(F^{-t}_X(F^s_Y(e^{tX}))))|_{s=0}|_{t=0}\\ &= \partial_t(\partial_s(F^{-t}_X(e^{tX}F^s_Y(e))))|_{s=0}|_{t=0} = \partial_t(\partial_s(e^{tX}e^{sY}e^{-tX}))|_{s=0}|_{t=0}.\end{aligned}$$
□

13.16 Remark.

(i) In a matrix group G any $B \in \mathfrak{g} \cong T_eG$ satisfies
$$e^{tX} = \exp(tB),$$
where exp is the usual exponential map for matrices: $\exp(tB)$ is a one parameter subgroup and $\exp(tB)'(0) = B$. (c.f. [3, 20.6]).

(ii) Using $[B, C] = BC - CB$ for $B, C \in T_eG$ this can be verified more easily.

13.17 Definition (invariant metric)**.** A Riemannian metric on G is

(i) *left-invariant*, if for any $g \in G$ the left translation L_g is an isomety.

(ii) *right-invariant*, if for any $g \in G$ the right translation R_g is an isometry.

(iii) *bi-invariant*, if it is left- and right-invariant.

13.18 Remark. Any Lie group with a left invariant metric is a homogenous space: For any $g, h \in G$ the map $L_{hg^{-1}}$ is an isometry mapping g to h. In particular G is complete (c.f. Lemma 12.1).

13.19 Definition (ad)**.** For any Lie algebra $\mathfrak{g}$ define $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ by $X \mapsto \mathrm{ad}_X$, where $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$, $Y \mapsto [X, Y]$. This is a Lie algebra homomorphism (here we use the notation $\mathfrak{gl}(\mathfrak{g})$ for the Lie algebra $\mathrm{End}(\mathfrak{g})$ together with the commutator).

13.20 Theorem (Curvature of left invariant metrics)**.** Let G be a Lie group with Lie algebra $\mathfrak{g}$, $\langle \_, \_ \rangle$ be a left invariant metric on G and let ∇ be the induced Levi-Civita connection. Then for any $X, Y, Z, W \in \mathfrak{g}$:

(i) $\nabla_X Y = \frac{1}{2}([X, Y] - \mathrm{ad}_X^t(Y) - \mathrm{ad}_Y^t(X))$, where ad_X^t is the adjoint endomorphism to ad_X w.r.t. $\langle \_, \_ \rangle$.

(ii) $Rm(X, Y, Z, W) = \langle R(X, Y)Z, W\rangle = \langle \nabla_X Z, \nabla_Y W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle - \langle \nabla_{[X,Y]}Z, W\rangle$.

(iii) $\langle R(X, Y)Y, X\rangle = |\nabla_X Y|^2 - |[X, Y]|^2 - \langle \nabla_X X, \nabla_Y Y\rangle - \langle \mathrm{ad}_Y^2(X), X\rangle$.

Proof.

(i) By the Koszul formula
$$2\langle \nabla_X Y, Z\rangle = X(\langle Y, Z\rangle) + Y(\langle Z, X\rangle) - Z(\langle X, Y\rangle) - \langle Y, \mathrm{ad}_X(Z)\rangle + \langle Z, \mathrm{ad}_X(Y)\rangle - \langle X, \mathrm{ad}_Y(Z)\rangle.$$
Notice that for a left invariant metric $\langle \_, \_ \rangle$, two left invariant vector fields $X, Y \in \mathfrak{g}$ and any $p \in G$, the following holds:
$$\langle X|_p, Y|_p\rangle_p = \langle (L_p)_*X|_e, (L_p)_*Y|_e\rangle_p = \langle X|_e, Y|_e\rangle_e.$$
Thus the function $p \mapsto \langle X|_p, Y|_p\rangle_p$ is constant. Therefore the Koszul formula collapses to
$$\begin{aligned}2\langle \nabla_X Y, Z\rangle &= -\langle Y, \mathrm{ad}_X(Z)\rangle + \langle Z, \mathrm{ad}_X(Y)\rangle - \langle X, \mathrm{ad}_Y(Z)\rangle \\ &= \langle [X, Y], Z\rangle - \langle \mathrm{ad}_X^t(Y), Z\rangle - \langle \mathrm{ad}_Y^t(X), Z\rangle.\end{aligned}$$

(ii) Since L_g is an isometry we obtain $\nabla_X Y \in \mathfrak{g}$ by the naturality of the Levi Civita connection (this can also be seen using (i)). Thus $X(\langle \nabla_Y Z, W\rangle) = 0$ as well and therefore
$$0 = \nabla_X(\langle \nabla_Y Z, W\rangle) = \langle \nabla_X \nabla_Y Z, W\rangle + \langle \nabla_Y Z, \nabla_X W\rangle$$
and $\langle \nabla_Y \nabla_X Z, W\rangle + \langle \nabla_X Z, \nabla_Y W\rangle = 0$. Thus
$$\begin{aligned}Rm(X, Y, Z, W) &= \langle R(X, Y)Z, W\rangle = \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z, W\rangle \\ &= -\langle \nabla_Y Z, \nabla_X W\rangle + \langle \nabla_X Z, \nabla_Y W\rangle - \langle \nabla_{[X,Y]}Z, W\rangle.\end{aligned}$$

(iii) Using (ii) and the symmetry $\nabla_Y X = \nabla_X Y - [X,Y]$ we obtain

$$\begin{aligned}\langle R(X,Y)Y,X\rangle &= \langle \nabla_X Y, \nabla_Y X\rangle - \langle \nabla_Y Y, \nabla_X X\rangle - \langle \nabla_{[X,Y]} Y, X\rangle \\ &= |\nabla_X Y|^2 - \langle \nabla_X Y, [X,Y]\rangle - \langle \nabla_Y Y, \nabla_X X\rangle - \langle \nabla_{[X,Y]} Y, X\rangle.\end{aligned}$$

Using (i) we obtain

$$\begin{aligned}&- \langle \nabla_X Y, [X,Y]\rangle - \langle \nabla_{[X,Y]} Y, X\rangle \\ &= -\frac{1}{2}\langle [X,Y],[X,Y]\rangle + \frac{1}{2}\langle \mathrm{ad}_X^t(Y),[X,Y]\rangle + \frac{1}{2}\langle \mathrm{ad}_Y^t(X),[X,Y]\rangle \\ &- \frac{1}{2}\langle [[X,Y],Y],X\rangle + \frac{1}{2}\langle \mathrm{ad}_{[X,Y]}^t(Y),X\rangle + \frac{1}{2}\langle \mathrm{ad}_Y^t([X,Y]),X\rangle \\ &= -\frac{1}{2}|[X,Y]|^2 + \frac{1}{2}\langle Y,[X,[X,Y]]\rangle + \frac{1}{2}\langle X,[Y,[X,Y]]\rangle \\ &- \frac{1}{2}\langle [[X,Y],Y],X\rangle + \frac{1}{2}\langle Y,[[X,Y],X]\rangle + \frac{1}{2}\langle [X,Y],[Y,X]\rangle \\ &= -|[X,Y]|^2 - \langle [Y,[Y,X]],X\rangle = -|[X,Y]|^2 - \langle \mathrm{ad}_Y^2(X),X\rangle.\end{aligned}$$

Combining both yields the statement.

□

13.21 Lemma. A left or right invariant metric $\langle \_,\_\rangle$ on G is bi-invariant if and only if for any $g \in G$ the map $\mathrm{Ad}_g := (L_g)_* \circ (R_{g^{-1}})_*$ is an isometry.

Proof.
"⇒": In case $\langle \_,\_\rangle$ is bi-invariant, the left- and right-translations are isometries. Thus the Ad_g are isometries.
"⇐": Let Ad_g be an isometry for all $g \in G$. Assume $\langle \_,\_\rangle$ is left invariant. We have to show, that it is right invariant as well. We calculate

$$\begin{aligned}&\langle (R_g)_*(X),(R_g)_*(Y)\rangle = \langle (\mathrm{Ad}_g \circ (R_g)_*(X)),(\mathrm{Ad}_g \circ (R_g)_*)(Y)\rangle \\ &= \langle ((L_g)_* \circ (R_{g^{-1}})_* \circ (R_g)_*(X)),((L_g)_* \circ (R_{g^{-1}})_* \circ (R_g)_*)(Y)\rangle = \langle (L_g)_*(X),(L_g)_*(Y)\rangle = \langle X,Y\rangle.\end{aligned}$$

In case $\langle \_,\_\rangle$ is right invariant, notice that Ad_g^{-1} is an isometry as well and calculate analogously:

$$\begin{aligned}&\langle (L_g)_*(X),(L_g)_*(Y)\rangle = \langle (((L_g)_* \circ (R_{g^{-1}})_*)^{-1} \circ (L_g)_*)(X),(((L_g)_* \circ (R_{g^{-1}})_*)^{-1} \circ (L_g)_*)(Y)\rangle \\ &= \langle ((R_g)_* \circ (L_g)_*^{-1} \circ (L_g)_*)(X),((R_g)_* \circ (L_g)_*^{-1} \circ (L_g)_*)(Y)\rangle = \langle (R_g)_*(X),(R_g)_*(Y)\rangle = \langle X,Y\rangle.\end{aligned}$$

□

13.22 Lemma (ad is skew)**.** Let $\langle \_,\_\rangle$ be a bi-invariant metric on G. Then ad is skew-symmetric, i.e.

$$\forall X,Y \in \mathcal{T}(G) : \langle \mathrm{ad}_X Y, Z\rangle = -\langle Y, \mathrm{ad}_X Z\rangle.$$

In other words $\mathrm{ad}_X^t = -\mathrm{ad}_X$.

Proof. We have just shown in Theorem 13.21, that

$$\forall X,Y \in \mathcal{T}(G) : \forall g \in G : \langle \mathrm{Ad}_g Y, \mathrm{Ad}_g Z\rangle = \langle Y,Z\rangle. \tag{13.2}$$

Notice that we may rephrase the statement of Lemma 13.15 by

$$\mathrm{ad}_X(Y) = \partial_t(\mathrm{Ad}_{e^{tX}}|_e(Y))|_{t=0}. \tag{13.3}$$

Near the identity we may write $g = e^{tX}$. Assume $X, Y, Z \in \mathfrak{g}$ and differentiate equation (13.2) in order to obtain
$$0 = \partial_t \langle \mathrm{Ad}_{e^{tX}} Y, \mathrm{Ad}_{e^{tX}} Z\rangle|_{t=0} \overset{(13.3)}{=} \langle \mathrm{ad}_X Y, Z\rangle + \langle X, \mathrm{ad}_X Z\rangle.$$
□

13.23 Theorem (Curvature of bi-invariant metrics)**.** Let G be a Lie group with a left invariant metric such that for all $X \in \mathfrak{g}$, the map ad_X is skew symmetric (c.f. 13.22) and let $X, Y, Z \in \}$.

(i) The associated Levi-Civita connection satisfies
$$\nabla_X Y = \frac{1}{2}[X, Y].$$

(ii) The Riemannian curvature is given by
$$R(X, Y)(Z) = -\frac{1}{4}[[X, Y], Z].$$

(iii) Its covariant derivative satisfies
$$\nabla R = 0,$$
so G is a locally symmetric space.

(iv) If X, Y is an ONB the sectional curvature of the plane they determine is
$$K(X, Y) = \frac{1}{4}|[X, Y]|^2.$$

(v) The Ricci curvature is given by
$$\mathrm{Ric}(X, Y) = -\frac{1}{4}\,\mathrm{tr}(\mathrm{ad}_X\,\mathrm{ad}_Y).$$

Proof.

(i) By Theorem 13.20 and the skew symmetry of ad_X we obtain:
$$\nabla_X Y = \frac{1}{2}([X, Y] - \mathrm{ad}^t_X(Y) - \mathrm{ad}^t_Y(X)) = \frac{1}{2}[X, Y].$$

(ii) Statement (i) and the Jacobi identity imply
$$R(X, Y)(Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z = \frac{1}{2}\nabla_X[Y, Z] - \frac{1}{2}\nabla_Y[X, Z] - \frac{1}{2}[[X, Y], Z]$$
$$= \frac{1}{4}[X, [Y, Z]] - \frac{1}{4}[Y, [X, Z]] - \frac{1}{2}[[X, Y], Z] = -\frac{1}{4}[[X, Y], Z].$$

(iii) Statment (ii) implies
$$\begin{aligned}
&8\nabla R(X, Y, Z, W) = 8(\nabla_W R)(X, Y, Z)\\
&= 8\nabla_W(R(X, Y, Z) - 8R(\nabla_W X, Y, Z) - 8R(X, \nabla_W Y, Z) - 8R(X, Y, \nabla_W Z)\\
&= 4[W, R(X, Y)Z] - 4R([W, X], Y, Z) - 4R(X, [W, Y], Z) - 4R(X, Y, [W, Z])\\
&= -[W, [[X, Y], Z]] + [[[W, X], Y], Z] + [[X, [W, Y]], Z] + [[X, Y], [W, Z]]\\
&= -[W, [[X, Y], Z]] \underline{+ [[[W, X], Y], Z] + [[[Y, W], X], Z]} + [[X, Y], [W, Z]]\\
&\overset{(1)}{=} -[W, [[X, Y], Z]] - [[[X, Y], W], Z] + [[X, Y], [W, Z]]\\
&= [[[X, Y], Z], W] + [[Z, W], [X, Y]] + [[W, [X, Y]], Z] \overset{(2)}{=} 0.
\end{aligned}$$

In step (1) we use the Jacobi identity at the underlined inner Lie bracket for W, X, Y and in (2) we use the Jacobi identity for $[X, Y], Z, W$.

(iv) Statement (ii), the definition of sectional curvature and the skew symmetry of ad_X implies

$$\begin{aligned} K(X,Y) &= Rm(X,Y,Y,X) = \langle R(X,Y)(Y), X\rangle = -\frac{1}{4}\langle [[X,Y],Y], X\rangle \\ &= \frac{1}{4}\langle \mathrm{ad}_Y([X,Y]), X\rangle = -\frac{1}{4}\langle [X,Y], \mathrm{ad}_Y(X)\rangle = \frac{1}{4}\langle [X,Y],[X,Y]\rangle. \end{aligned}$$

(v) Using (ii) we obtain

$$\mathrm{Ric}(X,Y) = \mathrm{Ric}(Y,X) = \mathrm{tr}(Z \mapsto R(Z,Y)X) = -\frac{1}{4}\,\mathrm{tr}(Z \mapsto [[Z,Y],X]) = -\frac{1}{4}\,\mathrm{tr}(Z \mapsto [X,[Y,Z]]).$$

□

13.24 Lemma. Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. For any $t \in \mathbb{R}$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{ad}"] \arrow[d, "e^t"] & \mathfrak{gl}(\mathfrak{g}) \arrow[d, "e^t"] \\ G \arrow[r, "\mathrm{Ad}"] & GL(\mathfrak{g}) \end{tikzcd}$$

commutes, i.e.

$$\forall X \in \mathfrak{g} : e^{t\,\mathrm{ad}_X} = \mathrm{Ad}_{e^{tX}}.$$

Here e^t denotes the resp. Lie group exponential maps for time t.

Proof. Fix $X \in \mathfrak{g}$ and consider the map $\alpha : \mathbb{R} \to GL(\})$, $t \mapsto \mathrm{Ad}_{e^{tX}}$. Clearly $\alpha(0) = \mathrm{id}_{\mathfrak{g}}$ and since Ad is a Lie group homomorphism, we obtain

$$\alpha(s+t) = \mathrm{Ad}_{e^{(s+t)X}} = \mathrm{Ad}_{e^{sX}e^{tX}} = \mathrm{Ad}_{e^{sX}} \circ \mathrm{Ad}_{e^{tX}} = \alpha(s)\alpha(t),$$

thus α is a one parameter subgroup of $GL(\mathfrak{g})$ through the identity. Furthermore

$$\forall Y \in \mathfrak{g} : \dot\alpha(0)(Y) = \partial_t (C_{e^{tX}})_*|_e|_{t=0}\partial_s(e^{sY})|_{s=0} = \partial_t\partial_s C_{e^{tX}}(e^{sY})|_{s=0}|_{t=0} \overset{13.15}{=} [X,Y] = \partial_t(e^{t\,\mathrm{ad}_X})|_{t=0}(Y).$$

□

13.25 Theorem (Characterization of bi-invariant metrics)**.** Let $(G, \langle\_,\_\rangle)$ be a connected Lie group with left-invariant metric. The following are equivalent:

(i) The metric $\langle\_,\_\rangle$ is bi-invariant.

(ii) For any $X \in \mathfrak{g}$ the map ad_X is skew-symmetric.

(iii) The one parameter subgroups $c(t) = e^{tX}$ are geodesics, i.e. $e^{tX} = \exp_e(tX)$.

(iv) The inversion $\sigma : G \to G$, $g \mapsto g^{-1}$, is an isometry of G.

(v) For any $g \in G$ the map Ad_g is an isometry.

Proof.
"(i)⇔(v)": This was shown in Lemma 13.21.
"(i)⇒(ii)": This was shown in Lemma 13.22.
"(ii)⇔(iii)": If $c = c_X$ is a such a one parameter subgroup, by definition we obtain

$$\dot c(t) = X|_{c(t)}.$$

Thus the geodesic ODE for c reads as

$$\nabla_X X = 0.$$

For a left-invariant metric Theorem 13.20 states

$$\nabla_X X = \frac{1}{2}([X,X] - \operatorname{ad}_X^t(X) - \operatorname{ad}_X^t(X)) = -\operatorname{ad}_X^t(X).$$

Thus we obtain the equivalence (always with $\forall X \in T_eG$)

$$c \text{ is geodesic } \iff \nabla_X X = 0 \iff \operatorname{ad}_X^t(X) = 0 \iff \operatorname{ad}_X \text{ is skew-symmetric,}$$

where the last equivalence follows from polarization via

$$\begin{aligned}
&\langle \operatorname{ad}_{Y+Z}^t(Y+Z), X\rangle = \langle Y+Z, \operatorname{ad}_{Y+Z}(X)\rangle = \langle Y+Z, \operatorname{ad}_Y(X) + \operatorname{ad}_Z(X)\rangle \\
&= \langle Y, \operatorname{ad}_Y^t(X)\rangle + \langle Y, \operatorname{ad}_Z(X)\rangle + \langle Z, \operatorname{ad}_Y(X)\rangle + \langle Z, \operatorname{ad}_Z(X)\rangle \\
&= \langle \operatorname{ad}_Y^t(Y), X\rangle - \langle Y, \operatorname{ad}_X(Z)\rangle - \langle Z, \operatorname{ad}_X(Y)\rangle + \langle \operatorname{ad}_Z^t(Z), X\rangle \\
&= -\langle \operatorname{ad}_X(Y), Z\rangle - \langle Y, \operatorname{ad}_X(Z)\rangle.
\end{aligned}$$

"(ii) and (iii) $\Rightarrow$ (v)": Since G is complete any $g \in G$ may be written as $g = \exp_e(tX)$, $X \in T_eG \cong \mathfrak{g}$. By (iii) and what we have just proven above $g = e^{tX}$. By Lemma 13.24, we obtain $\operatorname{Ad}_g = e^{t \operatorname{ad}_X}$ and by (ii)

$$\langle \operatorname{Ad}_g Y, \operatorname{Ad}_g Z\rangle = \langle e^{t\operatorname{ad}_X} Y, e^{t\operatorname{ad}_X} Z\rangle = \langle Y, (e^{t\operatorname{ad}_X})^t e^{t\operatorname{ad}_X} Z\rangle = \langle Y, e^{t(\operatorname{ad}_X + \operatorname{ad}_X^t)} Z\rangle = \langle X, Y\rangle$$

this is an isometry of $\mathfrak{g}$.
"(iv)$\Rightarrow$(i)": Since the metric is left invariant, statement (iv) together with the factorization

$$\forall g, h \in G : (\sigma \circ L_{g^{-1}} \circ \sigma)(h) = \sigma(g^{-1}h^{-1}) = hg = R_g(h)$$

implies (i).
"(i)$\Rightarrow$(iv)": Since $\sigma_*|_e = -\operatorname{id}$ (c.f. [3, 3-6]), this is certainly an isometry of T_eG. For any $g \in G$ we may transform the factorization above to $\sigma = \sigma^{-1} = R_{g^{-1}} \circ \sigma \circ L_{g^{-1}}$. Since $\langle \_, \_ \rangle$ is bi-invariant, we obtain alltogether that

$$\sigma_*|_g = R_{g^{-1}{}_*}|_e \circ \sigma_*|_e \circ L_{g^{-1}{}_*}|_g$$

is an isometry as well. □

13.26 Corollary (abelian)**.** Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra.

(i) If G is abelian, then $\mathfrak{g}$ is abelian.

(ii) If $\mathfrak{g}$ is abelian and G is connected, then G is abelian.

Proof.

(i) In case G is abelian, we may employ Lemma 13.15 above in order to obtain

$$\forall X, Y \in \mathfrak{g} : [X,Y] = \partial_t \partial_s e^{tX} e^{sY} e^{-tX}|_{s=0}|_{t=0} = \partial_t \partial_s e^{sY}|_{s=0} e^{tX} e^{-tX}|_{t=0} = \partial_t Y|_{t=0} = 0.$$

(ii) In case $\mathfrak{g}$ is abelian and G is connected we may write any $g, h \in G$ as $g = e^{tX}$, $h = e^{sY}$. Fix any $t \in \mathbb{R}$ and consider the 1PSGs $\alpha, \beta : \mathbb{R} \to G$, $\alpha(s) := e^{tX} e^{sY} e^{-tX}$, $\beta(s) := e^{sY}$. Differentiation yields

$$\dot\alpha(0) = \operatorname{Ad}_{e^{tX}}(Y) \overset{13.15}{=} e^{t\operatorname{ad}_X}(Y) = Y = \dot\beta(0)$$

and therefore $\alpha = \beta$, which implies $gh = hg$.

□

13.27 Corollary (Sectional curvature)**.** Let G be a connected Lie group with bi-invariant metric. The sectional curvatures K and the Ricci curvature Ric satisfy:

(i) $K \geq 0$.
(ii) $K = 0 \Leftrightarrow G$ is abelian.
(iii) Any $X \in \mathfrak{g}$ satisfies

$$\mathrm{Ric}(X,X) \geq 0 \qquad \text{and} \qquad \mathrm{Ric}(X,X) = 0 \Leftrightarrow X \in \mathfrak{z},$$

where $\mathfrak{z}$ is the center [16] of $\mathfrak{g}$. In case $\mathfrak{z} = \{0\}$, we always have $\mathrm{Ric}(X,X) > 0$, $X \neq 0$.

Proof.
(i) Theorem 13.23,(iv) directly implies $K \geq 0$.
(ii) Theorem 13.23,(iv) implies: If $K = K(X,Y) = \frac{1}{4}|[X,Y]|^2 = 0$ for every $X, Y \in \mathfrak{g}$, then $\mathfrak{g}$ is abelian. Conversely, if $\mathfrak{g}$ is abelian, then $K = 0$. So the statement follows from Corollary 13.26.
(iii) First a preliminary remark.
STEP 1: For any matrix $A \in \mathbb{R}^{n\times n}$ satisfying $A^t = -A$, we obtain

$$\mathrm{tr}(A^2) = (AA)^i_i = A^i_j A^j_i = -(A^i_j)^2 \leq 0$$

In particular $\mathrm{tr}(A^2) = 0 \Rightarrow A = 0$.
STEP 2: By Theorem 13.25 the map ad_X is skew symmetric for any $X \in \mathfrak{g}$. Thus $\mathrm{tr}(\mathrm{ad}_X^2) \leq 0$ and therefore Theorem 13.23,(v) implies

$$\forall X \in \mathfrak{g} : \mathrm{Ric}(X,X) = -\frac{1}{4}\,\mathrm{tr}(\mathrm{ad}_X^2) \geq 0.$$

Now let $X \in \mathfrak{g}$, such that $\mathrm{Ric}(X,X) = 0$. Then $\mathrm{tr}(\mathrm{ad}_X^2) = 0$ and thus $\mathrm{ad}_X = 0$, which is equivalent to $X \in \mathfrak{z}$.

□

13.28 Example.
(i) Let G be a Lie group of matrices. Then $-\mathrm{tr}(XY)$ defines a non degenerate bilinear form on $\mathfrak{g}$, such that ad_Z is skew symmetric for every $Z \in \mathfrak{g}$. In case $G \subset GL_n(\mathbb{R})$, we obtain the bi-invariant, but not necessarily Riemannian metric on G. In case $G \subset O(n)$ this metric is Riemannian:

$$-\mathrm{tr}(XX) = \mathrm{tr}(XX^t) \geq 0$$

(in case this is $= 0 \Leftrightarrow X = 0$). If $G \subset GL_n(\mathbb{C})$, then $-\mathrm{Re}\,\mathrm{tr}(X,Y)$ is a bi-invariant Semi-Riemannian metric on G.
(ii) On every Lie group we may define the *Killing form*

$$B(X,Y) := \mathrm{tr}(\mathrm{ad}_X \,\mathrm{ad}\, Y).$$

Since $\mathrm{ad}_{\mathrm{Ad}_g X} = \mathrm{Ad}_g \circ \mathrm{ad}_X \circ \mathrm{Ad}$ this is bi-invariant, but may be degenerate. A Lie group for which B is not degenerate, is called *semi-simple* and are very well understood. By Theorem 13.23 a semi-simple Lie group (G, B) automatically is a semi-Riemannian Einstein manifold, i.e. $\mathrm{Ric} = \mathrm{const}\, B$.

[16] Reminder: The center is defined by

$$\mathfrak{z} := \mathfrak{z}(\mathfrak{g}) := \{X \in \mathfrak{g} \mid \forall Y \in \mathfrak{g} : [X,Y] = 0\}$$

13.29 Theorem (Existence of bi-invariant metrics)**.** Let G be a compact Lie group. Then there exists a bi-invariant Riemannian metric on G.

Proof. Choose a left-invariant metric $\langle \_, \_ \rangle$ and a left-invariant volume form μ on G. For any two $X, Y \in \mathfrak{g}$ define the function $f(X,Y) : G \to \mathbb{R}$, $g \mapsto \langle \mathrm{Ad}_g(X), \mathrm{Ad}_g(Y) \rangle$. Define a scalar product on $\mathfrak{g}$ by

$$\langle\langle X, Y \rangle\rangle := \int_G \langle \mathrm{Ad}\, X, \mathrm{Ad}\, Y \rangle \mu = \int_G f(X,Y)\mu.$$

By Theorem 13.25 it suffices to show, that $\langle\langle \_, \_ \rangle\rangle$ is invariant under the adjoint representation. Now

$$\forall g \in G : (f(X,Y) \circ C_{h^{-1}}\mu)(g) = \langle \mathrm{Ad}_{h^{-1}gh}(X), \mathrm{Ad}_{h^{-1}gh}(Y) \rangle \mu_g = L^*_{h^{-1}}(f(X,Y) \circ R_h \mu)$$

and therefore the left-invariance of μ and the diffeomorphism invariance of the integral yields

$$\begin{aligned} \langle\langle \mathrm{Ad}_h(X), \mathrm{Ad}_h(Y) \rangle\rangle &= \int_G f(X,Y) \circ R_h \mu = \int_G L^*_{h^{-1}}(f(X,Y) \circ R_h \mu) = \int_G f(X,Y) \circ C_{h^{-1}} \mu \\ &\overset{(1)}{=} \int_G f(X,Y) \circ C_{h^{-1}} |\det \mathrm{Ad}_{h^{-1}}| \mu = \int_G f(X,Y) \mu = \langle\langle X, Y \rangle\rangle \end{aligned}$$

(1): Here we use the fact that $h \mapsto |\det \mathrm{Ad}_h^{-1}|$ is a continuous Lie group homomorphism $G \to (\mathbb{R}^+, \cdot)$. Therefore the image is a compact subgroup of $(\mathbb{R}^+, \cdot)$ and the only such subgroup is $\{1\}$. $\square$

13.1 The Unitary Group

As an example we discuss the unitary group

$$U_n := \{A \in GL_n(\mathbb{C}) \mid A\bar{A}^t = E\}$$

with its Lie algebra

$$\mathfrak{u_n} = T_E U_n = \{B \in \mathfrak{gl}_n(\mathbb{C}) \mid \bar{B}^t = -B\},$$

where $\mathfrak{gl}_n$ are the n by n matrices with the commutator as a Lie bracket. We define the Ad-invariant scalar product

$$\langle X, Y \rangle := -\operatorname{Re}\operatorname{tr}(XY) = \operatorname{Re}\operatorname{tr}(X\bar{Y}^t)$$

on $\mathfrak{u}_n$. This induces a bi-invariant Riemannian metric on U_n. On the other tangential spaces

$$T_A U_n = \{AB \mid B \in \mathfrak{u}_n\}$$

the metric is given by the same formula since for any $A \in U_n$

$$\operatorname{tr}(AX\overline{AX}^t) = \operatorname{tr}(AX\bar{Y}^t AA^{-1}) = \operatorname{tr}(X\bar{Y}^t).$$

We remark that the metric is the restriction of the canonical hermitian form on $\mathbb{C}^{n^2}$.
We choose an orthogonal basis for $(\mathfrak{u}_n, \langle \_, \_ \rangle)$ and therefore define $E_{ij} \in \mathbb{C}^{n \times n}$ to be the matrix $(E_{ij})^k_l := \delta_{ki}\delta_{lj}$. For $i > j$ the matrices

$$F_{ij} := E_{ij} - E_{ji} \qquad G_{ij} := i(E_{[ij} - E_{ji}) \qquad E_i := iE_{ii}$$

are an orthogonal basis for $\mathfrak{u}_n$. We remark that

$$E_{ij}E_{kl} = \delta_{jk}E_{il} \qquad |F_{ij}|^2 = |G_{ij}|^2 = 2 \qquad |E_i| = 1.$$

The set $\mathfrak{a} := \mathrm{Lin}\{E_i | 1 \le i \le n\}$ is a maximal abelian subalgebra of u_n. The F_{ij}, G_{ij} are common eigenvectors of the ad_A^2, $A = \sum_i \alpha_i E_i$:

$$\begin{aligned}
\mathrm{ad}_A(E_{jk}) &= \sum_i i\alpha_i(E_{ii}E_{jk} - E_{jk}E_{ii}) = \sum_i i\alpha_i(\delta_{ij}E_{ik} - \delta_{ki}E_{ji}) = i(\alpha_j - \alpha_k)E_{jk} \\
\mathrm{ad}_A(F_{ij}) &= i(\alpha_i - \alpha_j)E_{ij} - i(\alpha_j - \alpha_i)E_{ji} = (\alpha_i - \alpha_j)i(E_{ij} + E_{ji}) = (\alpha_i - \alpha_j)G_{ij} \\
\mathrm{ad}_A(G_{ij}) &= -(\alpha_i - \alpha_j)E_{ij} - (\alpha_j - \alpha_i)E_{ji} = -(\alpha_i - \alpha_j)(E_{ij} - E_{ji}) = -(\alpha_i - \alpha_j)F_{ij}.
\end{aligned}$$

Therefore the curvature may be calculated by

$$\begin{aligned}
R(F_{ij}, A)A &= -\frac{1}{4}\,\mathrm{ad}_A^2(F_{ij}) = \frac{1}{4}(\alpha_i - \alpha_j)^2 F_{ij} \qquad (13.4) \\
R(G_{ij}, A)A &= \frac{1}{4}(\alpha_i - \alpha_j)^2 G_{ij} \\
R(E_i, A)A &= 0.
\end{aligned}$$

13.30 Theorem. The sectional curvature K of U_n with respect to the bi-invariant metric $-\mathrm{Re}\,\mathrm{tr}(XY)$ satisfies the sharp estimate

$$0 \le K \le \frac{1}{2}.$$

Proof. Since any $A \in \mathfrak{u}_n$ may be transformed into normal form by conjugation with a suitable $g \in U_n$ (i.e. $gAg^{-1} \in \mathfrak{a}$), it suffices to consider planes $A \wedge X$, where $A = \sum_i \alpha_i E_i \in \mathfrak{a}$, $\sum_i \alpha_i^2 = 1$. But for those the estimate is clear due to (13.4). □

Next we determine the cut locus of U_n: Let $S \in U_n$. Choose a geodesic $e^{t\Sigma}$ between e and $e^{\Sigma} = S$, $\Sigma \in \mathfrak{u}_n$. Since Σ is skew-hermitian, there exists $T \in U_n$ such that $T\Sigma T^{-1} = A = i\,\mathrm{diag}(\alpha_1, \dots, \alpha_n)$, $\alpha_j \in \mathbb{R}$. Therefore $Te^{t\Sigma}T^{-1} = e^{tA}$ is a geodesic of the same length from e to $e^A = \mathrm{diag}(e^{i\alpha_1}, \dots, e^{i\alpha_n})$ and this normal form e^A of S is unique up to permutation. Therefore the lengths of geodesics from e to S are exactly $\sqrt{\sum_i \beta_i^2}$, where $B = i\,\mathrm{diag}(\beta_1, \dots, \beta_n)$ satisfies $e^B = e^A$.
Consequently $e^{t\Sigma}$ is minimizing between e and S if and only if the normal form A of Σ satisfies $\alpha_i \in [-\pi, \pi]$.

13.31 Theorem. The cut locus of e in U_n is

$$C(e) = \{S \in U_n \mid S \text{ has eigenvalue } -1\}.$$

Proof. If S does not have eigenvalue -1, there exists (up to permutation) precisely one $A = i\,\mathrm{diag}(\alpha_1, \dots, \alpha_n)$ satisfying $\alpha_i \in]-\pi, \pi[$ and e^A is conjugate to S. In that case $d(e, S) = \sqrt{\sum_i \alpha_i^2}$.
In case S does have eigenvalue -1, $S \in C(p)$ since $e^{i\pi} = e^{-i\pi} = -1$. □

13.32 Remark.
(i) $U_n = \mathbb{S}^1 \times SU_n$ as a product of groups and Riemannian manifolds.
(ii) $U_m \subset U_n$ totally geodesic, $m \le n$.
(iii) $O_n \subset U_n$ totally geodesic, since O_n is the fixed point set of the isometry $A \mapsto \bar{A}$.

13.2 The Heisenberg group H_3

$\mathfrak{h}_3 = \mathrm{Lin}(e_1, e_2, e_3)$, where

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Define a scalar product on $\mathfrak{h}_3$ by declaring e_1, e_2, e_3 to be an ONB. The Lie bracket is given by $[e_1, e_2] = e_3$ and $[e_i, e_j] = 0$ for any other $\{i, j\} \neq \{1, 2\}$. All ad^t are zero except

$$\mathrm{ad}^t_{e_1} e_3 = e_2 \qquad\qquad \mathrm{ad}^t_{e_2} e_3 = -e_1,$$

which implies that the connection is given by

$$\nabla_{e_1} e_2 = \frac{1}{2} e_3 \qquad \nabla_{e_1} e_3 = -\frac{1}{2} e_2 \qquad \nabla_{e_2} e_3 = \frac{1}{2} e_1$$

and all other values can be deduced from symmetry or are zero. Remember that the sectional curvature is given by Theorem 13.20, which implies that

$$K(e_1 \wedge e_2) = \frac{1}{4} - 1 = -\frac{3}{4} \qquad\qquad K(e_1 \wedge e_3) = \frac{1}{4} = K(e_2 \wedge e_3).$$

Therefore the scalar curvature is given by

$$S = \mathrm{Ric}(e_1) + \mathrm{Ric}(e_2) + \mathrm{Ric}(e_3) = -\frac{1}{2} - \frac{1}{2} + \frac{1}{2} = -\frac{1}{2}.$$

13.3 A realization of hyperbolic Space

We want to realize the hyperbolic space $\mathbb{R}H^{n+1}$ as a solvable group with left invariant metric. Choose Euclidean vector spaces $\mathfrak{a}$, x, $\dim \mathfrak{a} = 1$, $\dim x = n$ and define $s := \mathfrak{a} + x$ (as an orthogonal sum of Euclidean vector spaces). Then s becomes a solvable Lie algebra by defining $[\mathfrak{a}, \mathfrak{a}] := [x, x] := 0$ and $\mathrm{ad}_A |x = c\,\mathrm{id}$, where $A \in \mathfrak{a}$ (is a unit vector for example.). This implies for $X, Y \in x$

$$\mathrm{ad}^t_A X = cX \qquad\qquad \mathrm{ad}^t_X Y = -c\langle X, Y\rangle A$$

and all others are zero. Furthermore

$$\nabla_A = 0 \qquad\qquad \nabla_X = cA \wedge X,$$

where $U \wedge V$ is the skew-symmetric endomorphism $w \mapsto \langle w, v\rangle u - \langle w, u\rangle v$. This implies

$$R(A, X) = -c\nabla_X = -c^2 A \wedge X \qquad\qquad R(X, Y) = [\nabla_X, \nabla_Y] = -c^2 X \wedge Y.$$

Consequently the cuvature operator $R : \Lambda^2 s \to \Lambda^2 s$ is $-c^2\,\mathrm{id}$. So the simply connected Lie group $(S, \langle\_, \_\rangle)$ associated to $(s, \langle\_, \_\rangle)$ with left invariant metric is the space form with constant curvature $-c^2$.

The last two examples admit a proof of the following

13.33 Theorem (Milnor)**.** If G is non-abelian, then G admits a left-invariant metric of negative scalar curvature.

Proof.

Case 1: There exist $X, Y \in \mathfrak{g} : X, Y, [X, Y]$ are linearly independent. Complete them to a basis $b_1 := X, b_2 := Y, b_3 := [X, Y], b_4, \ldots, b_n$ of $\mathfrak{g}$. For any $\varepsilon > 0$ define a scalar product $\langle\_, \_\rangle_\varepsilon$ on $\mathfrak{g}$ by declaring

$$e_1 := \varepsilon b_1, e_2 := \varepsilon b_2, e_3 := \varepsilon^2 b_3, \ldots, e_n := \varepsilon^2 b_n$$

to be an ONB. The stucture constants α_{ijk} defined by $[e_i, e_j] = \sum_k \alpha_{ijk} e_k$ satisfy $\alpha_{123} = 1 = \alpha_{231}$ and $\alpha_{ijk} \leq \mathrm{const}\,\varepsilon$ otherwise. Thus $\lim_{\varepsilon\to 0} \alpha_{ijk}(\varepsilon)$ are the structure constants of a direct sum $\mathfrak{g}_0 = \mathfrak{h}_3 + \mathfrak{z}$, $\mathfrak{z} = \textit{center}$. Since the scalar curvature S depends continuously on the structure constants with respect to an ONB, we obtain

$$\lim_{\varepsilon\to 0} S(\mathfrak{g}, \langle\_, \_\rangle_\varepsilon) = S(\mathfrak{g}_0) = S(\mathfrak{h}_3) = -\frac{1}{2} < 0.$$

In particular $S < 0$, if $\varepsilon > 0$ is sufficiently small.

CASE 2: For any X, Y we have $\mathrm{Lin}(X, Y, [X, Y]) = \mathrm{Lin}(X, Y)$. Then

$$\mathrm{ad}_X Y = l(X)Y \mod \mathbb{R}x$$

for some linear functional $l \in \mathfrak{g}^*$. Thus

$$[X, Y] = l(X)Y - l(Y)X$$

and $\ker l$ is an $(n-1)$-dimensional abelian ideal in $\mathfrak{g}$. Choose a unit vector $A \in \ker l^\perp$. Then $\mathrm{ad}_A | \ker l = l(A)\,\mathrm{id}$. Thus we have the Lie algebra from the preceeding example with $c = l(A)$. In particular $\mathfrak{g}$ has negative scalar curvature.

□

13.4 The Heisenberg Group

As a last example we discuss the Heisenberg Group in more detail, a Lie group with a left invariant metric. We use slightly different coordinates as in Example 13.2.

13.34 Definition (Heisenberg Group)**.** For any $m \in \mathbb{N}$ the *Heisenberg group* is given by

$$H_{2m+1} := \left\{ \begin{pmatrix} 1 & x^t & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{(m+2)\times(m+2)} | x, y \in \mathbb{R}^m, z \in \mathbb{R}. \right\}.$$

It is an affine linear subspace of $M_{m+2}(\mathbb{R})$ and a subgroup of $GL_{m+2}(\mathbb{R})$. Therefore its Lie algebra is given by

$$\mathfrak{h}_{2m+1} = T_e H_{2m+1} = \left\{ \begin{pmatrix} 0 & x^t & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{(m+2)\times(m+2)} | x, y \in \mathbb{R}^m, z \in \mathbb{R} \right\}.$$

To simplify notation we employ the "exponential coordinates"

$$H \ni (x, y, z) := \exp(x, y, z) := \begin{pmatrix} 1 & x^t & z + \frac{1}{2}\langle x, y\rangle \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

in which multiplication looks like

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}\langle x, y'\rangle - \frac{1}{2}\langle x', y\rangle).$$

The Lie algebra may also be written as

$$\mathfrak{h} \ni (x, y, z) := \begin{pmatrix} 0 & x^t & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix},$$

in which the Lie bracket looks like

$$[(x, y, z), (x', y', z')] = (0, 0, \langle x, y'\rangle - \langle x', y\rangle).$$

A basis for $\mathfrak{h}$ is consequently given by

$$X_i := (e_i, 0, 0) \qquad Y_i := (0, e_i, 0) \qquad Z := (0, 0, 1),$$

where $1 \le i \le m$. The only non-zero Lie brackets are

$$[X_i, Y_i] = -[Y_i, X_i] = Z.$$

13.35 Remark.

(i) Observe that $[\mathfrak{h},\mathfrak{h}] = \mathrm{Lin}(Z) \neq 0$, $[\mathfrak{h},[\mathfrak{h},\mathfrak{h}]] = 0$, so $\mathfrak{h}$ is a second order nilpotent Lie algebra.

(ii) From the multiplication formula we may deduce $\exp(A)\exp(B) = \exp(A + B + \frac{1}{2}[A,B])$. This formula holds in every second order nilpotent Lie algebras and is a special case of the so called "Campbill-Baker-Hausdorff-Formula".

We would like to calculate the left-invariant vector fields corresponding to $X_1,\dots,X_m$, $Y_1,\dots,Y_m$, Z w.r.t. the chosen identification of H with $\mathbb{R}^{2m+1}$ (exponential coordinates): Let $p = (x,y,z) \in H$ and observe

$$X_i|_p = L_{p_*}X_i = \partial_t(x,y,z)\cdot(te_i,0,0)|_{t=0} = \partial_t(x+te_i, y, z - \frac{1}{2}ty_i)|_{t=0} = (e_i, 0, -\frac{1}{2}y_i).$$

Analogously

$$Y_i|_p = (0, e_i, \frac{1}{2}x_i) \qquad\qquad Z|_p = (0,0,1).$$

Definition of the metric: Let g be the left-invariant metric on H uniquely determind by requirering the left-invariant vector fields $X_1,\dots,X_m$, $Y_1,\dots,Y_m$, Z, to be orthonormal.

Levi-Civita Connection: The Levi-Civita connection with respect to this metric is given by

$$\nabla_{X_i}X_j = 0 = \nabla_{Y_i}Y_j, \qquad \nabla_{X_i}Y_j = \delta_{ij}\frac{1}{2}Z = -\nabla_{Y_j}X_i,$$
$$\nabla_{X_i}Z = -\frac{1}{2}Y_i = \nabla_Z X_i, \qquad \nabla_{Y_i}Z = \frac{1}{2}X_i = \nabla_Z Y_i, \qquad \nabla_Z Z = 0$$

Sectional Curvature: The only non-zero sectional curvatures of H are generated by

$$K(X_i,Y_i) = \frac{1}{4} - 1 - 0 = -\frac{3}{4} \qquad K(X_i,Z) = \frac{1}{4} - 0 - 0 = \frac{1}{4} = K(Y_i,Z)$$

Ricci Curvature: The Ricci curvature is given by

$$\mathrm{Ric}(X_i) = -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2} = \mathrm{Ric}(Y_i) \qquad \mathrm{Ric}(Z) = 2m\cdot\frac{1}{4} = \frac{m}{2}.$$

Scalar Curvature: The scalar curvature is given by

$$s = 2m\cdot(-\frac{1}{2}) + \frac{m}{2} = -\frac{m}{2} < 0.$$

Geodesics: Let γ be a unit speed geodesic in (H,g) satisfying $\gamma(0) = e = (0,0,0)$. We may decompose $\dot\gamma(t)$ into

$$\dot\gamma(t) = \sum_i a_i(t)X_i|_{\gamma(t)} + b_i(t)Y_i|_{\gamma(t)} + c(t)Z|_{\gamma(t)}.$$

Therefore the geodesic equation reads as

$$0 = \nabla_{\dot\gamma}\dot\gamma = \sum_i \dot a_i X_i + \dot b_i Y_i + \dot c Z + \sum_i a_i b_i(\underbrace{\nabla_{X_i}Y_i + \nabla_{Y_i}X_i}_{=0}) + \sum_i a_i c(\underbrace{\nabla_{X_i}Z + \nabla_Z X_i}_{=-Y_i}) + \sum_i b_i c(\underbrace{\nabla_{Y_i}Z + \nabla_Z Y_i}_{=X_i}).$$

By comparing the coefficients, we obtain the following system of ODE

$$\begin{pmatrix}\dot a\\ \dot b\\ \dot c\end{pmatrix} = \begin{pmatrix}-cb\\ ca\\ 0\end{pmatrix}$$

So $c = \text{const}$ and solutions are given by

$$\begin{pmatrix} a_i(t) \\ b_i(t) \end{pmatrix} = D^{tc} \begin{pmatrix} a_i(0) \\ b_i(0) \end{pmatrix}, \qquad D^{tc} := \begin{pmatrix} \cos(tc) & -\sin(tc) \\ \sin(tc) & \cos(tc) \end{pmatrix}.$$

Now let $\gamma(t) = (x(t), y(t), z(t)$ and obtain

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} a \\ b \\ c - \frac{1}{2}\sum_i a_i y_i + \frac{1}{2}\sum_i b_i x_i \end{pmatrix}.$$

Case 1 ($c = 0$): This implies $a = a(0)$, $b = b(0)$, $x(t) = ta(0)$, $y(t) = tb(0)$, $\dot{z} = 0$, $z = 0$ and therefore $\gamma(t) = (ta(0), tb(0), 0) = \exp(t\dot{\gamma}(0))$ is a one parameter subgroup which is a geodesic.
Case 2 ($c \neq 0$): In that case the solutions for x, y are given by

$$\begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} = \frac{D^{-90}(D^{tc} - E)}{c} \cdot \begin{pmatrix} a_i(0) \\ b_i(0) \end{pmatrix}$$

and therefore

$$\begin{aligned} \dot{z}(t) = c + \frac{1}{2c} \sum_i &(-\cos(tc)a_i(0) - \sin(tc)b_i(0))(a_i(0) - \cos(tc)a_i(0) + \sin(tc)b_i(0)) \\ + \sum_i &(\sin(tc)a_i(0) + \cos(tc)b_i(0))(-b_i(0) + \sin(tc)a_i(0) + \cos(tc)b_i(0)) \\ = c + \frac{1}{2c} \sum_i &(a_i(0)^2 + b_i(0)^2)(1 - \cos(tc)), \end{aligned}$$

which implies

$$z(t) = tc + \frac{1 - c^2}{2c^2}(tc - \sin(tc)).$$

13.36 Remark. If we had used the coordinates from 13.2 the function $z(t)$ would be even more complicated. This is one reason to use exponential coordinates.

Finally we would like to analyze lengths of geodesics having the same endpoint in $\exp(\mathbb{R}Z)$: Let $\lambda(t) := (0, 0, t)$ the particular geodesic through $(0, 0, 1)$. Let $c \neq 0$ and $a(0)$, $b(0)$, such that $|(a(0), b(0), c| = 1$, be arbitrary and let γ be the geodesic with initial velocity $(a(0), b(0), c)$. Now

$$\lambda(t_0) \in \lambda(\mathbb{R}) \Leftrightarrow t_0 = \frac{2k\pi}{c}, k \in \mathbb{Z}.$$

Such a t_0 satisfies $z(t_0) = 2k\pi + \frac{1-c^2}{2c^2}2k\pi = \frac{1+c^2}{2c^2}2k\pi$. Thus til the point $(0, 0, z(t_0))$ the "straight" geodesic λ has length $L_1 = \frac{2|k|\pi}{|c|}\frac{1+c^2}{2|c|}$ and γ has length $L_2 = \frac{2|k|\pi}{|c|}$. Since $\frac{1+c^2}{2|c|} > 1$, $0 < |c| < 1$, $L_1 > L_2$ and the "wriggled" geodesic γ is shorter.

A Appendix

A.1 Covariant Derivatives along fields of Endomorphisms

We are assuming that we already have established covariant derivaties of tensor fields on manifolds and along curves.

A.1 Remark. Let V be a real vector space of dimension n. Remember, that the map $\Phi : \mathrm{End}(V) \to T_1^1(M)$, defined by $\Phi(f) : V^* \times V \to \mathbb{R}$, $(\omega, X) \mapsto \omega(f(X))$, is an isomorphism (c.f. [2, 2.1]) For a smooth n-manifold M, this induces a diffeomorphism $\Phi : \mathrm{End}(M) \to \mathcal{T}_1^1(M)$.

A.2 Definition. Let $f \in \mathrm{End}(M)$ and $X \in \mathcal{T}(M)$. We call

$$\nabla_X f := \Phi^{-1}(\nabla_X(\Phi(f)),$$

the *covariant derivative of f in direction X*. By construction

$$\begin{array}{ccc} \mathrm{End}(M) & \xrightarrow{\Phi} & \mathcal{T}_1^1(M) \\ \Big\downarrow \nabla_X & & \Big\downarrow \nabla_X \\ \mathrm{End}(M) & \xrightarrow{\Phi} & \mathcal{T}_1^1(M) \end{array}$$

commutes. If $c : I \to M$ is a smooth curve, $f \in \mathrm{End}(c)$, $X \in \mathcal{T}(c)$, we say

$$(D_t f)(t) := f'(t) := \nabla_{\dot{c}(t)} f(t)$$

is the *covariant derivative of f along c.*

The only reason we consider this is, that we would like to derive some formulas, which are helpful when calculating with this derivative.

A.3 Theorem. Let $X, Y \in \mathcal{T}(M)$, $T \in \mathcal{T}_1^1(M)$, $f \in \mathrm{End}(M)$, sodass $\Phi(f) = T$. Then $f(Y) \in \mathcal{T}(M)$ and we obtain:

(i) Product Rule:
$$\nabla_X(f(Y)) = (\nabla_X f)(Y) + f(\nabla_X Y)$$

(ii) Let γ be a curve and let $Y \in \mathcal{T}(\gamma)$, $f \in \mathrm{End}(\gamma)$. Then
$$D_t(f(Y)) = (D_t f)(Y) + f(D_t Y),$$
where we are assuming that the covariant differential D_t is extended to tensor fields.

(iii) For any $g \in \mathrm{End}(M)$, we obtain the chain rule
$$\nabla_X(f(g(Y)) = (\nabla_X f)(g(Y)) + f((\nabla_X g)(Y)) + f(g(\nabla_X Y)).$$

(iv) In particular if f is a field of isomorphisms and $g := f^{-1}$, we obtain
$$(\nabla_X f^{-1})(f(Y)) = -f^{-1}((\nabla_X f)(Y)).$$

Analogous formulae hold when considering fields of endomorphisms along curves.

Proof. We choose a local frame $\{E_i\}$ and calculate using [2, 4.6]:

(i)

$$\begin{aligned}
&\nabla_X(f(Y)) - (\nabla_X f)(Y) - f(\nabla_X Y)\\
&= \nabla_X(T(E^i, Y)E_i) - (\nabla_X T)(E^i, Y)E_i - T(E^i, \nabla_X Y)E_i\\
&= \big(X(T(E^i, Y))E_i + T(E^i, Y)\nabla_X E_i\big) - \big(X(T(E^i, Y))E_i - T(\nabla_X E^i, Y)E_i - T(E^i, \nabla_X Y)E_i\big)\\
&- T(E^i, \nabla_X Y)E_i\\
&= T(E^i, Y)\nabla_X E_i + T(\nabla_X E^i, Y)E_i = T(E^i, Y)X^j\nabla_{E_j}E_i + X^j T(\nabla_{E_j}E^i, Y)E_i\\
&= T(E^i, Y)X^j\Gamma^k_{ji}E_k + X^j T(-\Gamma^i_{jk}E^k, Y)E_i = T(E^i, Y)X^j\Gamma^k_{ji}E_k - X^j\Gamma^k_{ji}T(E^i, Y)E_k = 0
\end{aligned}$$

(ii) Follows analogously.

(iii) Using (i) we obtain

$$\begin{aligned}
&\nabla_X(f(g(Y)) = (\nabla_X f)(g(Y)) + f(\nabla_X(g(Y))) = (\nabla_X f)(g(Y)) + f((\nabla_X g)(Y) + g(\nabla_X Y))\\
&= (\nabla_X f)(g(Y)) + f((\nabla_X g)(Y)) + f(g(\nabla_X Y)).
\end{aligned}$$

(iv) Using (iii) we obtain:

$$\begin{aligned}
&\nabla_X(f^{-1}(f(Y)) = (\nabla_X f^{-1})(f(Y)) + f^{-1}((\nabla_X f)(Y)) + f^{-1}(f(\nabla_X Y))\\
&\Leftrightarrow \nabla_X Y = (\nabla_X f^{-1})(f(Y)) + f^{-1}((\nabla_X f)(Y)) + \nabla_X Y\\
&\Leftrightarrow (\nabla_X f^{-1})(f(Y)) = -f^{-1}((\nabla_X f)(Y)).
\end{aligned}$$

□

Besides this abstract interpretation, there is a very easy way to calculate the differential.

A.4 Lemma (Covariant derivative in coordinates)**.** Let $c : I \to M$ be a unit speed curve and let $E_1 := \dot{c}, E_2, \dots, E_n$ be a parallel ONB along c. Let (u^j_i) be the coordinate matrix function of a field of endomorphisms $U \in \mathrm{End}(c)$ along c whith respect to this basis, i.e.

$$\forall t \in I : u^j_i(t) = \langle U_t(E_i(t)), E_j(t)\rangle.$$

Then

$$\forall t \in I : (u')^j_i(t) = (u^j_i)'(t),$$

i.e. the coefficients of U' are just the ordinary derivaties of the coefficients u of U.

Proof. Since $D_t E_i = D_t E_j = 0$ the product rule implies

$$\begin{aligned}
(u^j_i)'(r) &= \langle U_r(E_i), E_j(r)\rangle' = \langle D_t(U_r E_i), E_j(r)\rangle + \langle U_r(E_i), \underbrace{D_t E_j(r)}_{=0}\rangle\\
&= \langle U'_r(E_i), E_j(r)\rangle + \langle U_r(\underbrace{D_t E_i}_{=0}), D_t E_j(r)\rangle = \langle U'_r(E_i), E_j(r)\rangle = (u')^j_i(r).
\end{aligned}$$

□

The concept of covariant differentiation of fields of endomorphisms is of geometric intrest, because of the curvature endomorphism.

A.5 Lemma. Denote by $\mathcal{T}^3(M) \to \mathcal{T}(M) : (X, Y, Z) \to R(X, Y)Z$ the Riemannian curvature of M. By fixing two fields at two positions, this defines three smooth fields of endomorphisms

$$X \mapsto R_{Y,Z}(X) := R(X, Y, Z) \quad Y \mapsto R_{X,Z}(Y) := R(X, Y, Z) \quad Z \mapsto R_{X,Y}(Z) := R(X, Y, Z) \quad .$$

Their covariant differentials are given by

$$\begin{aligned}
\nabla_W R_{Y,Z}(X) &= \nabla_W(R(X, Y, Z)) - R(\nabla_W X, Y, Z) \\
\nabla_W R_{X,Z}(Y) &= \nabla_W(R(X, Y, Z)) - R(X, \nabla_W Y, Z) \\
\nabla_W R_{X,Y}(Z) &= \nabla_W(R(X, Y, Z)) - R(X, Y, \nabla_W Z).
\end{aligned}$$

Proof. This follows directly from Theorem A.3. □

A.6 Remark. This construction can be generalized. If we denote by $\mathrm{Mult}_l^k(V)$ the space of multilinear maps $(V^*)^l \times V^k \to V$, there is also a canonical isomorphism $\Phi : \mathrm{Mult}_l^k(V) \to T_{l+1}^k(V)$ (c.f. [2, 2.1] again). If $F \in \mathrm{Mult}_l^k(V)$ and $T \in T_{l+1}^k(V)$ such that $\Phi(F) = T$ we may use a basis $\{E_i\}$ of V and its corresponding dual basis $\{E^i\}$ to mutually identify $T \triangleq F$ (via Φ) using the equations

$$\begin{aligned}
T(\omega^1, \ldots, \omega^{l+1}, X_1, \ldots, X_k) &= \omega^{l+1}(F(\omega_1, \ldots, \omega_l, X_1, \ldots, X_k)) \\
F(\omega_1, \ldots, \omega_l, X_1, \ldots, X_k) &= T(E^i, \omega_1, \ldots, \omega_l, X_1, \ldots, X_k))E_i
\end{aligned}$$

This generalizes to manifolds and by forcing this diagramm

$$\begin{array}{ccc}
\mathrm{Mult}_l^k(M) & \xrightarrow{\Phi} & \mathcal{T}_l^k(M) \\
\downarrow{\scriptstyle \nabla_X} & & \downarrow{\scriptstyle \nabla_X} \\
\mathrm{Mult}_l^k(M) & \xrightarrow{\Phi} & \mathcal{T}_l^k(M)
\end{array}$$

to commute, we may also define covariant differentiation of (k, l)-multilinear fields in an entirely analogous fashion.

A.7 Lemma. In the sense of the above definition the covariant derivative of the Riemannian curvature $R \in \mathrm{Mult}^3(M)$ is given by

$$\nabla R(X, Y, Z, W) = \nabla_W(R(X, Y, Z)) - R(\nabla_W X, Y, Z) - R(X, \nabla_W Y, Z) - R(X, Y, \nabla_W Z).$$

Proof. By unwinding all the definitions we obtain

$$\begin{aligned}
&\nabla R(X, Y, Z, W) \triangleq \nabla R(X, Y, Z, W, E^i)E_i = (\nabla_W R)(X, Y, Z, E^i)E_i \\
&= W(R(X, Y, Z, E^i))E_i - R(X, Y, Z, \nabla_W E^i)E_i \\
&- R(\nabla_W X, Y, Z, E^i)E_i - R(X, \nabla_W Y, Z, E^i)E_i - R(X, Y, \nabla_W Z, E^i)E_i \\
&\triangleq E^i(\nabla_W(R(X, Y, Z, E^i))E_i \\
&- R(\nabla_W X, Y, Z) - R(X, \nabla_W Y, Z) - R(X, Y, \nabla_W Z).
\end{aligned}$$

□

A.2 Gauss' Lemma

There are various formulations of Gauss' lemma in the literature. We will prove one version of it as well as some small preliminary lemmas, which are sometimes useful themselves.

A.8 Lemma ($(\exp)_*$ and Jacobi fields)**.** Let $p \in M$, $w, v, Y \in T_pM$ $w = \|w\|v$ and let c be the unit speed geodesic through p with initial velocity $\dot{c}(0) = v$. Then

$$(\exp_p)_*|_w(Y) = \frac{1}{\|w\|} J(\|w\|),$$

where J is the Jacobi field along c satisfying $J(0) = 0$ and $J'(0) = Y$.

Proof. Define a geodesic variation H of c by

$$H(s,t) := \exp_p(t(v + sY)).$$

Its variation field

$$J(t) := \partial_s H(s,t)|_{s=0} = \partial_s(\exp_p(t(v + sY)))_{s=0}$$

is a Jacobi field ([2, 10.2]) and we obtain

$$\begin{aligned}(\exp_p)_*|_w(Y) &= \frac{1}{\|w\|}(\exp_p)_*|_{v\|w\|}(Y\|w\|) = \frac{1}{\|w\|}\partial_s \exp_p(v\|w\| + sY\|w\|)|_{s=0} \\ &= \frac{1}{\|w\|}\partial_s \exp_p(\|w\|(v + sY)|_{s=0} = \frac{1}{\|w\|} J(\|w\|).\end{aligned}$$

Obviously $J(0) = 0$ and furthermore

$$J'(0) = D_t J(0) = D_t \partial_s H(s,t)|_{s=0}|_{t=0} = D_s \partial_t \exp_p(t(v + sY))|_{t=0}|_{s=0} = D_s(v + sY)|_{s=0} = Y.$$

□

A.9 Lemma. Let $c : I \to M$ be a geodesic and let J be a Jacobi field along c. Then the function $t \mapsto \langle J(t), \dot{c}(t)\rangle$ is a polynomial of degree 1. More precisely:

$$\langle J(t), \dot{c}(t)\rangle = \langle D_t J(0), \dot{c}(0)\rangle t + \langle J(0), \dot{c}(0)\rangle$$

Proof. Since c is a geodesic, we have $D_t \dot{c} \equiv 0$. Using compatibility with the metric and that J solves the Jacobi equation, we obtain

$$\langle J, \dot{c}\rangle'' = \langle D_t^2 J, \dot{c}\rangle = -\langle R(J, \dot{c})(\dot{c}), \dot{c}\rangle = -Rm(J, \dot{c}, \dot{c}, \dot{c}) = 0,$$

where the last equality follows from the symmetries of the curvature tensor (c.f. [2, 7.4]). Thus there are $a, b \in \mathbb{R}$, such that $\langle J(t), \dot{c}(t)\rangle = at + b =: p(t)$. We obtain

$$b = p(0) = \langle J(0), \dot{c}(0)\rangle, \qquad a = p'(0) = \langle D_t J(0), \dot{c}(0)\rangle.$$

□

A.10 Theorem (Gauss' Lemma)**.** Let $p \in M$, $X \in \mathcal{E}_p \subset T_pM$ and $Y \in T_pM$. Then

$$\langle \exp_{p_*}|_X(X), \exp_{p_*}|_X(Y)\rangle = \langle X, Y\rangle.$$

Proof. In case $X = 0$, the statement is trivial, so let $X \neq 0$. Let c_X be the geodesic through p with initial velocity $X \in T_pM$. Then

$$\exp_{p_*}|_X(X) = \partial_t(\exp_p(X + tX)|_{t=0} = \partial_t(\exp_p(tX))|_{t=1} = \dot{c}_X(1).$$

By Lemma A.8 we obtain

$$\exp_{p_*}|_X(Y) = \frac{1}{\|X\|} J_Y(\|X\|),$$

where J_Y is the Jacobi field along $c_{\frac{X}{\|X\|}}$ satisfying $J_Y(0) = 0$ and $D_t J_Y(0) = Y$. Using $c_{\frac{X}{\|X\|}}(t\|X\|) = c_X(t)$, we obtain alltogether

$$\begin{aligned}
\langle \exp_{p_*}|_X(X), \exp_{p_*}|_X(Y)\rangle &= \langle \dot{c}_X(1), \frac{1}{\|X\|} J_Y(\|X\|)\rangle = \langle \dot{c}_{\frac{X}{\|X\|}}(\|X\|)\|X\|, \frac{1}{\|X\|} J_Y(\|X\|)\rangle \\
&\overset{A.9}{=} \langle \dot{c}_{\frac{X}{\|X\|}}(0), D_t J_Y(0)\rangle \|X\| + \langle J_Y(0), \dot{c}_{\frac{X}{\|X\|}}(0)\rangle = \langle X, Y\rangle.
\end{aligned}$$

□

A.3 Technical Lemmata

A.11 Lemma. Let $D \subset \mathbb{R}$ be open, $f : D \to \mathbb{R}$ be smooth and $a \in D$. Suppose

$$\forall 0 \leq k \leq n-1 : f^{(k)}(a) = 0,$$

but $f^{(n)}(a) \neq 0$. Then there exists a smooth function $g : D \to \mathbb{R}$, such that

$$\forall x \in D : f(x) = (x-a)^n g(x),$$

where $g(a) \neq 0$.

Proof. This is just a weaker formulation of Taylor's formula: Although f is not analytic it has a representation

$$\begin{aligned}
f(x) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-a)^{n+1} \\
&= (x-a)^n \underbrace{\left(\frac{f^{(n)}(a)}{n!} + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-a)\right)}_{=:g(x)},
\end{aligned}$$

where $\xi(x) \in [x, a]$ bzw. $\xi(x) \in [a, x]$. □

A.12 Corollary. Let $c : [0, R] \to M$ be a curve and let $X \in \mathcal{T}(c)$ be a vector field such that $X(0) = 0$ and $X'(0) \neq 0$. Then there exists $Y \in \mathcal{T}(c)$ such that

$$\forall t \in [0, R] : X(t) = tY(t),$$

where $Y(0) = X'(0)$.

Proof. Let E_1 be the parallel translate of $X'(0) \neq 0$ and choose $E_2, \ldots, E_n$, such that $E_1, \ldots, E_n$ is a parallel ONB along c. We obtain

$$\begin{aligned}
0 = X(0) = X^i(0)E_i(0) &\Longrightarrow X^i(0) = 0 \\
E_1(0) = X'(0) = \dot{X}^i(0)E_i(0) + X^i(0)\dot{E}_i(0) &\Rightarrow X^i(0) = \delta_1^i.
\end{aligned}$$

Therefore Lemma A.11 above yields functions $G^i : [0, R] \to \mathbb{R}$ such that $G^i(0) \neq 0$ and $X^i(t) = tG^i(t)$. Thus

$$X(t) = X^i(t)E_i(t) = tG^i(t)E_i(t) =: tY(t).$$

Clearly $X'(0) = Y(0)$. □

A.4 Law of Cosines

A.13 Theorem (Law of Cosines)**.** Let $\kappa \in \mathbb{R}$ and let $\Delta = (\gamma_0, \gamma_1, \gamma_2)$ be a triangle in M_κ^2 with side lengths l_i and angles α_i, $i \in \{1,2,3\}$. Taking all indices modulo 3, the *law of cosines* holds:

(i) If $\kappa = 0$
$$l_i^2 = l_{i+1}^2 - 2l_{i+1}l_{i+2}\cos(\alpha_i).$$

(ii) If $\kappa \neq 0$
$$\mathrm{cs}_\kappa(l_i) = \mathrm{cs}_\kappa(l_{i+1})\,\mathrm{cs}_\kappa(l_{i+2}) + \mathrm{sgn}(\kappa)\kappa\,\mathrm{sn}_\kappa(l_{i+1})\,\mathrm{sn}_\kappa(i_{i+2})\cos(\alpha_i).$$

Here the functions cs_κ, sn_κ are taken from 2.7.

Proof. We just quote this from [6, p.138]. □

A.14 Theorem (Angle sum identity)**.** If $\kappa \neq 0$ the cs_κ satisfies
$$\mathrm{cs}_\kappa(\alpha_1 + \alpha_2) = \mathrm{cs}_\kappa(\alpha_1)\,\mathrm{cs}_\kappa(\alpha_2) - \kappa\,\mathrm{sn}_\kappa(\alpha_1)\,\mathrm{sn}_\kappa(\alpha_2).$$

A.15 Theorem. Let M be a Riemannian manifold with metric g. Assume there is a second metric $\tilde{g}$ and a constant $\lambda \in \mathbb{R}_{>0}$, such that $\tilde{g} = \lambda g$. Then we obtain the following transformation laws:

(i) Length:
$$\forall p \in M : \forall v \in T_pM : \|v\|_{\tilde{g}} = \sqrt{\lambda}\|v\|_g.$$

(ii) Balls
$$\forall p \in M : \forall v \in T_pM : \forall R \geq 0 : \tilde{B}_R(v) = B_{\frac{R}{\sqrt{\lambda}}}(v).$$

(iii) Distance:
$$\forall p, q \in M : \tilde{d}(p,q) = \sqrt{\lambda}d(p,q).$$

(iv) Levi/Civita-Connection:
$$\forall X, Y \in \mathcal{T}(M) : \tilde{\nabla}_X Y = \nabla_X Y.$$

(v) Riemannian Curvature Endomorphism:
$$\forall X, Y, Z \in \mathcal{T}(M) : \tilde{R}(X,Y)Z = R(X,Y)Z.$$

(vi) Curvature Tensor:
$$X, Y, Z, W \in \mathcal{T}(M) : \tilde{R}m(X,Y,Z,W) = \lambda R(X,Y,Z,W).$$

(vii) Sectional Curvature
$$\tilde{K}(X,Y) = \frac{1}{\lambda}K(X,Y)$$

List of Symbols

1PSG one parameter subgroup, page 84

Ad Adjoint representation of a Lie group, page 86

ad the ad map in a Lie group, page 85

C_g conjugation with g, page 82

$C(p)$ cut locus, page 13

cs_κ standard solution of the Jacobi equation, page 10

ct_κ standard solution of the Riccati equation, page 10

c_v geodesic through $p := \pi(v)$ with initial velocity $v \in T_pM$

$d\mathscr{L}^n$ integration with respect to the Lebesgue measure in $\mathbb{R}^n$

D_T tangential cut ball, page 15

e^tX the exponential map of a Lie group, page 84

$\mathrm{End}(c)$ the smooth endomorphism fields along a curve c

$\mathrm{End}(M)$ the smooth endomorphism fields on M

$\mathcal{E}_p$ domain of definition for $\exp_p$

F_X^t maximal flow of the vector field X, page 83

$\mathfrak{g}$ a Lie algebra, page 82

G usually a Lie group or the group of deck transformations

g a Riemannian metric on M

II the second fundamental form

$\mathrm{Isom}(M)$ the isometry group of M

$\mathrm{Jac}\, F$ the Jacobian of F, page 34

$K(X \wedge Y)$ the sectional curvature of the plane spanned by X, Y

L_g left-translation with g, page 82

Lip Lipschitz constant

M smooth Riemannian n-manifold with metric g

m_κ a distance modifying function, page 24

$\mathrm{Mult}_l^k(M)$ the smooth (k, l)-multilinear fields M, page 99

NM the normal bundle of M

R_g right-translation with g, page 82

R_v curvature endomorphism along c_v , page 5

sn_κ standard solution of the Jacobi equation, page 10

S_p the geodesic reflection, page 71

t_0 cut point, page 12

t_1 conjugate point, page 12

$\mathcal{T}_l^k(M)$ the smooth tensor fields of type (k, l) on M

$T_l^k M$ the bundle of tensors of type (k, l) on M

$\mathcal{T}(M)$ the smooth vector fields on M

TM the tangential bundle of M

$U(n)$ the unitary group, page 67

$U_v(t)$ second fundamental form of geodesic sphere, page 43

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